Radiation conditions and spectral theory for 2-body Schrödinger operators with "oscillating" long-range potentials III

By

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Introduction

In the recent works ([1], [2], [3]) we have developed a spectral theory for the Schrödinger operators \(-\mathcal{A}+V(x)\), in an exterior domain \(\Omega\) of \(\mathbb{R}^n\), with some real "oscillating" long-range potentials \(V(x)\). Roughly speaking, \(V(x)\) is called "oscillating" long-range if it behaves as \(r=|x|\to\infty\) like

\[
V(x) = O(1), \quad \partial_r V(x) = O(r^{-1}) \quad (\partial_r = \partial/\partial r) \quad \text{and}
\]

\[
\partial_r^2 V(x) + a V(x) = O(r^{-1-\delta}) \quad \text{for some } a \geq 0 \quad \text{and} \quad \delta > 0.
\]

For example, the potential

\[
V(x) = \frac{c \sin br}{r} \quad (b, c \text{ are non-zero real})
\]

satisfies the above conditions with \(a=b^2\) and \(\delta=1\).

In this paper we shall modify our previous results to the case of potentials which consist of the sum of several "oscillating" long-range potentials. Note that the last condition of (0.1) is not satisfied by the potential

\[
V(x) = \frac{c_1 \sin b_1 r}{r} + \frac{c_2 \sin b_2 r}{r}
\]

\((b_1, b_2, c_1, c_2 \text{ are non-zero real})\) unless \(b_1=b_2\). So the results of [2] and [3] are not directly applied to this type of potentials, and it is necessary to make some modification.

For this purpose we return to the semi-abstract theory developed in the first half of [2], where we gave a sufficient condition under which the principle of limiting absorption are justified for the exterior boundary-value problem

\[
\begin{cases}
\{ -\mathcal{A}+V(x)-\xi \} u = f(x) & \text{in } \Omega \\
Bu = \begin{cases} u \quad \text{or} \\
\nu \cdot F u + d(x) u \end{cases} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(0.2)
Here ζ is a complex number, ν=(ν₁, ⋯, νₙ) is the outer unit normal to the boundary ∂Ω, F is the gradient in ℝⁿ and d(x) is a real-valued smooth function on ∂Ω.

The condition is summarized as follows:

Assumption 1 ([2]). There exist real constants δ>0, Aδ and a real function γ(λ) of λ>Λδ such that

\begin{equation}
0<γ(λ)<\min\{4δ, 2\}
\end{equation}

and the following growth property holds: Let \( u \in H^{1}_{loc}(\overline{Ω}) \) satisfy

\begin{equation}
(−Δ+V(x)−λ)u=0 \quad \text{in} \quad Ω
\end{equation}

with λ>Λδ. If we have the inequality

\begin{equation}
\int_{B(R_0)} (1+r)^{−1+β} |u|^2 dx < ∞
\end{equation}

for some β>γ(λ)/2 and \( R_0 > 0 \), where \( B(R_0) = \{ x \in Ω ; |x| > R_0 \} \), then \( u \) must identically vanish in Ω.

Assumption 2 ([2]). Let

\begin{equation}
\Pi_δ^z = \{ζ=λ±ir ∈ C ; λ > Λ_δ \quad \text{and} \quad r ≥ 0\}.
\end{equation}

and let \( K^z \) be a compact set in \( Π_δ^z \). Then there exists an \( R_1 = R_1(K^z) > R_0 \) and a complex-valued function \( k(x, ζ) = k(x, λ \pm ir) \) which is continuous in \( (x, ζ) ∈ B(R_1) × K^z \) and satisfies the following conditions: for any \( (x, ζ) ∈ B(R_1) × K^z \)

\begin{align}
(A2-1) \quad &|V(x)−ζ+∂_k(x, ζ)+\frac{n−1}{r}k(x, ζ)−k(x, ζ)| ≤ C_1 r^{−1−α}, \\
(A2-2) \quad &|k(x, ζ)| ≤ C_2, \\
(A2-3) \quad &\mp\text{Im} \quad k(x, λ \pm ir) ≥ C_3, \\
(A2-4) \quad &\text{Re} \quad k(x, ζ)−\frac{n−1−β}{2r} ≥ C_4 r^{−1}. \\
(A2-5) \quad &|v(x)−k(x, ζ)| ≤ C_5 r^{−1−α} \quad (x = x/|x|).
\end{align}

Here \( C_j = C_j(K^z) > 0 (j=1\sim5) \) and \( β = β(K^z) > 0 \) is chosen as follows:

\begin{equation}
(A2-6) \quad γ(λ)/2 < β < 2δ \quad \text{for any} \quad ζ=λ \pm ir ∈ K^z \quad \text{and} \quad β ≤ 1.
\end{equation}

We shall show that the above assumptions can be verified for a class of potentials consisting of the sum of several “oscillating” long-range potentials.

The main results of this paper will be summarized in §1 in three theorems. Theorem 1 which asserts the growth property of solutions of (0.4) is a consequence of [1]. Theorem 2 summarizes results concerning the principle of limiting absorption. We shall prove it in §2. Theorem 3 summarizes results concerning spectral representations for the selfadjoint realization of \(-Δ+V(x)\) in the Hilbert space \( L^2(Ω) \). An outline of the proof will be given in §3 (we can...
follow the same line of proof of [3]). Finally, in § 4 we shall give several examples.

§ 1. Conditions and results

Let \( Q \) be an infinite domain in \( \mathbb{R}^n \) with smooth compact boundary \( \partial Q \) lying inside some sphere \( S(R_0) = \{ x ; |x| = R_0 \} \). We consider in \( Q \) the Schrödinger operator \(-\Delta + V(x)\), where \( \Delta \) is the Laplacian and \( V(x) \) is a potential function of the form

\[
V(x) = V_1(x) + V_s(x) = \sum_{j=1}^{m} V_{1j}(x) + V_s(x).
\]

\( V_s(x) \) is a short-range potential and the \( V_{1j}(x) \) are "oscillating" long-range potentials. More precisely, we assume:

(V1) \( V_1(x) \) is a real-valued function belonging to a Stummel class \( Q_\mu(\mu > 0) \) and \( V_s(x) \) is a real-valued bounded measurable function in \( Q \). Moreover, the unique continuation property holds for both \(-\Delta + V(x)\) and \(-\Delta + V_1(x)\).

