Leray-Volevich’s system and Gevrey class

By

Kunihiko KAJITANI

(Communicated by Prof. S. Mizohata, April 14, 1980)

§1. Introduction

We consider the Cauchy problem for hyperbolic systems with multiple characteristics of constant multiplicity. Let \( \Omega \) be a band \([0, T] \times \mathbb{R}^n \) in \( \mathbb{R}^{n+1} \). We consider the following equations in \( \Omega \),

\[
\sum_{\xi=1}^{N} a_{\xi}(x, D)u^{\xi}(x) = f^{\xi}(x), \quad p = 1, \ldots, N,
\]

where \( x = (x_0, x_1, \ldots, x_n) = (x_0, x') \in \Omega \) and \( a_{\xi}(x, D) \) differential operators of order \( m_{\xi} \) of which coefficients are in the Gevrey class \( \mathcal{r}(\Omega)(s \geq 1) \).

We use the notation as follows,

\[
D = (D_0, \ldots, D_n), \quad D_\alpha = -\sqrt{-1} \frac{\partial}{\partial x_k},
\]

\[
\alpha = (\alpha_0, \ldots, \alpha_n), \quad \alpha_k \text{ integers},
\]

\[
D_\alpha = D_\alpha^{(s_1)} \cdots D_\alpha^{(s_n)}, \quad |\alpha| = \sum \alpha_k,
\]

\[
\xi = (\xi_0, \xi_1, \ldots, \xi_n); \quad \text{dual variables of } x,
\]

and \( \mathcal{r}_s(\Omega) \) consists of all functions \( f \) such that there exists positive constants \( C \) and \( A \) satisfying for any \( \alpha \),

\[
|D_\alpha f(x)| \leq CA^{m_{\alpha}}|\alpha|!^s, \quad x \in \Omega.
\]

We correspond the polynomial \( a_{\xi}(x, \xi) \) in \( \xi \) to a differential operator \( a_{\xi}(x, D) \). We denote by \( a_{\xi}(x, \xi) \) the homogeneous part of degree \( m_{\xi} \) of \( a_{\xi}(x, \xi) \). We define the total order \( m \) of \( \{a_{\xi}(x, D)\} \) such that

\[
m = \max_{\pi} \sum_{\xi=1}^{N} m_{\xi}(p),
\]

where \( \pi \) runs over all permutations of \([1, \ldots, N]\). Then it follows from Volevich’s lemma [16] that there exists a pair of integers \( (t_p, s_p) \), \( p = 1, \ldots, N \), such that

\[
m_{\xi} \leq t_q - s_p, \quad (\xi, q) \in [1, \ldots, N]^{s},
\]

\[
m = \sum_{p=1}^{N} (t_p - s_p),
\]

where \( \sum_{p=1}^{N} m_{\xi}(p) \) is the degree of \( a_{\xi}(x, D) \).
where for convinience we define $m_{p}^\xi=-\infty$ if $a_{p}^\xi(x,D)\equiv 0$. We denote by $\bar{a}(x,\xi)$ the homogeneous part of degree $m$ of $\det (a_{p}^\xi(x,\xi))$. We call $\bar{a}(x,\xi)$ the characteristic polynomial of the system $a_{p}^\xi(x,D)$. The pair of weights $\{t_{p}, s_{p}\}$ satisfying the property (1.2) is not uniquely determined for $\{m_{p}^\xi\}$. But the weights $\{t_{p}, s_{p}\}$ is fixed from now on. We call the Leray-Volevich’s system of weights $\{t_{p}, s_{p}\}$, a system of differential operators $a_{p}^\xi(x,D)$ of which orders satisfies the property (1.2).

We assume that the intial plane $\{x_{0}=0\}$ is not characteristic with respect to $a(x,\xi)$, that is, $\bar{a}(x,\xi)\neq 0$ for $\xi=(1,0,\cdots,0)$ and $x \in \mathcal{O}$. For the equations (1.1) of the Leray-Volevich’s system of weights $\{t_{p}, s_{p}\}$, we can give $t_{p}$ numbers of the intial data (cf. [5]),

$$\frac{d}{dx_{h}}u_{p}|_{x_{0}=0}=w_{p}(x_{p}), \quad h=0,1,\cdots,t_{p}-1. \tag{1.3}$$

If $s_{p}>0$, the data $\{f_{p}, w_{p}\}$ must satisfy the following compatibility conditions,

$$f_{p}=\sum_{q=1}^{t_{p}}a_{p}^\xi(x,D)w^{q}=O(x_{p}^{s_{p}}), \quad p=1,\cdots,N, \tag{1.4}$$

where $\{w_{p}\}$ are the functions in $\gamma_{i}(\mathcal{O})$ such that

$$\frac{d}{dx_{h}}w_{p}|_{x_{0}=0}=w_{p}, \quad h=0,1,\cdots,t_{p}-1, \tag{1.5}$$

$$p=1,\cdots,N.$$ The compatibility conditions (1.4) do not depend on the choice of $\{w_{p}\}$ satisfying (1.5). We assume that the characteristic polynomial $\bar{a}(x,\xi)$ of $\{a_{p}^\xi(x,D)\}$ is of constant multiplicity, that is,

