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Some types of derivations and their applications to field theory

By

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Dedicated to Professor Y. Nakai for his sixtieth birthday

Let k be a field. Let $X_1, X_2, \cdots, X_n$ be elements in an over-field of k such that $\frac{\partial}{\partial X_i}$ have meaning. We consider derivations of $k(X_1, X_1, \cdots, X_n)$ of the forms

$$d=\frac{\partial}{\partial X_1}+\frac{1}{X_1}\cdot\frac{\partial}{\partial X_2}+\frac{1}{X_2}\cdot\frac{\partial}{\partial X_3}+\cdots+\frac{1}{X_{n-1}}\cdot\frac{\partial}{\partial X_n} \qquad (1)$$

and

$$d=\frac{\partial}{\partial X_1}+\frac{1}{X_1}\cdot\frac{\partial}{\partial X_2}+\frac{1}{X_1X_2}\cdot\frac{\partial}{\partial X_3}+\cdots+\frac{1}{X_1\cdots X_{n-1}}\cdot\frac{\partial}{\partial X_n} \qquad (2)$$

Using (1) or (2) in the case $\mathrm{ch}(k)=0$ and (2) in the case $\mathrm{ch}(k)=p\neq 0$, we prove the only if part of the following theorem. The if part is well-known (e.g. Heerema-Deveney [1]).

Theorem 1. *Let K/k be a finitely generated field extension. Then K/k is regular if and only if there exists a derivation of K whose field of constants is k when* $\mathrm{ch}(k)=0$ *and there exists a higher derivation of infinite rank of K whose field of constants is k when* $\mathrm{ch}(k)=p\neq 0$.

Next, using (2), we give an alternative proof of the only if part of the following Weisfeld's theorem in [4].

Theorem 2. *Let K/k be a purely inseparable field extension of finite exponent r where* $\mathrm{ch}(k)=p\neq 0$. *Then, K/k is modular if and only if there exists a higher derivation of rank m with $p^{r-1}<m\leqq p^r$, whose field of constants is k.*

Weisfeld defined and used the notion of non-integrable elements for higher derivations but in our method, we only use non-integrable elements of ordinary derivations. Our proof is much simpler than Weisfeld's one.

Finally we give a proof of the only if part of the following theorem, using Miyanishi's idea in [2] and Zerla's result in [5].

Theorem 3. *Let K/k be a finitely generated field extension where* $\mathrm{ch}(k)=p\neq 0$. *Then, K/k is regular if and only if there exists an iterative higher derivation of infinite rank, whose field of constants is k.*

Although the only if part of Theorem 3 is stronger than that of Theorem 1 for $\mathrm{ch}(k)=p\neq 0$, our method used in the latter is direct and common for other purposes (Theorem 1 for $\mathrm{ch}(k)=0$ and Theorem 2).

In working on these problems, Prof. Nakai, himself, gave the author precious advices, to whom the author would like to express his thanks.

1. Preliminaries.

In this section, k, K and L denote fields. k is a subfield of K and L is an overfield of K.

Lemma 1. *Let L/K be separably algebraic. Let $D=\{D^i\}_{1\leq i<m}$ be a higher derivation of K where m may be ∞ and let k be the field of constants of D. Assume that k is algebraically closed in L. Denote by D' the unique extension of D to L. Then k is the field of constants of D'.*

Proof. Let $y\in L$ be such that $D'^i y=0$ for $1\leqq i<m$. Since

$$D'^i y^l=\sum D'^{i_1}yD'^{i_2}y\cdots D'^{i_l}y \qquad (i_1+i_2+\cdots+i_l=i,\ i_k\geqq 0),$$

we have $D'^i y^l=0$ and therefore $D'^i(\alpha y^l)=(D^i\alpha)y^l$ for $\alpha\in K$, $1\leqq i<m$, $0\leqq l$. Let $f(Y)=Y^n+\alpha_1Y^{n-1}+\cdots+\alpha_n$ ($\alpha_j\in K$, $1\leqq j\leqq n$) be a minimal polynomial of y over K. Then we have

$$0=D'^i(f(y))=(D^i\alpha_1)y^{n-1}+(D^i\alpha_2)y^{n-2}+\cdots+D^i\alpha_n\,.$$

Hence $D^i\alpha_1=D^i\alpha_2=\cdots=D^i\alpha_n=0$ ($1\leqq i<m$) and we have $\alpha_j\in k$ ($1\leqq j\leqq n$). Hence y is algebraic over k and $y\in k$ by our assumption.