(V2) For some \( 0 < \delta_i \leq 1 (i = 0, 1, 2) \) and \( 0 < \varepsilon_j \leq 1 (j = 1, \ldots, m) \),

(i) \( V_{1j}(x) = 0(r^{\gamma_j}) \),

(ii) \( \partial_\nu V_{1j}(x) = 0(r^{\gamma_j}) \),

(iii) \( \partial_\nu^2 V_{1j}(x) + a_j(r) V_{1j}(x) = 0(r^{-\delta_j}) \),

(iv) \( (\mathcal{P} - \varepsilon \partial_\nu) V_{1j}(x) = 0(r^{-\delta_j}) \),

(v) \( (\mathcal{P} - \varepsilon \partial_\nu) \partial_\nu V_{1j}(x) = 0(r^{-\delta_j}) \),

(vi) \( (\mathcal{P} - \varepsilon \partial_\nu)^2 (\mathcal{P} - \varepsilon \partial_\nu) V_{1j}(x) = 0(r^{-\delta_j}) \),

(vii) \( V_s(x) = 0(r^{-\delta_j}) \)

as \( r = |x| \to \infty \), where the \( a_j(r) (j = 1, \ldots, m) \) are non-negative functions of \( r > R_0 \) satisfying

\[
a_i(r) \geq a_2(r) \geq \cdots \geq a_m(r) \geq 0,
\]

\[
a_j(r) = 0(r^{-\gamma_j}), \quad a_j'(r) = \frac{d}{dr} a_j(r) = 0(r^{-\gamma_j})
\]

and

\[
a_j''(r) = \frac{d^2}{dr^2} a_j(r) = 0(r^{-\delta_j}),
\]

with \( \gamma_j = \max \{ 0, 1 + \delta_j - \varepsilon_j - \min \{ \varepsilon_j \} \} \) and \( \mu_j \) such that \( \mu_j \geq \delta_j \) and \( \mu_j \geq 1 - \varepsilon_j \).

Remark 1.1. (V2-i) is stronger than the corresponding condition required in [2] and [3] (cf., (0.1)). (V2-vi) is used only to show (e) of Theorem 3 stated below (cf., [3]).

Lemma 1.1. If \( \delta_i \) and \( \{ \varepsilon_j \} \) satisfy \( \max \{ \varepsilon_j \} - \min \{ \varepsilon_j \} \leq 1 - \delta_i , 0 < \delta_i , \varepsilon_j \leq 1 , \) and

\[
a_j(r) = 0(r^{-\gamma_j}), \quad a_j'(r) = 0(r^{-1}) \text{ and } a_j''(r) = 0(r^{-2}),
\]
Proof. Obvious from $2-2\varepsilon_j \geq 1+\delta_1-\varepsilon_j - \min \{\varepsilon_j\}, 1 \geq \delta_1, 1 > 1 - \varepsilon_j$ and $2 > 1 + \delta_1 - \varepsilon_j$. q.e.d.

**Lemma 1.2.** (cf., [2]; Remark 8.5). If $\delta_i$ and $\{\varepsilon_j\}$ are as above, and

$$a_j(r) = 0(r^{-2+\varepsilon_j})$$

for $a_j(r)$ satisfies the condition (1.3).

Proof. Obvious from $2-2\varepsilon_j \geq 1+\delta_1-\varepsilon_j - \min \{\varepsilon_j\}, 1 \geq \delta_1, 1 > 1 - \varepsilon_j$ and $2 > 1 + \delta_1 - \varepsilon_j$. q.e.d.

In the following we put $\delta = \min \{\delta_0, \delta_1, \delta_2\}$ and

$$a^*_j = \lim_{r \to \infty} a_j(r)(\geq 0).$$

Then we have by (1.2)

$$a^*_j \geq a^*_1 \geq \cdots \geq a^*_m \geq 0.$$

For $V_{i,j}(x)$ satisfying (V2-i) and (V2-ii) we put

$$E(\gamma) = \frac{1}{\gamma} \lim_{r \to \infty} \sup \{\sum_{j=1}^m r \partial_{x_{1,j}} V_{i,j}(x)\}$$

Then obviously (cf., [2]; Lemma 8.2) $E(\gamma) \geq \lim_{r \to \infty} V_i(x) = 0$ and we have for $0 < \gamma \leq 2$,

$$0 \leq E(2) \leq E(\gamma) \leq \frac{1}{\gamma} \lim_{r \to \infty} \sup \{\sum_{j=1}^m r \partial_{x_{1,j}} V_{i,j}(x)\} \leq \frac{1}{\gamma} \sum_{j=1}^m \lim_{r \to \infty} \sup |r \partial_{x_{1,j}} V_{i,j}(x)| < \infty.$$

The following theorem is already proved in [1] (Theorems 1, 2 and Remark 2).

**Theorem 1.** Suppose that $V(x)$ satisfies (V1), (V2-i), (V2-ii) and (V2-vii). Then we have:

(a) Let $\lambda > E(2)$ and $u \in H^1_{rad}(\Omega)$ be a not identically vanishing solution of $\{ -D + V(x) - \lambda \} u = 0$ in $\Omega$. Then for any $\tilde{\gamma}$ such that

$$0 < \tilde{\gamma} < 2 \quad \text{and} \quad E(2) \leq E(\tilde{\gamma}) < \lambda,$$

we have

$$\lim_{R \to \infty} R^{-1+\tilde{\gamma}/2} \int_{R \rho < |x| < R} |u(x)|^2 dx \to \infty.$$  

(b) Any selfadjoint realization of $-D + V(x)$ in $L^2(\Omega)$ has no eigenvalue in $(E(2), \infty)$.

We put for $\sigma > 0$

$$A_\sigma = \frac{1}{\min \{4\sigma, 2\}} \max_{1 \leq i \leq m} \lim_{r \to \infty} \sup \left\{ \sum_{j=1}^m r \partial_{x_{1,j}} V_{i,j}(x) \right\} + \frac{1}{4} a^*_1.$$
and

\[ P^*_\delta = \{ \zeta = \lambda \pm i \tau : \lambda > A_\delta \text{ and } \frac{1}{2} (\lambda - \frac{1}{4} a^*_j) \geq \tau \geq 0 \} . \]

For \( \zeta \in P^*_\delta \) let

\begin{align*}
\eta_j(r) &= \frac{4\zeta}{4\zeta - a_j(r)} \quad \text{and} \\
\eta_j(r) \cdot V_\lambda(x) &= \sum_{j=1}^n \eta_j(r)V_{ij}(x).
\end{align*}

**Lemma 1.3.** Let \( K^* \) be any compact set of \( P^*_\delta \). Then there exist \( R_\delta = R_\delta(K^*) > R_0 \) and \( C = C(K^*) > 1 \) such that

\[ 0 \leq \pm \text{Im} \{ \zeta - \eta(r) \cdot V_\lambda(x) \} \leq C, \]

\[ C^{-1} \leq \text{Re} \{ \zeta - \eta(r) \cdot V_\lambda(x) \} \leq C \]

for any \( (x, \zeta) \in B(R_\delta) \times K^* \).