$$\bar{a}(x,\xi)=\prod_{i=1}^{d}(\bar{\xi}_{i,0}(x,\xi)-\lambda_{i}(x,\xi))^{m_{i}}, \tag{1.6}$$

where $m_{i}$ are constant integers and

$$\inf_{\Omega \times \mathbb{R}^{n}-0} |\lambda_{i}(x,\xi')-\lambda_{j}(x,\xi')|\neq 0.$$ Then we note that the characteristic roots $\lambda_{i}(x,\xi')$ are in $\gamma_{i}(\Omega \times \mathbb{R}^{n})$ and in particular analytic in $\xi'$. It follows from Matsuura’s lemma [13] that we can factorize

$$\bar{a}(x,\xi)=a_{1}(x,\xi)^{p_{1}}a_{2}(x,\xi)^{p_{2}}\cdots a_{r}(x,\xi)^{p_{r}},$$

where each $a_{j}(x,\xi)$ and the product $a_{1}(x,\xi)a_{2}(x,\xi)\cdots(a_{r}(x,\xi))$ are strictly hyperbolic polynomials. To study the Cauchy problem (1.1) and (1.3) with the compatibility conditions (1.4), we reduce a Leray-Valevich’s system to a system with a diagonal principal part by a transformation of unknown functions. To do so, we introduce the cofactor operator of a system $\{a_{p}^\xi\}$. We denote by $h_{p}^\xi(x,\xi)$ the homogeneous part of degree $t_{p}-s_{p}$ of $a_{p}^\xi(x,\xi)$ that is,

$$h_{p}^\xi(x,\xi)=\begin{cases} a_{p}^\xi(x,\xi), & \text{if } m_{p}^\xi=t_{p}-s_{p}, \\ 0, & \text{if } m_{p}^\xi<t_{p}-s_{p}. \end{cases}$$
Denote by \( G^p(x, \xi) \) the cofactor of \( h^p(x, \xi) \). Then the degree of \( G^p(x, \xi) \leq m-(t_q-s_p) \) and
\[
\sum_{r=1}^{N} h^p(x, \xi) G^p(x, \xi) = \delta^p a(x, \xi).
\]
Hence we have
\[
\sum_{r=1}^{N} a^r(x, D) G^p(x, D) = \delta^p a(x, D) - b^p(x, D),
\]
where \( a(x, D) = a_1(x, D)^{r_1} \cdots a_r(x, D)^{r^r} \), each \( a_i(x, \xi) \) and its products \( \prod a_i(x, \xi) \) are strictly hyperbolic polynomials, and
\[
(1.7) \quad \text{order } b^p(x, D) \leq m-1+s_q-s_p.
\]
We call the above system of differential operators a Leray-Volevich's system with diagonal principal part of constant multiplicity of order \( m \). In the equations (1.1), we put
\[
u^p(x) = \sum_{q=1}^{N} G^p(x, D) v^q(x), \quad p=1, \cdots, N.
\]
Then we can see that it is sufficient to solve (1.1) and (1.3) that we can solve the following Cauchy problem (c.f. [17])
\[
(1.8) \begin{cases}
 a(x, D) v^p(x) - \sum_{q=1}^{N} b^p_q(x, D) v^q(x) = f^p(x), & p=1, \cdots, N, \\
 D^h b^p |_{x_0=0} = g^p_h(x'), & h=0, 1, \cdots, m-1.
\end{cases}
\]
Our aim is to construct a fundamental solution for the system (1.8). We factorize the principal part \( d(x, \xi) \) of \( a(x, D) \) as follows
\[
d(x, \xi) = \prod_{i=1}^{d} (\xi - \lambda^{(i)}(x, \xi')).
\]
Denote by \( \phi^{(i)} \) a phase function associated with \( \lambda^{(i)} \), that is, \( \phi^{(i)} = \lambda^{(i)}(x, \phi^{(i)} ) \), \( \phi^{(i)} \neq 0 \). Let \( m^p_{(i)} \) be integers satisfying for each \( (p, q, l) \)
\[
(1.9) \quad e^{-i \phi^{(i)}(l)} b^p(x, D) e^{i \phi^{(i)}(l)} f = O(p_m^{p_{(i)}(l)}), \quad p \to \infty.
\]
That it follows from (1.7) that we have
\[
(1.10) \quad m^p_{(i)} \leq m-1+s_q-s_p.
\]
We define the rational numbers \( q^{(i)} \) as
\[
(1.11) \quad q^{(i)} = \max_s \left\{ \sum_{p=1}^{N} m^p_{(i)p}/N + m^{(i)} - m, \quad l=1, \cdots, d \right\},
\]
where \( \pi \) stands for a permutation of \( [1, \cdots, N] \). Then using again the Volvich's lemma, we can find the rational numbers \( \{ n^{(i)} \} \) such that
\[
m^p_{(i)} \leq m - m^{(i)} + q^{(i)} + n^{(i)} - n^{(i)}.
\]
Remark. In the case of $q^{(i)} = 0 (i = 1, \ldots, d)$, we can solve the Cauchy problem (1.8) in the sense of $C^\infty$-class (c.f. [9]). When $q^{(i)} \neq 0$, we cannot solve (1.8) in $C^\infty$-class. Then we treat the Cauchy problem (1.8) in the Gevrey class $\gamma_s(Q)$.