Lemma 2. *Let d be a derivation of K. Let k be the field of constants of d. Let H be an overfield of k contained in L, such that K and H are linearly disjoint over k. We denote by $\bar{d}$ an extension of d to $K\cdot H$ such that $\bar{d}a=0$ for $a\in H$. Then*

1) *$k\cdot H$ is a field of constants of $\bar{d}$.*
2) *If $\lambda\in K$ and $\lambda\notin d(K)$, we have $\lambda\notin\bar{d}(K\cdot H)$.*

Proof. If H is algebraic over k, our assertions follow from linear disjointness, directly. If $H=k(x)$, a purely transcendental extension, we easily obtain 1) and 2) embedding H in $k((x))$. Our proof is reduced to these two cases.

Lemma 3. *Assume that* $\mathrm{ch}(K)=0$ *and that L/K is algebraic. Let d be a derivation of K and let d' be the unique extension of d to L. Then, if $\lambda\in K$ and $\lambda\notin d(K)$, we have $\lambda\notin d'(L)$.*

Proof. Assume that there exists $y\in L$ with $d'y=\lambda$. Let $f(Y)=Y^n+\alpha_1Y^{n-1}+\cdots+\alpha_n$ be the minimal polynomial of y over K. Then it holds that

$$\begin{aligned}0=d'f(y)&=\{(d\alpha_1)y^{n-1}+(d\alpha_2)y^{n-2}+\cdots+d\alpha_n\}\\&\quad+\{ny^{n-1}+(n-1)\alpha_1y^{n-2}+\cdots+\alpha_{n-1}\}\lambda\\&=(d\alpha_1+n\lambda)y^{n-1}+\cdots\end{aligned}$$

Hence $d\alpha_1+n\lambda=0$ and $\lambda=d\left(\frac{-\alpha_1}{n}\right)\in d(K)$, a contradiction.

2. Proof of Theorem 1 for ch$(K)=0$.

We can reduce our assertion to the case where $K=k(x_1, x_2, \cdots, x_n)$, a purely transcendental extension of k with indeterminates $x_1, x_2, \cdots, x_n$, by Lemma 1. Therefore, our assertion follows from the following two lemmas.

Lemma 4. *Let x be an algebraically independent element over a field L of characteristic* 0. *Let d be a derivation of L whose field of constants is k. Let $\bar{d}$ be an extension of d to $L(x)$ such that $\bar{d}x=0$. Let ε be a derivation of $L(x)$ over L such that $\varepsilon(x)=1$. Let $\lambda\in L$ and $\lambda\notin d(L)$. We put $d'=\bar{d}+\lambda\varepsilon$.* Then, the *field of constants of d' is k.*

Proof. Let $u(x)\in L(x)$ such that $d'u(x)=0$. Then,

$$0=d'u(x)=\bar{d}u(x)+\lambda u'(x)\quad\cdots(a).$$

We put $u(x)=\frac{h(x)}{g(x)}$ with $g(x), h(x)\in L[x]$ and $(g(x), h(x))=1$. If $u=u(x)\in L$, we have $du=0$ and $u\in k$. Therefore we assume that $u(x)\notin L$. Let $\xi_1, \xi_2, \cdots\xi_l$ be roots of the equation $u'(x)=0$ in the algebraic closure $\bar{L}$ of L which are not roots of $g(x)=0$. We put $c_i=u(\xi_i)$ and take $c\in k$ such that $c\neq c_i$ for $1\leqq i\leqq l$. Since $u(x)\notin L$, there exists at least a solution $\xi\in\bar{L}$ of the equation $h(x)=cg(x)$. Then $g(\xi)\neq 0$, because $(g(x), h(x))=1$. Let d^* be the unique extension of d to $L(\xi)$. Then it holds that

$$0=d^*(u(\xi))=(\bar{d}u(x))_{x=\xi}+(u'(x))_{x=\xi}d^*\xi\quad\cdots(b).$$

By (a) and (b) we have $d^*\xi=\lambda$ which contradicts Lemma 3.