**Proof.** It follows from (1.7) and (1.8) that

\[ \text{Im} \{ \zeta - \eta(r) \cdot V_\lambda(x) \} = \text{Im} \zeta \left\{ 1 + \sum_{j=1}^n \frac{4a_j(r)V_{ij}(x)}{|4\zeta - a_j(r)|^2} \right\}, \]

\[ \text{Re} \{ \zeta - \eta(r) \cdot V_\lambda(x) \} = \text{Re} \zeta - V_\lambda(x) - \sum_{j=1}^n \text{Re} \frac{a_j(r)V_{ij}(x)}{4\zeta - a_j(r)}. \]

Here by (v2-i), (1.5) and (1.6),

\[ \lim_{r \to \infty} V_{ij}(x) = 0 \quad \text{and} \quad \lim_{r \to \infty} \frac{1}{4} a^*_j \geq \frac{1}{4} \lim_{r \to \infty} a_j(r) \quad \text{for any } j. \]

Thus, noting the boundedness of \( a_j(r) \), we have the assertion of the lemma.

**Lemma 1.4.** We have for any \( \lambda > A_\delta \),

\[ 0 < \gamma(\lambda) \leq \min \{ 4\delta, 2 \}, \]

\[ E(\gamma(\lambda)) \leq \frac{1}{\gamma(\lambda)} \max_{l \leq m} \limsup_{r \to \infty} \left\{ \sum_{j=1}^n \gamma \partial_r V_{ij}(x) \right\} + \frac{1}{4} a^*_j \]

\[ \leq \frac{1}{2} (A_\delta + \lambda) \leq \lambda. \]

**Proof.** The first assertion is obvious from the definition of \( \gamma(\lambda) \). If \( A_\delta = 0 \), the second assertion is also obvious since \( a^*_j = 0 \) and \( \limsup_{r \to \infty} \left\{ \sum_{j=1}^n r \partial_r V_{ij}(x) \right\} = 0 \) \((l=1, \ldots, m)\). On the other hand, if \( A_\delta > 0 \), we have noting (1.6)
\[
\frac{1}{\gamma(\lambda)} \max_{1 \leq j \leq m} \limsup_{r \to 0} \left\{ \sum_{j=1}^{\infty} r \partial_j V_{ij} \right\} + \frac{1}{4} a^*_r = \frac{\lambda^* + \lambda}{2\lambda} \left( A_0 - \frac{1}{4} a^*_r \right) + \frac{1}{4} a^*_r \leq \frac{1}{2} (A_0 + \lambda). \]

The lemma is proved.

For \( \mu \in \mathbb{R} \) and \( G \subseteq \Omega \), let \( L^+_\mu(G) \) denote the space of all functions \( f(x) \) such that

\[
\| f \|_{\mu, \mathbb{R}} := \int_\mathbb{R} (1 + r)^{2\mu} |f(x)|^2 \, dx < \infty.
\]

If \( \mu=0 \) or \( G=\Omega \), the subscript \( \mu \) or \( G \) will be omitted. Let \( k(x, \zeta) \) \( (x, \zeta) \in \mathbb{B}(R_0) \times K^* \), be defined by

\[
k(x, \zeta) = -i \sqrt{\zeta - \eta(r)} \cdot V_1(x) + \frac{n-1}{2r} + \frac{-\partial \{ \eta(r) \cdot V_1(x) \}}{4(\zeta - \eta(r)) \cdot V_1(x)},
\]

where \( R_0 = R_3(\zeta) \) and we take the branch \( \text{Im} \sqrt{-1} \geq 0 \).

**Definition.** For solutions \( u \in H^1_{\text{loc}}(\Omega) \) of (0.2) with \( \zeta \in \Pi_0^\delta \), the outgoing (+) [or incoming (−)] radiation condition at infinity is defined by

\[
u \in L^2_{(-1, \alpha/2)}(\Omega) \quad \text{and} \quad \partial_\nu + \kappa(x, \zeta) u \in L^2_{(-1, \alpha/2)}(\mathbb{B}(R_0)),
\]

where \( \alpha = \alpha(\zeta), \beta = \beta(\zeta) \) is a pair of positive constants such that

\[
0 < \alpha \leq \beta \leq 1, \quad \frac{1}{2} \gamma(\text{Re} \zeta) < \beta < 2\delta \quad \text{and} \quad \alpha \leq 2\delta - \beta.
\]

A solution \( u \) of (0.2) which also satisfies the radiation condition (1.12.) [or (1.12.)] is called an outgoing [incoming] solution.

We are now ready to state two theorems concerning the principle of limiting absorption and spectral representations for the Schrödinger operator \(-D + V(x)\).

**Theorem 2.** Suppose that \( V(x) \) satisfies (V1), (V2-i)~(V2-v) and (V2-vii). Then we have:

(a) Let \( K^* \) be a compact set of \( \Pi_0^\delta \), let \( \gamma(K^*) = \max_{\zeta \in K^*} \gamma(\lambda) (\lambda = \text{Re} \zeta) \), and let \( \alpha = \alpha(K^*), \beta = \beta(K^*) \) be a pair satisfying (1.13) with \( \gamma(\lambda) \) replaced by \( \gamma(K^*) \). Then for any \( \zeta = \lambda \pm i\tau \in K^* \) and \( f \in L^2_{(-1, \beta/2)}(\mathbb{B}(R_0)) \), (0.2) has a unique outgoing [incoming] solution \( u = u^-(x, \lambda \pm i\tau) = R_{\alpha, \beta} f \), which also satisfies the inequalities

\[
\| u \|_{(-1, \alpha/2)} \leq C \| f \|_{(1, \beta/2)} , \quad \| \Gamma u + \kappa(x, \lambda \pm i\tau) u \|_{(-1, \beta/2) B(R_0)} \leq C \| f \|_{(1, \beta/2)} ,
\]

where \( C = C(K^*) > 0 \) and \( R_{\alpha} = R_\alpha(K^*) \geq R_\alpha(K^*) \) are independent of \( f \).