We return to (1.9) and expand it in a power of $\rho$,

\begin{equation}
\tag{1.12}
e^{t\rho\phi^{(i)}}h_\rho(e^{t\rho\phi^{(i)}}f) = \sum_{k=0}^{m^{(i)}} \rho^k b^{(i)-k}_\rho h_\rho(x, D)f,
\end{equation}

where $b^{(i)}_\rho(x, D)$ is a differential operator and denote by $d^{(i)}_k$ its order and without loss of generality we may put

\[ m^{(i)}_k = m^{(i)} + q^{(i)} + n^{(i)}_q - n^{(i)}_p. \]

Then (1.7) implies

\begin{equation}
\tag{1.13}
m^{(i)}_k - k + d^{(i)}_k \leq m - 1 + s_q - s_p.
\end{equation}

We define

\[ d^{(i)}_k = \max_{\pi} \sum_{p=1}^N d^{(p)}_{\kappa^{(i)}_k}/N, \]

\[ \kappa^{(i)} = \inf_{q^{(i)} - k \geq 0} m^{(i)}_k - d^{(i)}_k. \]

Then from (1.13) we have

\[ d^{(i)}_k \leq m^{(i)} - 1 - q^{(i)} + k, \]

which implies

\begin{equation}
\tag{1.14}
\kappa^{(i)} \geq \inf_k \frac{q^{(i)} + 1 - k}{q^{(i)} - k} = \frac{q^{(i)} + 1}{q^{(i)}} > 1.
\end{equation}

Moreover we note that (1.18) and valevich's lemma imply

\begin{equation}
\tag{1.15}
d^{(i)}_k \leq d^{(i)}_k + s^{(i)}_q - s^{(i)}_p
\leq m^{(i)} - \kappa^{(i)}(q^{(i)} - k) + s^{(i)}_q - s^{(i)}_p,
\end{equation}

where $s^{(i)}_p = s_p - n^{(i)}_p$. The number $\kappa^{(i)}$ given by (1.14) is same one which is introduced in [7], [10] and [3] in the case of $N = 1$. We call the fundamental solution of the Cauchy problem (1.8) a distribution satisfying

\begin{equation}
\tag{1.16}
\begin{cases}
a(x, D)K^{p}(x, y) = \sum_{q=1}^N b_q(x, D)K^{q}(x, y), & p = 1, \ldots, N, \\
D^h_k K^{p} |_{x = y} = \delta(x' - y')\delta^{h}_{n-1}, & h = 0, 1, \ldots, m - 1.
\end{cases}
\end{equation}

When we regard $K^{p}(x_0, y_0)$ as an operator from $\gamma_s(R^n_+)$ to $\gamma_s(R^n_+)$, we write $K(x_0; y_0)$, that is,

\[ (K^{p}(x_0; y_0)u)(x) = \int_{R^n} K^{p}(x, y)u(y')d'y', \]

for $u \in \gamma_s(R^n)$.

**Theorem 1.1.** Let $a(x, D)\delta^p + b_q(x, D)$ be a Leray-Volevich's system with diagonal principal part of constant multiplicity of order $m$ and the order of $b_q$...
satisfies (1.7) for the weight \((s_p, s_0)\). We assume that the order \(d_q^{(t)}\) of the operator \(b_q^{(t)}\) given in (1.12) satisfies (1.15) and assume

\[
s^{(t)}_q - s^{(t)}_0 \leq q^{(t)}, \quad \text{if } q^{(t)} \neq 0
\]

for any \((p, q)\). Then we can construct the fundamental solution \(K^p(x_0; y_0)\) of (1.16) as follows,

\[
K^p(x_0; y_0) = W^p(x_0; y_0) + \int_{y_0}^{x_0} W^p(x; t) F^p(t; y_0) dt,
\]

where

\[
W^p(x_0; y_0) = \sum_{l=1}^{d} e^{i(q^{(t)}_l x - y; \xi')} w^{(t)} p(x, y_0; \xi') d\xi',
\]

and \(q^{(t)}\) the phase function associated with \(\lambda^{(t)}\) such that \(q^{(t)}_0 = \lambda^{(t)}(x, y_0, e')\) at \(x_0 = y_0\), and \(F^p(x_0; y_0)\) is a solution of the integral equation

\[
F^p(x_0; y_0) = - R^p(x_0; y_0) - \int_{y_0}^{x_0} R^p(x; t) F^p(t; y_0) dt,
\]

where

\[
R^p(x_0; y_0) = a(x, D) W^p(x_0; y_0) - \sum_{q=1}^{N} b^p(x, D) W^q(x_0, y_0)
\]

\[
= \sum_{l=1}^{d} e^{i(q^{(t)}_l x - y; \xi')} d\xi',
\]

and the amplitude functions \(w^{(t)} p(x, y_0; \xi')\) and \(r^{(t)} p(x, y_0; \xi')\) are estimated by

\[
|D^q_x D^p_y w^{(t)}| \leq C_1 A^{(n+\beta)} \exp\{A_1 |x_0 - y_0| \langle \xi' \rangle^{11/\gamma^{(t)}} \} \langle \xi' \rangle^{m^{(t)}_p - |\beta| + a_0(1/\gamma^{(t)}) n + |\beta| n},
\]

\[
|D^q_x D^p_y r^{(t)}| \leq C_2 A^{(n+\beta)} \exp\{A_2 |x_0 - y_0| \langle \xi' \rangle^{11/\gamma^{(t)}} \} \langle \xi' \rangle^{m^{(t)}_p - |\beta| + a_0(1/\gamma^{(t)}) n + |\beta| n}
\]

for \(\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)\), \(\beta = (\beta_0, \ldots, \beta_n)\) and \(\mu = 1, 2, 3, \ldots, m^{(t)} - m - n^{(t)}_q\).