Lemma 5. *Let $x_1, x_2, \cdots, x_n$ be algebraically independent elements over a field k of characteristic* 0. *Put $K=k(x_1, x_2, \cdots, x_n)$. Let d be a derivation of the form* (1) *or* (2) *with $X_i=x_i$. Then we have*

$$\frac{1}{x_n}\notin d(K)\ \textit{in case}\ (1)\ \textit{and}\ \frac{1}{x_1x_2\cdots x_n}\notin d(K)\ \textit{in case}\ (2).$$

Proof. We give a proof for case (1). Case (2) can be proved in the same way. We put $K'=k(x_1, x_2, \cdots, x_{n-1})$ and $d=d_1+\frac{1}{x_{n-1}}\cdot\frac{\partial}{\partial x_n}$. Then $K=K'(x_n)$ and, d and d_1 map $K'[x_n]$ into $K'[x_n]$. Let $u\in K$ such that $du=\frac{1}{x_n}$. We put $u=\frac{h}{g}$ with $g, h\in K'[x_n]$ and $(g, h)=1$. Since $du=\frac{(dh)g-hdg}{g^2}=\frac{1}{x_n}$, we have $x_n|g$. Hence there exists a positive integer l such that $g=x_n^l\bar{g}$ where $\bar{g}\in K'[x_n]$ and $\bar{g}(0)\neq 0$. Then we have

$$
\begin{aligned}
du&=\frac{1}{x_n^{2l}\bar{g}^2}\{x_n^l(d_1h\cdot\bar{g}-hd_1\bar{g})+\frac{1}{x_{n-1}}(h_{x_n}x_n^l\bar{g}-hx_n^l\bar{g}_{x_n}-lhx_n^{l-1}\bar{g})\}\\
&=\frac{1}{x_n^{l+1}\bar{g}^2}\left[x_n\{(d_1h\cdot g-hd_1g)+\frac{1}{x_{n-1}}(h_{x_n}\bar{g}-h\bar{g}_{x_n})\}-\frac{1}{x_{n-1}}lh\bar{g}\right]\\
&=\frac{1}{x_n}.
\end{aligned}
$$

Since $l+1\geqq 2$, it holds that $x_n|lh\bar{g}$, a contradiction.

3. Proofs of Theorem 1 for $\mathrm{ch}(k)=p\neq 0$ and Theorem 2.

Lemma 6. *Let L/k be a field extension of characteristic $p\neq 0$. Let x be an element of an overfield of L. Assume that x is not separably algebraic over k, and L and $k(x)$ are linearly disjoint over k. Let d be a derivation of L whose field of constants is k (, therefore $k\supset L^p$). We take $\bar{d}$, ε, λ and d' as in Lemma 4. Then the field of constants of d' is $k(x^p)$.*

Proof. Assume that $d'f(x)=0$ for $f(x)\in L(x)$. We express $f(x)=f_0+f_1x+\cdots+f_{p-2}x^{p-2}+f_{p-1}x^{p-1}$, where $f_i\in L(x^p)$ $1\leqq i\leqq p-1)$. We denote by d itself the extension of d to $L(x^p)$ such that $dx^p=0$. Then we have

$$
\begin{aligned}
0=d'u(x)=&df_0+(df_1)x+\cdots+(df_{p-2})x^{p-2}+(df_{p-1})x^{p-1}\\
&+\lambda(f_1+2f_2x+\cdots+\overline{p-1}f_{p-1}x^{p-2}).
\end{aligned}
$$

Hence we have $df_{p-1}=0$, $df_{p-2}+\lambda\overline{p-1}f_{p-1}=0, \cdots, df_0+\lambda f_1=0$. From $df_{p-1}=0$ and by Lemma 2, 1) we have $f_{p-1}\in k(x^p)$. If $f_{p-1}\neq 0$, $d(-f_{p-2}/\overline{p-1}f_{p-1})=\lambda$ contradicts Lemma 2, 2). Therefore $f_{p-1}=0$. Using the same reasoning and by induction we prove $f_1=f_2=\cdots=f_{p-1}=0$ and finally $f_0\in k(x^p)$.

Lemma 7. *Let L/k be a field extension of characteristic $p\neq 0$. Let $x_1, x_2, \cdots, x_n$ be elements in an overfield of L such that the x_i are not separably algebraic over k $(1\leqq i\leqq n)$. We put $x_0=1$. Let d be a derivation of L over k. Assume that $L(x_0, x_1, \cdots, x_{i-1})$ and $k(x_i)$ are linearly disjoint over k for $i=1, 2\cdots, n$ and $1\notin d(L)$. Then for the derivation*

$$d'=d+\frac{1}{x_0}\cdot\frac{\partial}{\partial x_1}+\frac{1}{x_0x_1}\cdot\frac{\partial}{\partial x_2}+\cdots+\frac{1}{x_0x_1\cdots x_{n-1}}\cdot\frac{\partial}{\partial x_n}$$