(b) Let \( R^+_\alpha : L^1_{-\alpha/2}(\Omega) \to L^1_{-\beta/2}(\Omega) \) be the adjoint of \( R_\alpha \). Then

\[
R^+_\alpha f = R_{\alpha, \beta} f \quad \text{for} \quad f \in L^1_{(-1, \beta/2)}(\Omega).
\]

(c) \( u = R_\alpha f \) is continuous in \( L^1_{(-1, \alpha/2)}(\Omega) \) with respect to \( (\zeta, f) \in K^* \times L^1_{(1, \beta/2)}(\Omega) \).
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(d) Let \( L \) be the selfadjoint operator in \( L^2(\Omega) \) defined by
\[
\begin{aligned}
D(L) &= \{ u \in H^2(\Omega) ; \ Bu \big|_{\partial \Omega} = 0 \} \\
Lu &= -Lu + V(x)u \quad \text{for} \quad u \in D(L),
\end{aligned}
\]
and let \( \mathcal{E}(\lambda) ; \lambda \in \mathbb{R} \) be its spectral measure. Then for any Borel set \( e \subseteq (A_\delta, \infty) \), \( f \in L^p_{1+\beta}/S_{1+\beta}(\Omega) \) and \( g \in L^p_{1+\alpha}/S_{1+\alpha}(\Omega) \) (\( \alpha, \beta \) is chosen as above with \( K^* = \overline{e} \)) we have
\[
(\mathcal{E}(\lambda)f, g) = \frac{1}{2\pi i} \int_{\mathbb{R}} \langle (R_{2-i} - R_{2-i}f), g \rangle d\lambda,
\]
where \( (\ , \ ) \) denotes the inner product in \( L^2(\Omega) \), or more generally, the duality between \( L^p_{1+\alpha}/S_{1+\alpha}(\Omega) \) and \( L^p_{1+\alpha}/S_{1+\alpha}(\Omega) \). Namely, the part of \( L \) in \( \mathcal{E}(\lambda; (A_\delta, \infty))L^2(\Omega) \) is absolutely continuous with respect to the Lebesgue measure.

Theorem 3. Suppose that \( V(x) \) satisfies (V1) and (V2) with \( \delta > 1/2 \) (\( s = 1, 2 \)). Then we have:

(a) For any \( \varepsilon > 0 \) and \( \lambda \geq A_\delta + \varepsilon \) there exist bounded linear operators \( \mathcal{F}_\varepsilon(\lambda) : L^2_{1+\beta}/S_{1+\beta}(\Omega) \to L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \) such that each \( \mathcal{F}_\varepsilon(\lambda)f \in L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \) depends continuously on \( (\lambda, f) \subseteq (A_\delta + \varepsilon, \infty) \times L^2_{1+\beta}/S_{1+\beta}(\Omega) \), and the following relations hold:
\[
(\mathcal{F}_\varepsilon(\lambda)f, \mathcal{F}_\varepsilon(\lambda)g)_{L^2((A_\delta + \varepsilon, \infty) \times S^{n-1})} = \frac{1}{2\pi i} \int (R_{2+i}f - R_{2-i}f, g) \, d\lambda,
\]
\[
(\mathcal{E}(\lambda)f, g) = \frac{1}{2\pi i} \int (\mathcal{F}_\varepsilon(\lambda)f, \mathcal{F}_\varepsilon(\lambda)g)_{L^2((A_\delta + \varepsilon, \infty) \times S^{n-1})} \, d\lambda, \quad e \subseteq (A_\delta + \varepsilon, \infty).
\]

(b) The operators \( \mathcal{F}_\varepsilon : L^2_{1+\beta}/S_{1+\beta}(\Omega) \to L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \) defined by \( \mathcal{F}_\varepsilon f(x) = [\mathcal{F}_\varepsilon(\lambda)f(x)](\cdot) \) can be uniquely extended by continuity to partial isometric operators from \( L^2(\Omega) \) into \( L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \) with initial set \( \mathcal{E}(\lambda; (A_\delta + \varepsilon, \infty))L^2(\Omega) \).

(c) For any bounded Borel function \( b(\lambda) \) on \( (A_\delta + \varepsilon, \infty) \), the following relation holds:
\[
\mathcal{F}_\varepsilon b(L) = b(\lambda) \mathcal{F}_\varepsilon.
\]

(d) Let \( \mathcal{F}_\varepsilon^* : L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \to L^2(\Omega) \) be the adjoint operators of \( \mathcal{F}_\varepsilon \). Then each \( \mathcal{F}_\varepsilon^* \) admits the representation
\[
\mathcal{F}_\varepsilon^* f = \text{strong lim} \sum_{\lambda \in \mathbb{R}} \mathcal{F}_\varepsilon(\lambda)^* f(\lambda) d\lambda \quad \text{in} \quad L^2(\Omega)
\]
for any \( f \in L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \), where \( \mathcal{F}_\varepsilon(\lambda)^* : L^2(S^{n-1}) \to L^2_{1+\beta}/S_{1+\beta}(\Omega) \) is the adjoint of \( \mathcal{F}_\varepsilon(\lambda) \).

(e) Let \( \delta = \min\{\delta, 2\delta_0 - 1\} \). Then each \( \mathcal{F}_\varepsilon \) maps \( \mathcal{E}(\lambda; (A_\delta + \varepsilon, \infty))L^2(\Omega) \) onto \( L^2((A_\delta + \varepsilon, \infty) \times S^{n-1}) \), that is, \( \mathcal{F}_\varepsilon \) restricted on \( \mathcal{E}(\lambda; (A_\delta + \varepsilon, \infty))L^2(\Omega) \) is a unitary operator.

Remark 1.2. In [3] we neglect the fact that the construction of \( \mathcal{F}_\varepsilon(\lambda) \) depends in general on \( \varepsilon \). So the above is a correction of [3]. Note that for two \( \varepsilon, \varepsilon' > 0 \) and \( \lambda > A_\delta + \max\{\varepsilon, \varepsilon'\} \), the corresponding operators \( \mathcal{F}_\varepsilon(\lambda) \) and \( \mathcal{F}_{\varepsilon'}(\lambda) \) are unitary equivalent to each other in \( L^2(S^{n-1}) \).
§ 2. Proof of Theorem 2

As we see in [2] (Theorems 1–5), all the assertions of Theorem 2 hold true under Assumptions 1 ([2]) and 2 ([2]) stated in the introduction of this article. So to complete the proof, we have only to check that these assumptions are satisfied by $A_\delta$, $\gamma(\lambda)$ and $k(x, \zeta)$ given in the previous section.

Assumption 1 ([2]) directly follows from Theorem 1 and the definition of $A_\delta$ and $\gamma(\lambda)$. In fact, let $\lambda > A_\delta$ and $u$ satisfy (0.4) and (0.5) for some $\beta > \gamma(\lambda)/2$ (without loss of generality we can assume $\beta \leq 1$). Then we have

$$R^{-1+\gamma(\lambda)/2} \int_{R_0 | x | < R} |u(x)|^2 dx \leq R^{-\beta + \gamma(\lambda)/2} \int_{R_0 | x | < R} r^{-1+\beta} |u(x)|^2 dx \to 0 \text{ as } R \to \infty.$$ 

Since $\tilde{\gamma} = \gamma(\lambda)$ satisfies the condition of Theorem 1 (a) by Lemma 1, this and Theorem 1 (a) lead $u=0$.