\section*{§ 2. Asymptotic solution of fundamental solution}

We note that the distribution \(\delta(x' - y')\) is represented by

\[
\delta(x' - y') = \frac{1}{(2\pi)^n} \int_{R^n} e^{i\rho' \cdot \xi'} d\xi'.
\]

Then we can obtain the asymptotic solutions of (1.16) by integrating in \(\xi'\) the following solutions

\[
a(x, D) u^p(x, y; \xi') = \sum_{q=1}^{N} b^p(x, D) u^q(x, y; \xi'), \quad p = 1, \ldots, N,
\]

\[
D^h_{x_0} u^p|_{x_0 = y_0} = \frac{1}{(2\pi)^n} \int_{R^n} e^{i\rho' \cdot \xi'} \delta_m \cdot d\xi', \quad h = 0, 1, \ldots, m-1,
\]

where \(\delta_m\) is the \(m\)-dimensional measure on \(\mathbb{R}^m\).
where $\delta_{j}$ is Kroner's delta. We seek for $u^{p}(x, y, \xi')$ as forms

$$u^{p}(x, y; \xi') = \sum_{k=0}^{\infty} u^{p}_{k}(x, y; \xi')$$

where $u^{p}_{k}$ is a solution such that

$$\begin{cases} 
  a(x, D)v = 0, \\
  D_{h}^{2}v |_{x_{0}=y_{0}} = \frac{1}{(2\pi)^{n}} e^{i(x'-y').\xi'} \delta_{m-1}^{h}, & h = 0, 1, \ldots, m-1,
\end{cases}$$

and for $k \geq 1$, $u^{p}_{k}$ satisfies

$$\begin{cases} 
  a(x, D)u^{p}_{k} = \sum_{j=1}^{\infty} b_{k}(x, D)u^{p}_{k-1}, \\
  D_{h}u^{p}_{k} |_{x_{0}=y_{0}} = 0, & h = 0, 1, \ldots, m-1.
\end{cases}$$

We construct an asymptotic solution of (2.2) as follows

$$v(x, y; \xi') = \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} e^{i\phi^{(j)}(x, D)_{m-1}^{h}}(x, y; \xi'),$$

where $\phi^{(j)}(x, y, \xi')$ is the phase function associated with $\lambda^{(j)}$ such that

$$\phi^{(j)}(x, y, \xi') = \lambda^{(j)}(x) \phi^{(j)}(x, D),$$

and

$$\phi^{(j)} |_{x_{0}=y_{0}} = (x'-y', \xi').$$

We note that $\phi^{(j)}(x, y, \xi')$ is a homogenous function of degree one in $\xi'$. For a differential operator $P(x, D)$ of order $m$ and for a function $\phi$ we define the differential operators $\sigma_{\mu}(P, \phi)$ such that for $\rho > 0$

$$e^{-\rho \phi}P(x, D)(e^{-\rho \phi}f) = \sum_{\mu=0}^{m} \rho^{m-k} \sigma_{\mu}(P, \phi)f.$$

Then the principal part of $\sigma_{\mu}(P, \phi)$ is given by

$$\sum_{|\alpha|=\mu} \frac{1}{\alpha!} \left( \frac{\partial}{\partial \xi} \right)^{\alpha} P(x, \phi_{x})D^{\alpha},$$

where $P(x, \xi)$ is the principal part of $P(x, D)$.

**Lemma 2.1** (c.f. [2]). Let $P(x, D)$ be a differential operator of order $m$. Assume that for a phase function $\phi^{(j)}$ associated with $\lambda^{(j)}$

$$\sigma_{\mu}(P, \phi^{(j)}) = 0, \quad \text{for } \mu = 0, 1, \ldots, r-1.$$

Then we have

$$\sigma_{j}(P, \phi^{(j)}) = \sum_{j=0}^{r} a_{j}(x)H^{(j)}(x, D)^{j},$$

where

$$H^{(j)}(x, D) = D_{0} - \sum_{j=1}^{n} \lambda_{j}^{(j)}(x, \phi^{(j)}_{x})D_{j}. $$
Now we return to (2.2). Noting that $a(x, D)$ is a product of strictly hyperbolic operators, we have

$$e^{-iq_{1}(t)}a(x, D)(e^{iq_{1}(t)}f) = \sum_{j=0}^{m_{\mu}} \rho^{m_{\mu} - p} \sigma_{\mu + m_{\mu}}(a, \tilde{\varphi}^{(t)})f,$$

where $\rho = |\xi'|$ and $\tilde{\varphi}^{(t)} = \varphi^{(t)}/|\xi'|$. In particular, we have by Lemma 2.1,

$$\sigma_{m_{\mu}}(a, \tilde{\varphi}^{(t)}) = \sum_{j=0}^{m_{\mu}} a_{q_{j}}^{(t)}(x)H^{(t)}(x, D)^{j},$$

(2.6)