of $L(x_0, x_1, \cdots, x_n)$, we have

$$\frac{1}{x_0 x_1 \cdots x_n} \notin d'(L(x_0, x_1, \cdots, x_n)).$$

Proof. We use induction on n. The assertion is true for $n=0$ by our assumption. Assume $n>0$. We put $L'=L(x_0, x_1, \cdots, x_{n-1})$ and $d'=d_1+\frac{1}{x_0 x_1 \cdots x_{n-1}} \cdot \frac{\partial}{\partial x_n}$. Let $u \in L(x_0, x_1, \cdots, x_n)$ be such that $d'u=\frac{1}{x_0 x_1 \cdots x_n}$. We write

$$u=u_0+u_1 x_n+\cdots+u_{p-1} x_n^{p-1}, \qquad u_i \in L'(x_n^p).$$

Then

$$\begin{aligned} d'u &= d_1 u_0+(d_1 u_1) x_n+\cdots+(d_1 u_{p-2}) x_n^{p-2}+(d_1 u_{p-1}) x_n^{p-1} \\ &\quad +\frac{1}{x_0 \cdots x_n}(u_1+2 u_2 x_n+\cdots+\overline{p-1} u_{p-1} x_n^{p-2}) \\ &= \frac{x_n^{p-1}}{x_0 x_1 \cdots x_{n-1} \cdot x_n^p}. \end{aligned}$$

Hence $d_1 u_{p-1}=\frac{1}{x_0 x_1 \cdots x_{n-1} \cdot x_n^p}$ and $d_1(x_n^p u_{p-1})=\frac{1}{x_0 x_1 \cdots x_{n-1}}$ which contradicts the induction assumption by Lemma 2, 2).

Hereafter we write as $p(r)$ instead of writing as p^r.

Proof of Theorem 1 *for* $\mathrm{ch}(k)=p \neq 0$. We can reduce our assertion to the case where $K=k(x_1, x_2, \cdots, x_n)$, a purely transcendental extension of k with indeterminates $x_1, x_2, \cdots, x_n$, by Lemma 1. We define a higher derivation $D=\{D^i\}_{1 \leq i<\infty}$ by

$$D^1 x_1=1, \quad D^2 x_1=D^3 x_1=\cdots=0$$

$$D^1 x_2=\frac{1}{x_1}, \quad D^2 x_2=D^3 x_2=\cdots=0$$

$$D^1 x_3=\frac{1}{x_1 x_2}, \quad D^2 x_3=D^3 x_3=\cdots=0$$

....

$$D^1 x_n=\frac{1}{x_1 x_2 \cdots x_{n-1}}, \quad D^2 x_n=\cdots=0.$$

D is determined by an embedding ϕ of K in the power series ring $K[[T]]$ such that $\phi(x_1)=x_1+T$, $\phi(x_2)=x_2+\frac{1}{x_1}T$, $\cdots$, $\phi(x_n)=x_n+\frac{1}{x_1 \cdots x_{n-1}}T$, that is, for an arbitary $f(x_1, x_2, \cdots, x_n) \in K$, $f+(D^1 f)T+(D^2 f)T^2+\cdots$ is the T-adic expansion of $f\left(x_1+T, x_2+\frac{1}{x_1}T, \cdots, x_n+\frac{1}{x_1 \cdots x_{n-1}}T\right)$. In order to prove our assertion, we have only to show that the field of constants of $\{D^1, D^2, \cdots, D^{p(r)-1}\}$ is $kK^{p(r)}$ for every $r \geqq 0$. We use induction on r. It is trivial for $r=0$. Assume that it is true for r. Then $D^1, D^2, \cdots, D^{p(r)-1}$ are zero maps on $kK^{p(r)}$ and $D^{p(r)}$ maps $kK^{p(r)}$ into itself, because $\psi(f(x_1^{p(r)}, x_2^{p(r)}, \cdots, x_n^{p(r)}))=f\left(x_1^{p(r)}+T^{p(r)}, x_2^{p(r)}+\frac{1}{x_1^{p(r)}}T^{p(r)}, \cdots, x_n^{p(r)}+\frac{1}{x_1^{p(r)}\cdots x_n^{p(r)}}T^{p(r)}\right)\in kK^{p(r)}[[T^{p(r)}]]$. Therefore the restriction of $D^{p(r)}$ in $kK^{p(r)}$ is a derivation and equals

$$\frac{\partial}{\partial(x_1^{p(r)})}+\frac{1}{x_1^{p(r)}}\cdot\frac{\partial}{\partial(x_2^{p(r)})}+\cdots+\frac{1}{x_1^{p(r)}\cdots x_{n-1}^{p(r)}}\cdot\frac{\partial}{\partial(x_n^{p(r)})}\,.$$