Next let us verify Assumption 2 ([2]) for $k(x, \zeta)$ restricted in $(x, \zeta) \in B(R_\delta) \times K^*$, where $K^*$ is any compact set in $\Pi_{2s}^*$, and $R_\delta = R_\delta(K^*)$ is what is given in Lemma 1.3. We choose $\gamma(K^*)$ and $\beta(K^*)$ as in (a) of Theorem 2. Then (A2-6) is obviously satisfied by this $\beta = \beta(K^*)$. Further, the continuity of $k(x, \zeta)$ in $(x, \zeta) \in B(R_\delta) \times K^*$ is easily known by Lemma 1.3. So it remains only to verify (A2-1)–(A2-5).

For this purpose we prepare a lemma.

**Lemma 2.1.** There exist some constant $C > 0$ such that for any $(x, \zeta) \in B(R_\delta) \times K^*$ we have

$$|\gamma_j(r)V_{1j}(x)\{1-\gamma_k(r)\}V_{1k}(x)| \leq Cr^{-1-\delta} (j, k = 1, \ldots, m),$$

$$|\partial_{s_0} \{\gamma(r) \cdot V_1(x)\}| \leq Cr^{-1},$$

$$|\partial_{s_0}^2 \{\gamma_j(r)V_{1j}(x)\} + \gamma_j(r)a_j(r)V_{1j}(x)| \leq Cr^{-1-\delta},$$

$$|(\mathcal{F} - \tilde{x}\partial_{s_0})\{\gamma(r) \cdot V_1(x)\}| \leq Cr^{-1-\delta},$$

$$|(\mathcal{F} - \tilde{x}\partial_{s_0})\partial_{s_0} \{\gamma(r) \cdot V_1(x)\}| \leq Cr^{-1-\delta}.$$

**Proof.** The above inequalities respectively follow from (V2-i)–(V2-v) if we note that

$$\gamma_j(r) = 0(1), 1 - \gamma_j(r) = 0(r^{-s_j}), \gamma_j'(r) = 0(r^{-s_j}) \quad \text{and}$$

$$\gamma''(r) = 0(r^{-1-\delta_1+i\delta_2}) + 0(r^{-2\mu_0}) = 0(r^{-1-\delta+i\delta_2}). \quad \text{q. e. d.}$$

Now, (A2-2) and (A2-3) easily follow from Lemma 1.3 and Lemma 2.1 since we have

$$|k(x, \zeta)| \leq |\sqrt{\zeta - \gamma(r)} \cdot V_1(x)| + \frac{n-1}{2} + \left| \frac{-\partial_{s_0} \{\gamma(r) \cdot V_1(x)\}}{4(\zeta - \gamma(r)) \cdot V_1(x)} \right|,$$

$$\mp \text{Im } k(x, \zeta) = \pm \text{Re } \sqrt{\zeta - \gamma(r)} \cdot V_1(x) \mp \text{Im } \left[ \frac{-\partial_{s_0} \{\gamma(r) \cdot V_1(x)\}}{4(\zeta - \gamma(r)) \cdot V_1(x)} \right].$$
Note that
\[
(P - \dot{x}, \partial_x)(x, \zeta) = \frac{i(P - \dot{x}, \partial_x)(\eta \cdot V_1)}{2\sqrt{\zeta - \eta \cdot V_1}} + \frac{(P - \dot{x}, \partial_x)(\eta \cdot V_1)}{4(\zeta - \eta \cdot V_1)^2} - \frac{\partial_i(\eta \cdot V_1)(P - \dot{x}, \partial_x)(\eta \cdot V_1)}{4(\zeta - \eta \cdot V_1)^3}.
\]

Then (A2-5) also follows from Lemma 2.1.

Next we prove (A2-1). It follows from (1.11) that (cf., Appendix of [2])
\[
V(x) - \zeta + \eta + \frac{n-1}{r} - k - k^2
\]
\[= V_1 - \eta \cdot V_1 + \frac{-\partial_1(\eta \cdot V_1)}{4(\zeta - \eta \cdot V_1)} + \frac{(n-1)(n-3)}{4r^2} - \frac{5(\partial_i(\eta \cdot V_1))^2}{16(\zeta - \eta \cdot V_1)^2} + V_i.
\]

Since we have
\[
V_1 - \eta \cdot V_1 + \frac{-\partial_1(\eta \cdot V_1)}{4(\zeta - \eta \cdot V_1)} = \frac{1}{4(\zeta - \eta \cdot V_1)} \left[ \sum_{j=1}^n \left( \frac{4\zeta(1 - \eta_j)V_{ij} - \partial_1(\eta_j V_{ij})}{4(1 - \eta_j)\eta_j V_{ij}} \right) \right]
\]
and
\[
4\zeta(1 - \eta_j)V_{ij} - \partial_1(\eta_j V_{ij}) = - \left( \partial_1(\eta_j V_{ij}) + \eta_j a_j(r) V_{ij} \right),
\]

(A2-1) follows from Lemma 2.1 and (V2-vii).

Lastly, we prove (A2-4).
\[
\text{Re} k(x, \zeta) - \frac{n-1 - \beta}{2r} = \frac{1}{4} \left\{ \text{Re} \left[ -\frac{\partial_1(\eta \cdot V_1)}{\zeta - \eta \cdot V_1} \right] + \frac{r}{\zeta} \right\} + \text{Im} \sqrt{\zeta - \eta \cdot V_1} + \frac{2\beta - \gamma}{4r}.
\]

Since $2\beta > \gamma$ by definition and $\text{Im} \sqrt{\zeta - \eta \cdot V_1} \geq 0$, we have only to show that there exists an $R_x = R_x(K^*) \supseteq R_x$ such that
\[
(2.1) \quad \text{Re} \left[ -\frac{\partial_1(\eta \cdot V_1)}{\zeta - \eta \cdot V_1} \right] + \frac{r}{\zeta} \geq 0 \quad \text{for any} \quad (x, \zeta) \in B(R_x) \times K^*.
\]