Hence inserting $v(x, y; \xi')$ into (2.2) we obtain

$$a(x, D)v(x, y, \xi') = \sum_{j=0}^{d} \sum_{p=0}^{m_{\mu}} \sum_{j=0}^{m_{\mu}} \sigma_{m_{\mu} + p}(a, \tilde{\varphi}^{(t)}) \rho^{-p - j} v_{j}^{(t)}.$$ 

Therefore we have

$$\sum_{j=0}^{m_{\mu}} \sigma_{m_{\mu} + p}(a, \tilde{\varphi}^{(t)}) v_{j}^{(t)} = 0, \quad j=0, 1, \ldots, l=1, 2, \ldots, d.$$ 

From the initial condition of (2.2) it follows

$$D_{0}^{h}v|_{x_{0}=y_{0}} = e^{t <x'-y', \xi'>} \sum_{j=0}^{m_{\mu}} \sigma_{j}(a, \tilde{\varphi}^{(t)}) v_{j}^{(t)} \rho^{m_{\mu} - p - j}$$

$$= e^{t <x'-y', \xi'>} \sum_{j=0}^{m_{\mu}} \sigma_{j}(a, \tilde{\varphi}^{(t)}) v_{j}^{(t)} \rho^{m_{\mu} - j}$$

$$= - \frac{1}{(2\pi)^{h}} e^{t <x'-y', \xi'>} \sigma_{m_{\mu} - 1}^{(t)}.$$

Hence

$$\sum_{j=0}^{d} \sum_{p=0}^{h} \sigma_{j}(a, \tilde{\varphi}^{(t)}) v_{j}^{(t)} \rho^{m_{\mu} - j} = \begin{cases} 1, & h=m-1, j=-1 \\ 0, & \text{otherwise}. \end{cases}$$

(2.8)

Noting that the principal part of $\sigma_{j}(a, \tilde{\varphi}^{(t)})$ is \(\binom{h}{\mu}(\lambda^{(t)})^{h-p} D_{0}^{h}\), we can solve (2.8) as the linear equations of \(D_{0}^{h}v_{j}^{(t)} = h=0, 1, \ldots, m_{\mu} - 1, l=1, \ldots, d\). For, the determinant of Van der Monde

$$\binom{h}{\mu}(\lambda^{(t)})^{h-p} D_{0}^{h}$$

does not vanish. Therefore we obtain $l=0, 1, \ldots, d$, $p=0, 1, \ldots, m_{\mu} - 1, j=0, 1, \ldots, m_{\mu} - 1,$

$$D_{0}^{h}v_{j}^{(t)} = \sum_{h=0}^{m_{\mu} - 1} c_{h}^{(t)} f_{j}^{(t)} + \sum_{l=0}^{d} \sum_{p=0}^{m_{\mu} - 1} N_{h+p, \mu}^{(t)} v_{j}^{(t)} - D_{0}^{h}v_{j}^{(t)}$$

$$+ \sum_{j=0}^{m_{\mu} - 1} N_{h+p, \mu}^{(t)} v_{j}^{(t)} - D_{0}^{h}v_{j}^{(t)}, \quad j \geq m_{\mu} - 1,$$

where $f_{j}^{(t)} = 1 \text{ if } h=m-1 \text{ and } j=1$, $f_{j}^{(t)} = 0$ if otherwise, and $N_{h+p, \mu}^{(t)}$ are differential operators of order $\mu$ with respect to $D_0$ and \(\{c_{h+p, \mu}^{(t)}\}\) is the inverse matrix of

$$\binom{h}{\mu}(\lambda^{(t)})^{h-p}.$$
Thus we can determine \( \{ v_{(j)} \} \) successively by (2.7) and (2.9). We define \( u_{(j)} \) by,
\[
u_{(j)} = \sum_{\ell = 0} e^{i \nu_{(j)} \phi_{(j)}} \sum_{\ell = 0} b_{(j)} - m + m_{(j)} + \frac{n_{(j)}}{p}, j \geq n_{(j)}^{(j)}, \quad j < n_{(j)}^{(j)}.
\]
where \( n_{(j)}^{(j)} = \max_{q} n_{(j)}^{(j)} - n_{(j)}^{(j)} \),
\[
u_{(j)}^{(j)} = \begin{cases} \nu_{(j)}^{(j)}, & j \geq n_{(j)}^{(j)} \\ 0, & j < n_{(j)}^{(j)} \end{cases}
\]
Next we seek for \( u_{(j)} \) the solution of (2.3) as
\[
u_{(j)}(x, y; \xi') = \sum_{l = 1}^{d} e^{i \nu_{(j)} \phi_{(j)}} \rho^{-m + m_{(j)} + \frac{n_{(j)}}{p} + \nu_{(j)}^{(j)} - j} u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x),
\]
Then inserting \( u_{(j)}^{(j)} \) into (2.3), we have by virtue of (1.12),
\[
u_{(j)}^{(j)} = \sum_{\ell = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
On the other hand we have
\[
u_{(j)}^{(j)}(x, y; \xi') = \sum_{l = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
Therefore we obtain
\[
u_{(j)}^{(j)}(x) = \sum_{\ell = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
As the initial condition we have
\[
u_{(j)}^{(j)}(x, y; \xi') = \sum_{l = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
which implies that
\[
u_{(j)}^{(j)}(x, y; \xi') = \sum_{l = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
Hence we have analogously to (2.9)
\[
u_{(j)}^{(j)}(x, y; \xi') = \sum_{l = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
Thus we can seek for \( u_{(j)}^{(j)} \) by solving (2.12) and (2.13). We reduce (2.12) to simple forms by a canonical transformation \( x' = \dot{x}'(x, \xi'), x_0 = x_0 \),
\[
u_{(j)}^{(j)}(x, y; \xi') = \sum_{l = 1}^{d} \sum_{\ell = 0}^{m_{(j)}} \phi_{(j)}^{(j)} + \nu_{(j)}^{(j)} + \nu_{(j)}^{(j)} - j u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x, D) u_{(j)}^{(j)}(x).
\]
Then we have for any function $f(x)$