Therefore by Lemma 6 and Lemma 7, the field of constants of $D^{p(r)}$ in $kK^{p(r)}$ is $kK^{p(r+1)}$. For $f(x_1^{p(r+1)}, x_2^{p(r+1)}, \cdots, x_n^{p(r+1)})\in kK^{p(r+1)}$, $\psi(f)=f(x_1^{p(r+1)}+T^{p(r+1)}, x_2^{p(r+1)}+\frac{1}{x_1^{p(r+1)}}T^{p(r+1)}, \cdots, x_n^{p(r+1)}+\frac{1}{x_1^{p(r+1)}\cdots x_{n-1}^{p(r+1)}}T^{p(r+1)})\in kK^{p(r+1)}[[T^{p(r+1)}]]$. $D^1=D^2=\cdots=D^{p(r+1)-1}=0$ in $kK^{p(r+1)}$. Hence we are done.

Before proceeding to the proof of Theorem 2, we give

Lemma 8. *Let k be a field of characteristic $p\neq 0$. Let I be a well-ordered set without maximal element. We denote by $\iota+1$ the successor of $\iota\in I$. Let L/k be a purely inseparable extension of k and $L=\bigotimes_{\iota\in I} k(\mathcal{V}_\iota)$ where $\mathcal{V}_\iota\notin k$. We define a derivation of L over k such that $d\mathcal{V}_\iota=\mathcal{V}_\iota\mathcal{V}_{\iota+1}$. Then the constant field of d is $k(L^p)=\bigotimes_{\iota\in I} k(\mathcal{V}_\iota^p)$ and $1\notin d(L)$.*

Proof is easy. Hence we omit it.

Proof of Theorem 2. Let A be a subbasis of K. As in Weisfeld [3], we split A into $B=\{u_\iota\}_{\iota\in I}$ and $\{x_1, x_2, \cdots, x_n\}$ such that $\exp u_\iota<\exp x_i$ for every $\iota\in I$ and $i=1, 2, \cdots, n$ and there are infinitely many elements of exponent $q=\max_{\iota\in I}\exp u_\iota$ if $B\neq\emptyset$. Then well-order I so that this ordering keeps the orders of the $\exp u_i$. We put $e_\iota=\exp u_\iota$ and $e_i=\exp x_i$ ($\iota\in I$, $1\leqq i\leqq n$). We may assume that $r=e_1\geqq e_2\geqq\cdots\geqq e_n$. We consider a higher derivation $D=\{D^i\}_{1\leqq i<m}$ which is determined by the embedding ψ of K into $K[[T]]/(T^m)$, a power series ring over K modulo (T^m), such that

$$\psi(x_i)=x_i+\frac{1}{x_0^{p(e_0-e_i)}x_1^{p(e_1-e_i)}\cdots x_{i-1}^{p(e_{i-1}-e_i)}}T^{p(r-e_i)}\qquad(1\leqq i\leqq n)$$

$$\psi(u_\iota)=u_\iota+u_\iota u_{\iota+1}^{p(e_{\iota+1}-e_1)}T^{p(r-e_\iota)}\qquad(\iota\in I)\,,$$

where we put $x_0=1$ and $e_0=r+1$. Our assertion follows from the following fact. For every non-negative integer $l\leqq r$ the field of constants of $\{D^1, D^2, \cdots, D^{p(l)-1}\}$ is

$$K_l=k(\{u_\iota\}_{\iota\in I}, x_n, \cdots, x_j)\,(x_{j-1}^{p(e_{j-1}-\overline{r-l})}, \cdots, x_0^{p(e_0-\overline{r-l})})$$

if there exists j ($1\leqq j\leqq n$) such that $e_j<r-l\leqq e_{j-1}$ and otherwise

$$K_l=k(\{u_\iota\}_{\iota\in I_1},\ \{u_\iota^{p(e_\iota-\overline{r-l})}\}_{\iota\in I_2},\ x_n^{p(e_n-\overline{r-l})},\ \cdots,\ x_0^{p(e_0-\overline{r-l})}),$$

where I_1 is a subset of I consisting of elements ι such that $e_\iota<r-l$ and I_2 is the complements of I_1 in I, provided that K_l is the field of constants of $\{D^1, D^2, \cdots, D^m\}$ when $l=r$.