We have
\[
\frac{r |\zeta - \eta \cdot V_1|^2}{\gamma |\zeta|^2} \left\{ \text{Re} \left[ -\frac{\partial_1(\eta \cdot V_1)}{\zeta - \eta \cdot V_1} \right] + \frac{r}{\zeta} \right\}
\]
\[= \frac{|\zeta - \eta \cdot V_1|^2}{|\zeta|^2} \text{Re} \left[ \frac{\zeta - (r/\gamma) \partial_1(\eta \cdot V_1) - \eta \cdot V_1}{\zeta - \eta \cdot V_1} \right]
\]
\[= 1 - \frac{1}{\gamma} \text{Re} \left\{ \sum_{j=1}^n \frac{r \partial_j V_{ij}(x)}{\zeta - (1/4) a_j(r)} \right\} + |\zeta|^{-r} I;
\]
\[I = \text{Re} \left\{ \sum_{j=1}^n \frac{r \eta_j V_{ij}}{\zeta - (1/4) a_j(r)} + \frac{r}{\zeta} \partial_i(\eta \cdot V_i) \eta \cdot V_i - 2 \text{Re} \left[ \frac{\zeta \eta \cdot V_i}{|\zeta \cdot V_i|^2} + |\eta \cdot V_i|^2 \right] \right\}.
\]

Noting $K^* \subseteq \Pi^*_R$, we can choose a constant $\epsilon > 0$ to satisfy
We fix such an $\varepsilon$. Since $\zeta = \lambda + \iota \varepsilon \in K^* \equiv \Pi^*_A$ satisfies \((1/2)(\lambda - (1/4)a^*_I) \geq \varepsilon \geq 0\), there exists an $R_\varepsilon = R_\varepsilon(K^*) \geq R_4$ such that for any $(x, \zeta) \in B(R_\varepsilon) \times K^*$,

$$0 < \text{Re} \left( \frac{1}{\zeta^{(1/4)}a_m(r)} \right) \leq \text{Re} \left( \frac{1}{\zeta^{(1/4)}a_{m-1}(r)} \right) \leq \ldots \leq \text{Re} \left( \frac{1}{\zeta^{(1/4)}a_1(r)} \right) \leq \frac{1 + \varepsilon}{\lambda - (1/4)a^*_I},$$

\((2.3)\)

where in the last inequality \((2.4)\) we have used \((1.7)\), Lemmas 1.3, 2.1 and the fact that $\mu_j > 1 - \varepsilon_j$ in \((1.3)\). Note here

$$\lim \sup \max_{r \to \infty} \max_{i \leq m} \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x) = \max \lim \sup \max_{r \to \infty} \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x).$$

Then we see that there exists an $R_\varepsilon = R_\varepsilon(K^*) \geq R_4$ such that for any $x \in B(R_\varepsilon)$

$$\max \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x) = \max \lim \sup \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x) + \gamma \varepsilon \min_{\zeta \in K^*} (\lambda - (1/4)a^*_I),$$

\((2.5)\)

\((2.3), (2.5)\) and Abel’s theorem imply that (cf., also \((1.5)\))

$$\frac{1}{\gamma} \text{Re} \left( \sum_{j=1}^{n} \frac{r a_j \partial_r V_{ij}(x)}{\zeta^{(1/4)}a_j(r)} \right) \leq \text{Re} \left( \frac{1}{\zeta^{(1/4)}a_1(r)} \right) \frac{1}{\gamma} \max \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x) \leq \frac{1 + \varepsilon}{(\lambda - (1/4)a^*_I) \gamma} \left( \max \lim \sup \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x) + \gamma \varepsilon \min_{\zeta \in K^*} \left( \lambda - \frac{1}{4}a^*_I \right) \right).$$

Since

$$\frac{1}{\gamma} \max \lim \sup \sum_{j=1}^{n} r a_j \partial_r V_{ij}(x) \leq \frac{1}{2} \left( \lambda + A_\delta - \frac{1}{2} a^*_I \right)$$

by Lemma 1.4, we have from \((2.4), (2.6)\) and \((2.2)\)

$$1 - \frac{1}{\gamma} \text{Re} \left( \sum_{j=1}^{n} \frac{r a_j \partial_r V_{ij}(x)}{\zeta^{(1/4)}a_j(r)} \right) \geq |\zeta|^{-\varepsilon} I \geq 1 - \left( \frac{\lambda + A_\delta - (1/2)a^*_I}{2\lambda - (1/2)a^*_I} \right)(1 + \varepsilon) - \varepsilon(3 + \varepsilon) \geq 0$$

for any $(x, \zeta) \in B(R_\varepsilon) \times K^*$. This proves \((2.1)\), and we have \((A2-4)\).

**Remark 2.1.** The condition \((1.2)\) on $[a_j(r)]$ is used only to show \((2.6)\)

(Abel’s theorem). Note that Assumptions 1 and 2 can be verified without \((1.2)\) if we replace $A\delta$ (see \((1.6)\)) by the following

$$A\delta = \min \left( \frac{1}{4\delta}, 2 \right) \sum_{j=1}^{n} \lim \sup_{r \to \infty} \max \{r a_j \partial_r V_{ij}(x)\} + \frac{1}{4} \max \{a_j^*\}.$$
§ 3. Sketch of proof of Theorem 3

On the bases of the principle of limiting absorption (Theorem 2), we can prove Theorem 3 by the same argument as in [3]. Here, in this section, we shall sketch an outline of the proof.

First we restrict ourselves to the case $V_s(x) = 0$. Then $A_3 = A_{1/2}$ since we have assumed $\delta_1 > 1/2 (l = 1, 2)$ in (V2) (we can choose $\delta_1 = 1$ in this case). For $f \in L_c^2(\Omega)$ and $\lambda > A_{1/2} + \varepsilon (\varepsilon > 0)$, let $R_{1, \lambda \pm i0}f$ be the outgoing [incoming] solution of (0.2) with $V(x) = V_1(x)$ and $\zeta = \lambda \pm i0$. We choose $R_1 = R_3(\varepsilon) > R_0$ so large that

$$
\lambda - \eta(r) \cdot V_1(x) \geq C > 0 \quad \text{for} \quad (x, \lambda) \in B(R_1) \times (A_{1/2} + \varepsilon, \infty),
$$

and define the function $\rho(x, \lambda \pm i0) = \rho(x, \lambda \pm i0; \varepsilon)$ as follows:

$$
(3.1) \quad \rho(x, \lambda \pm i0) = \int R_1^0 k(s, x, \lambda \pm i0) ds
$$

$$
= \mp i \int R_1^0 \sqrt{\lambda - \eta(s) \cdot V_1(s, x)} ds + \frac{m - 1}{2} \log r + \frac{1}{4} \log (\lambda - \eta(r) \cdot V_1(x)).
$$