$$\langle H'(x, D_x)f \rangle_{x=(x_0, x')} = D_{x_0}(f(x_0, x'\langle z_0, z' \rangle)).$$

Hence it follows from (2.6) that

$$\sigma_{m(\ell)}(a, \tilde{\eta}^{(\ell)}(y)) \mid x=(x_0, x') = \sum_{j=1}^{m(\ell)} a_j^{(\ell)}(y) D_{x_0}^{(\ell)}(A_{m(\ell)}^{(\ell)}(y, \xi'^{\ell}, D_{y})).$$

We put

$$U_{\ell}^{(\ell)}(y) = u_{j}^{(\ell)}(y) \mid x=(x_0, x'),$$

$$A_{\ell}^{(\ell)}(y, \xi'^{\ell}, D_y) = \sigma(a, \tilde{\eta}^{(\ell)}(y)) \mid x=(x_0, x'),$$

$$B_{\ell}^{(\ell)}(y, \xi'^{\ell}, D_y) = b_{j}^{(\ell)}(x, D) \mid x=(x_0, x').$$

Then (1.12) is reduced to

$$A_{m(\ell)}^{(\ell)}(y, \xi'^{\ell}, D_y) U_{\ell}^{(\ell)} = F_{\ell}^{(\ell)} + G_{\ell}^{(\ell)},$$

where

$$F_{\ell}^{(\ell)} = \sum_{j=1}^{m(\ell)} A_{m(\ell), \rho}^{(\ell)}(y, \xi'^{\ell}, D_y) U_{\ell}^{(\ell)},$$

$$G_{\ell}^{(\ell)} = \sum_{q=1}^{N} \sum_{p=1}^{m(\ell)} B_{q, p}^{(\ell)}(y, \xi'^{\ell}, D_y) U_{\ell}^{(\ell)},$$

and

order $\quad A_{\ell}^{(\ell)}(y, D) \leq j,$

order $\quad B_{\ell}^{(\ell)} \leq d_{\mu}^{(\ell)} + s_{\mu}^{(\ell)} - s_{\mu}^{(\ell)} \leq m_{\ell}^{(\ell)} - k_{\ell}^{(\ell)}(q_{\ell}^{(\ell)} - \mu) + s_{\mu}^{(\ell)} - s_{\mu}^{(\ell)}.$

As the initial conditions, we have by (2.12)

$$D_{y}^{\ell} U_{\ell}^{(\ell)}(y, \xi'^{\ell}, y_0) = \sum_{j=1}^{d} \sum_{k=1}^{m(\ell)} M_{\ell, \rho}^{(\ell)}(y, \xi'^{\ell}, y_0) U_{\ell}^{(\ell)}(y, \xi'^{\ell}, y_0) + \sum_{\ell=1}^{d} \sum_{\rho=1}^{m(\ell)} \sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \sum_{\nu=1}^{m(\ell)} M_{\ell, \rho}^{(\ell)}(y, \xi'^{\ell}, y_0) U_{\ell}^{(\ell)}(y, \xi'^{\ell}, y_0)$$

where $M_{\ell, \rho}^{(\ell)} = N_{\ell, \rho}^{(\ell)} |_{x=(x_0, z_0)}$ is a differential operator of order $\mu$. Then we have

**Theorem 2.1.** There exist positive constants $C$, $A$ and $\delta$ independent of $k$, $j$, $\alpha$ and $\beta$ such that for any $j$, $k$, $\alpha$ and $\beta$, $\|z_0 - y_0\| \leq \delta$, $|\xi'| = 1$, $z' \in K$, a compact set in $R^n$, 

$$|D_{y}^{\ell} D_{x}^{\ell} U_{\ell}^{(\ell)}(z, y_0, \xi'^{\ell})| \leq C_{j+k+1} A^{(\alpha + \beta + j + k\xi'^{\ell} - q^{(\ell)} - s_{\mu}^{(\ell)})} \times \sum_{\mu=1}^{\mu} \frac{(|z_0 - y_0| A)}{\mu} \left[ \frac{\alpha + \beta + j k^{(\ell)} - k q^{(\ell)} - \mu + s_{\mu}^{(\ell)}}{\mu} \right],$$

where the summation in $\mu$ is from $[m_{\ell}^{(\ell)} - \alpha_0]_+$ to $[jm_{\ell}^{(\ell)} + k q^{(\ell)} - [m_{\ell}^{(\ell)} - \alpha_0]_+ 1]$. 