We prove this fact by induction on l. It is trivial when $l=0$. Assume that it is true for an l $(0\leqq l\leqq r)$. Then

$$\phi(x_i^{p(e_i-\overline{r-l})})=x_i^{p(e_i-\overline{r-l})}+\frac{1}{x_0^{p(e_0-\overline{r-l})}\cdots x_{i-1}^{p(e_{i-1}-\overline{r-l})}}T^{p(l)},$$

where $0\leqq i\leqq j-1$ in the first case and $0\leqq i\leqq n$ in the second case and

$$\phi(u_\iota^{p(e_\iota-\overline{r-l})})=u_\iota^{p(e_\iota-\overline{r-l})}+u_\iota^{p(e_\iota-\overline{r-l})}u_{\iota+1}^{p(e_{\iota+1}-\overline{r-l})}T^{p(l)}$$

for $\iota\in I_2$ in the second case and

$$\phi(x_i)=x_i+\frac{1}{x_0^{p(e_0-e_i)}\cdots x_{i-1}^{p(e_{i-1}+e_i)}}T^{p(r-e_i)},\qquad r-e_i>l$$

and

$$\phi(u_\iota)=u_\iota+u_\iota u_{\iota+1}^{e_{\iota+1}-e_\iota}T^{p(r-e_\iota)},\ r-e_\iota>l$$

for other i's and ι's. Therefore $D^{p(l)}$ induces a derivation of K_l over $k(\{u_\iota\}_{\iota\in I}, x_n, \cdots, x_j)$ in the first case and over $k(\{u_\iota\}_{\iota\in I_1})$ in the second case. We can apply Lemma 6, Lemma 7 and Lemma 8 to this induced derivation and it is easily seen that the field of constants of this derivation is K_{l+1} and $D^{p(l)}, \cdots, D^{p(l+1)-1}$ ($D^{p(l)}, \cdots, D^m$ in case $l=r-1$) are zero maps on K_{l+1}.

Remark. In this section, we used derivations of the form (2) but we cannot replace it by those of the form (1), which will be shown by the following counter example.

Example. Let k be a filed of characteristic $p\neq 0$. $x, y, z,$ are independent variables over k.
For the quation

$$\frac{\partial f}{\partial x}+\frac{1}{x}\cdot\frac{\partial f}{\partial y}=\frac{1}{y},$$

we have solutions $f\in k(x, y)$ such that

$$f=c+\frac{(p-1)!}{y^p}x+\frac{(p-2)!}{y^p}xy+\cdots+\frac{1!}{y^p}xy^{p-2}+\frac{0!}{y^p}xy^{p-1}$$

with $c\in k$. And the $u=f-z\in k(x, y, z)$ are solutions for

$$\frac{\partial u}{\partial x}+\frac{1}{x}\cdot\frac{\partial u}{\partial y}+\frac{1}{y}\cdot\frac{\partial u}{\partial z}=0.$$

4. Proof of theorem 3.

Zerla proved in [5] our assertion when the cardinality of k is countable. Therefore we may assume that k contains infinitely many algebraically independent elements $\{\alpha_{ij} | 2 \leqq i \leqq n, 0 \leqq j < \infty\}$ over the prime field, where n=trans. $\deg_k K$. Our assertion is also reduced to the case where K is a purely transcendental extension $K = k(x_1, x_2, \cdots, x_n)$ over k (e.g. [5], Lemma 5). We use Miyanishi's idea in [2]. In the quotient $k((x))$ of the formal power series ring over k, n elements $y_1 = x$, $y_i = \sum_{j=0}^{\infty} \alpha_{ij} x^{p(j)}$ $(2 \leqq i \leqq n)$ are algebraically independent over k. Let χ be an embedding of K in $k((x))$ such that $\chi(x_i) = y_i$ $(1 \leqq i \leqq n)$. We consider the standard iterative higher derivation D of $k((x))$, which is determined by the embedding ψ of $k((x))$ in the formal power series ring $k((x))[[T]]$ such that $\psi(c) = c$ $(c \in k)$ and $\psi(x) = x + T$. Then the field of constants of D is k. D induces an iterative higher derivation of K through χ whose field of constants is k.

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