Then as we see in Propositions 1.2 and 2.1 of [3], there exists a sequence $r_p \rightarrow 0$ (as $p \rightarrow \infty$) such that

$$
\left\{ \frac{1}{\sqrt{\pi}} e^{i(x, p \cdot \lambda \pm i0)} \left[ R_{1, \lambda \pm i0} f_p \right] (r_p) \right\}
$$

strongly converges in $L^2(S^{n-1})$. We define the operator $\mathcal{F}_{1, \varepsilon}(\lambda) : L^2(\Omega) \rightarrow L^2(S^{n-1})$ as follows:

$$
(3.2) \quad \mathcal{F}_{1, \varepsilon}(\lambda)f = \lim_{p \rightarrow \infty} \frac{1}{\sqrt{\pi}} e^{i(x, p \cdot \lambda \pm i0)} \left[ R_{1, \lambda \pm i0} f_p \right] (r_p) \quad \text{in} \quad L^2(S^{n-1}).
$$

Then $\mathcal{F}_{1, \varepsilon}(\lambda)$ is independent of the choice of $\{r_p\}$, and becomes a bounded operator from $L_c^2(\Omega)$ to $L_c^2(S^{n-1})$ which depends continuously on $\lambda > A_{1/2} + \varepsilon$ ([3], Lemma 2.4). Further, we have for $f \in L_c^2(\Omega)$ and $\lambda > A_{1/2}$ ([3], Proposition 1.3)

$$
(3.3) \quad \| \mathcal{F}_{1, \varepsilon}(\lambda)f \|_{L^2(S^{n-1})} = \frac{1}{2\pi^2} (R_{1, \lambda \pm i0} f - R_{1, \lambda - i0} f, f).
$$

By use of this $\mathcal{F}_{1, \varepsilon}(\lambda)$, the operator $\mathcal{F}_{\delta}(\lambda) : L_{c(1+\delta)}^2(\Omega) \rightarrow L^2(S^{n-1})(\delta = \min \{\delta_0, \delta_1, \delta_2\})$ will be defined by

$$
(3.4) \quad \mathcal{F}_{\delta}(\lambda) = \mathcal{F}_{1, \varepsilon}(\lambda) \left[ 1 - V_2 R_{1, \lambda \pm i0} \right], \quad \lambda > A_{1/2} + \varepsilon (\geq A_{1/2} + \varepsilon).
$$

In order to verify that (3.4) is well defined as a bounded operator from $L_{c(1+\delta)}^2(\Omega)$ to $L^2(S^{n-1})$, we have to show that for any $\lambda > A_{1/2} + \varepsilon$, $\mathcal{F}_{1, \varepsilon}(\lambda)$ can be extended to a bounded operator from $L_{c(1+\delta)}^2(\Omega)$ to $L^2(S^{n-1})$, where $\beta$ is any constant satisfying (1.13). This is possible by (3.3) and Theorem 2 (a) with $V(x) = V_1(x)$.

Now, as is proved in Theorems 2.1 and 3.1 of [3], we can have the assertions (a)~(d) of Theorem 3 with this $\mathcal{F}_{\delta}(\lambda)$.

To establish the assertion (e), we follow the argument of the proof of Theorem 4.1 of [3]. Namely, if $\tilde{f} \in L^2((A_{1/2} + \varepsilon) \times S^{n-1})$ is orthogonal to the
range of $\mathcal{F}_s$, we can have for any smooth $\phi \in L^2(S^{n-1})$,
\begin{equation}
(\bar{f}(\lambda, \cdot), \phi)_{L^2(S^{n-1})} = 0 \quad \text{a.e.} \quad \lambda > A_3 + \varepsilon ,
\end{equation}
where $\delta = \min\{\delta, 4\delta_2 - 2\}$. Note that to obtain (3.5) we have used the following relation satisfied by any smooth $\phi \in L^2(S^{n-1})$ and $\lambda > A_3 + \varepsilon$:
\begin{equation}
(\mathcal{F}_1, (\lambda) f, \phi)_{L^2(S^{n-1})} = \lim_{p=\infty} \left( \frac{1}{\sqrt{\pi}} e^{\delta (r_p, \lambda z/\delta)} [R_{1, \lambda z/\delta} f](r_p, \cdot), \phi \right)_{L^2(S^{n-1})}
\end{equation}
(cf., [3]; Lemma 3.2 and Proposition 1.4).

§ 4. Examples

I. We consider potentials of the form
\begin{equation}
V(x) = \sum_{j=1}^{m} \frac{c_j \sin b_j r^{-j}}{r^j} + 0(r^{-1-\delta_0}) \quad \text{near infinity},
\end{equation}
where $b_j, c_j$ are non-zero real, $0 < \varepsilon_m \leq \varepsilon_{m-1} \leq \ldots \leq \varepsilon_1 \leq 1$ and $0 < \delta_0 \leq 1$. If $\varepsilon_k = \varepsilon_{k+1} = \ldots = \varepsilon_{k+p}$, the order of summation is chosen like $|b_k| \geq |b_{k+1}| \geq \ldots \geq |b_{k+p}|$. We put
\begin{equation}
V_{1j}(x) = V_{1j}(r) = \frac{c_j \sin b_j r^{-j}}{r^j}, \quad a_j(r) = \varepsilon_j^2 b_j^2 r^{-2+2\varepsilon_j}.
\end{equation}
Then it follows that
\begin{equation}
V_{1j}(r) = 0(r^{-1}),
\end{equation}
\begin{equation}
V_{1j}'(r) = \frac{\varepsilon_j b_j c_j \cos b_j r^{-j}}{r} + 0(r^{-1-\varepsilon_j}) = 0(r^{-1}),
\end{equation}
\begin{equation}
V_{1j}''(r) = \frac{-\varepsilon_j^2 b_j^2 c_j \sin b_j r^{-j}}{r^{2-\varepsilon_j}} + 0(r^{-1}) = -a_j(r) V_{1j}(r) + 0(r^{-1}).
\end{equation}
Thus, choosing $\delta_1 = 1 - \varepsilon_1 + \varepsilon_m$ and $\delta_2 = 1$, we see that $V_{1j}(r)$ satisfies (V2-i)\sim (V2-iii) and $a_j(r)$ satisfies (1.2) and (1.3) (see Lemma 1.1). Note that in this case, (V2-iv)\sim (V2-vi) are trivially satisfied by $V_{1j}(r)$. Since $\delta = \delta = \min\{\delta_0, 1 - \varepsilon_1 + \varepsilon_m\}$ and
\begin{equation}
\lim_{r \to \infty} r V_{1j}(r) = \varepsilon_j |b_j c_j|,
\end{equation}
it follows from (1.4), (1.5) and (1.6) that
\begin{equation}
E(2) \leq \frac{1}{2} \sum_{j=1}^{m} \varepsilon_j |b_j c_j|,
\end{equation}
\begin{equation}
A_2 = A_2 \leq \frac{1}{\min\{4\delta, 2\}} \sum_{j=1}^{m} \varepsilon_j |b_j c_j| + \frac{1}{4} \varepsilon_j^2 b_j^2 \lim_{r \to \infty} r^{-2+2\varepsilon_j}.
\end{equation}
Namely, we have the following results for the potential (4.1): $(E(2), \infty)$ is contained in the continuous spectrum of $L = -\Delta + V(x)$ (Theorem 1). $(A_3, \infty)$ is contained in the absolutely continuous spectrum of $L$ (Theorem 2). If $1/2 > \varepsilon_1 - \varepsilon_m$, for any $\varepsilon > 0$ there exists a unitary operator $\mathcal{F}_s$ (depending on $\varepsilon$) from $C((A_3 + \varepsilon, \infty))L^2(Q)$ onto $L^2((A_3 + \varepsilon, \infty) \times S^{n-1})$ which diagonalizes $L$ (Theorem 3).
In (4.3) we have used the fact that \( \varepsilon_i = \max \{ \varepsilon_j \} \) and \( |b_j| = \max \{ |b_k| : \varepsilon_k = \varepsilon_i \} \).