\[\text{Leray-Volevich's systems and Gevrey class}\]
$[h]_e = \begin{cases} [h], & h > 0 \\ 0, & h \leq 0 \end{cases}$

$[h]_s = \begin{cases} [h], & h > 1 \\ 1, & h \leq 1 \end{cases}$

Therefore, noting that $u_{k,j}^{p_1}(x, y_0, \xi')$ is homogeneous of degree zero, we obtain

**Theorem 2.2.** The solutions $u_{k,j}^{p_1}(x, y_0, \xi')$ of (2.12)$_{k,j}$ and (2.13)$_{k,j}$ satisfies

$$|D_x^k D_{\xi'}^\beta u_{k,j}^{p_1}(x, y_0, \xi')| \leq C_1^{\beta + k + 1} A_{\alpha}^{1 + \beta} I_{1}^{1 + \beta} + C_1^{\beta + k + 1} A_{\alpha}^{1 + \beta} I_{1}^{1 + \beta}$$

for $|x_0 - y_0| \leq \delta, \xi' \in \mathbb{R}^n \setminus 0$, where $\mu = [m(t) - \alpha]_+, \cdots, [j m(t) + k \kappa(t) q(t) + [m(t) - \alpha]_+] + s_p^{(1)}$, and $C_1$ and $A_1$ are independent of $j, k, \alpha$ and $\beta$.

§ 3. Successive estimates of asymptotic solution

We start with a lemma to be used often in our reasoning which proof refers to [14] and [8].

**Lemma 3.1.** Let $p_1$ and $p_2$ be non negative integers, $\gamma > 1$ and $s \geq 1$. For any multi integer $\alpha = (\alpha_1, \cdots, \alpha_n)$, we have

$$\sum_{\alpha' + \alpha'' = \alpha} \left( \frac{\alpha}{\alpha'} \right)^{\gamma - |\alpha'| - |\alpha''|} (|\alpha'| + p_1)!(|\alpha''| + p_2)!^s$$

$$\leq \frac{\gamma}{\gamma - 1} (|\alpha| + p_1 + p_2)!^s,$$

where $\left( \frac{\alpha}{\alpha'} \right) = \frac{\alpha_1!}{(\alpha_1 - \alpha'_1)!\alpha'_1!} \cdots \frac{\alpha_n!}{(\alpha_n - \alpha'_n)!\alpha'_n!}.$

Leibniz formula and Lemma 3.1 imply.

**Lemma 3.2.** Let $P(x, D) = \sum_{|\alpha| \leq d} a_\alpha(x)D^\alpha$ be a differential operator in $\mathbb{R}^n$, $p_1$ and $p_2$ non negative integers and $\gamma > 1$. Assume the coefficients of $P(x, D)$ satisfy

$$|D^\alpha a_\beta(x)| \leq C_\gamma (\gamma - 1) A^{\alpha}(|\alpha| + p_1)!^s,$$

$$|D^\alpha u(x)| \leq C A^{\alpha}(|\alpha| + p_2)!^s,$$

for $x \in K$ a compact set in $\mathbb{R}^n$. Then we have

$$|D^\alpha P(x, D) u(x)| \leq C C_\gamma A^{\alpha}(|\alpha| + p_1 + p_2)!^s, \quad x \in K,$$
where \( n_d = (n^d - 1)(n - 1)^{-1}(\gamma - 1)^{-1} \).

**Lemma 3.3.** Let \( X_j(x, D) = \sum_{i=1}^{n} a_{j,i}(x) D_i + A_{j,0}(x) \) be first order differential operators \((j=1, \ldots, N)\). Assume that the coefficients \( a_{j,i}(x) \) of \( X_j(x, D) \) and \( u(x) \) satisfy (3.2) and (3.3) with \( p_1 = 0 \) and \( p_2 = 0 \) respectively. Then we have

\[
|D^n u(x) - \sum_{|\gamma| = n} A_{\gamma} |u|^{p} (1 + \cdot + \gamma)| \leq C \left(C_0^{\bar{n}_1} \right)^{\bar{n}_1} (1 + \cdot + \gamma)^{!},
\]

for \( x \in K \) and for \((j_1, \ldots, j_l) \subseteq \{1, 2, \ldots, N\} \), where \( \bar{n}_1 = (n+1)(\gamma-1)^{-1} \).

Let \( \phi=(\phi_1(y), \ldots, \phi_m(y)) \) be a mapping from \( R^m \) to \( R^n \). Noting that

\[
D_\phi(u(\phi(y))) = (X_{\bar{n}_1} X_{\bar{n}_2} \cdots X_{\bar{n}_m})u(\phi(y)),
\]

where \( X_j = \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial y_j} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_j} \), we obtain by virtue of Lemma 3.3.

**Lemma 3.4.** Assume that \( \phi_1(y) \) satisfy

\[
|D^n \phi_1(y)| \leq C_0 A_1^{\alpha | \alpha |} \left| \alpha \right| !',
\]

for \( y \in \Omega_1 \) a compact set in \( R^m \). Then for \( u(x) \) satisfying (3.3), if \( A > A_0 \), we have

\[
|D^n u(\phi(y))| \leq C(2^C_0 \bar{m}_1 A_0+1)^{\alpha | \alpha |} A^{\alpha | \alpha |} \left( | \alpha | + p \right)^{!},
\]

for \( y \in \Omega_1 \), where \( \bar{m}_1 = (m+1)(A/A_0 - 1)^{-1} A_0^{-1} \).