Remark 4.1. If \( \varepsilon_i = 1 \), then we have

\[
A_0 = A_0 \leq \frac{1}{\min \{4\delta, 2 \}} \sum_{j=1}^{m} \varepsilon_j |b_j c_j| + \frac{1}{4} b_i^2
\]

(cf., [2]; Example II-1). On the other hand, if \( \varepsilon_j < 1 \) for any \( j \), we have

\[
A_0 = A_0 \leq \frac{1}{\min \{4\delta, 2 \}} \sum_{j=1}^{m} \varepsilon_j |b_j c_j|
\]

(cf., [2]; Example III).

II. We consider a more general case:

\[
V(x) = \sum_{j=1}^{m} c_j(x) \sin b_j r^{s_j} + 0(r^{-1-\delta_0})
\]

near infinity, where \( b_j, \varepsilon_j \) and \( \delta_0 \) are as given above and \( c_j(x) \) is a real-valued function such that

\[
\mathcal{F} \mathcal{T} e_j(x) = 0(r^{-1-\varepsilon_i}) (l = 0, 1, 2).
\]

We put

\[
V_{ij}(x) = c_j(x) \sin b_j r^{s_j}, \quad a_j(r) = \varepsilon_j^2 b_j^2 r^{-2s_j}.
\]

Then, choosing \( \delta_i = 1 - \varepsilon_i + \varepsilon_m \) and \( \delta_0 = \varepsilon_m \), we see that \( V_{ij}(x) \) satisfies (V2-i) \( \sim \) (V2-vi) and \( a_j(r) \) satisfies (1.2) and (1.3). In this case, we have \( \delta = \min \{ \delta_0, \varepsilon_m \} \) and

\[
E(2) \leq \frac{1}{2} \sum_{j=1}^{m} \varepsilon_j |b_j c_j^*|; \quad c_j^* = \limsup_{r \to \infty} r^{-s_j} e_j(x),
\]

\[
A_0 \leq \frac{1}{\min \{4\delta, 2 \}} \sum_{j=1}^{m} \varepsilon_j |b_j c_j^* + \frac{1}{4} \varepsilon_j^* b_i^2 \lim_{r \to \infty} r^{-2s_j+1}.
\]

Namely, Theorems 1 and 2 hold with the above \( E(2) \) and \( A_0 \), respectively. In order to apply Theorem 3 we have to assume

\[
\delta = \varepsilon_m > 1/2.
\]

Then we see that for any \( \varepsilon > 0 \) there exists a partial isometric operator \( \mathcal{X}_\varepsilon \) (depending on \( \varepsilon \)) from \( \mathcal{C}(A_0 + \varepsilon, \infty) L^2(\Omega) \) to \( L^2(A_0 + \varepsilon, \infty) \times S^{n-1} \) which diagonalizes \( L \), and maps \( \mathcal{C}(A_0 + \varepsilon, \infty) L^2(\Omega) \) onto \( L^2(A_0 + \varepsilon, \infty) \times S^{n-1} \), where

\[
A_0 \leq \frac{1}{\min \{4\delta, 2 \}} \sum_{j=1}^{m} \varepsilon_j |b_j c_j^* + \frac{1}{4} \varepsilon_j^* b_i^2 \lim_{r \to \infty} r^{-2s_j+1},
\]

with \( \delta = \min \{ \delta, 2\delta_0 - 1 \} = \min \{ \delta_0, 2\varepsilon_m - 1 \} \).

Remark 4.2. In general \( A_0 \geq A_0 \). However, if \( \delta_0 = \varepsilon_m \geq 3/4 \), we have \( A_0 = A_0 \) (cf., [3]; Corollary 5.1).

III. The above results can be applied to potentials of the form

\[
V(x) = c(x) \sin b r^s + 0(r^{-1-\delta_0}) (b \neq 0, 0 < \delta_0 \leq 1, 0 < \varepsilon \leq 1)
\]
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near infinity, where $c(x)$ is a real-valued function satisfying

\begin{equation}
F^{l}c(x) = 0 (r^{-l-1}) \quad (l=0, 1, 2).
\end{equation}

In fact,

\begin{align*}
\sin^p br^t &= c_0 + \sum_{k=1}^{P} (c_k \sin kbr^t + d_k \cos kbr^t)
\end{align*}

for suitable constants $c_0$, $c_k$ and $d_k (k=1, \ldots, p)$. So, if we put

\begin{align*}
V_{1j}(x) &= c(x) \{c_{j-p+1} \sin (p-j+1)br^t + d_{j-p+1} \cos (p-j+1)br^t\} (j=1, \ldots, p), \\
V_{1(p+1)}(x) &= c_0 c(x), \\
a_j(r) &= \varepsilon^j (p-j+1) br^{r-b+1} (j=1, \ldots, p), \\
ap_{p+1}(r) &= 0,
\end{align*}

$V_{1j}(x) (j=1, \ldots, p+1)$ satisfies \(V2-i\) and \(a_j(r) (j=1, \ldots, p+1)\) satisfies (1.2) and (1.3).

**Remark 4.3.** Potentials of the form

\begin{equation}
V(x) = c_1 \sin (\log r) + c_2 r^{-s} \sin br^t + 0(r^{-1-s})
\end{equation}

\((bc_1c_2 \neq 0, 0 < \varepsilon \leq 1)\) are not covered by our theory (see \(V2-i\)), though each potential \(\sin (\log r)\) or \(r^{-s} \sin br^t\) is in the framework of our “oscillating” long-range potentials ([2], [3]).

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