We consider an ordinary differential equation in \( y_0 \) with parameters \( y' = (y_1, \ldots, y_n) \),

\[
\sum_{j=0}^{m} a_j(y) D_j u(y) = f(y), \quad (a_m(y) = 1),
\]

(3.4)

\[
D^h \left| y_{0} = 0 \right. = u_h(y'), \quad h = 0, 1, \ldots, m-1.
\]

We assume that the coefficients satisfy

\[
|D^n a_j(y)| \leq C_0 A_j^{\alpha | \alpha |} \left| \alpha \right| !',
\]

for \( y \in K \), a compact set in \( R^{n+1} \), and that \( f(y) \) and \( u_k(y') \) are estimated by respectively,

\[
|D^n f(y)| \leq CA^{m+1} \sum_{\mu=0}^{m_0} (A | y_0 |)^{\mu} \left( | \alpha | + m + \mu + p \right)^{!},
\]

\( y \in K \),

(3.6)

\[
|D^n u_k(y')| \leq CA^{1+|h|} \left( | \alpha | + h + p \right)^{!}, \quad y' \in K \setminus \{y_0 = 0\}.
\]

We denote by \([h] \) the largest integer which does not exceed \( h \) and

\[
[h]_1 = \begin{cases} [h], & h > 0 \\ [h], & h \leq 0 \end{cases}, \quad [h]_1 = \begin{cases} [h], & h > 1 \\ 1, & h \leq 1. \end{cases}
\]
Then we have

**Proposition 3.5.** Assume that \( f(y) \) and \( u_h(y) \) satisfy (3.6) and (3.7) respectively. Then if \( A > 2^{s+s}C_0A_0 \), the solution \( u(y) \) of the equation (3.4) can be estimated by

\[
|D^n u(y)| \leq C \hat{C} A^{|\alpha|} \sum_{\mu=0}^{m-|\alpha|+1} \frac{(A |y_0|)^\mu}{\mu!} (|\alpha| + \mu + p)!^s,
\]

for \( y \in K \), where \( \hat{C} = m \frac{A}{2^{s+s}C_0A_0} \left( \frac{A}{2^{s+s}C_0A_0} - 1 \right)^{-2} \).

**Proof.** We reduce (3.4) to the first order system as follows,

\[
w_i = D_i^{-1} u, \quad i = 1, \ldots, m, \\
D_0 w_i = w_{i+1}, \quad i = 1, \ldots, m-1,
\]

(3.9)

\[
D_0 w_m = D_0^n u = - \sum_{j=0}^{m-1} a_j w_{i+1}.
\]

Denote

\[
M = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
-a_0 & \cdots & -a_{m-1}
\end{pmatrix}, \\
w = \begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_m
\end{pmatrix}, \\
F = \begin{pmatrix}
0 \\
0 \\
\vdots \\
f
\end{pmatrix}.
\]

Then we can rewrite (3.9),

\[
D_0 w = Mw + F.
\]

Putting \( S = \exp \left\{ \int_0^w M(t, y') dt \right\} \), we can represent

\[
w(y) = S(y) \left\{ w(0, y') + \int_0^w S(t, y')^{-1} F(t, y') dt \right\}.
\]

(3.10)

Denote by \( a_{q}^{(p)}(y) \) the \((p, q)\)-element of \( S(y)^{-1} \). Then by virtue of Lemma 3.3 with \( A = 2A_0 \) and \( \gamma = 2 \),

\[
|D^n a_{q}^{(p)}(y)| \leq (2^{s+s}C_0A_0)^{a_1} |y_0| \gamma^n, \quad y \in K.
\]

Therefore using again Lemma 3.2, we obtain from (3.6),

(3.11)

\[
|D^n S^{-1} F| = \max_t |D^n (a_{q}^{(p)}(y) f(y))| \\
\leq \frac{C \gamma_1}{\gamma_1 - 1} A^{m+a_1} \sum_{\mu=0}^{m} \frac{(A |y_0|)^\mu}{\mu!} (|\alpha| + m + \mu + p)!^s,
\]

where we have assumed that \( \gamma_1 = A(2^{s+s}C_0A_0)^{-1} \gamma > 1 \). We define

\[
T u(y) = \sqrt{-1} \int_0^w u(t, y') dt.
\]
Then we have
\[ D_0^i u = T D_0^{i+1} u + D_0 u(0, y') , \quad i=0, 1, \ldots . \]
Hence we obtain
\[ u(y) = T^{m-1} D_0^{m-1} u + \sum_{i=0}^{m-1} \frac{y_i}{i!} D_0^i u(0, y') . \]
(3.12)

On the other hand the \( m \)-th component of \( u \) is given by (3.10) as follows,
\[ w_m = D_0^{m-1} u(y) = \sum_{i=1}^{m} a_i^{(m)}(y) \{ D_0^{i-1} u(0, y') + T a_i^{(m)} f \} . \]
Hence we have from (3.4) and (3.12)
\[ (3.13) \quad u(y) = T^{m-1} (\sum_{i=1}^{m} a_i^{(m)}(y) \{ u_{i-1}(y') + T a_i^{(m)} f \}) + \sum_{i=1}^{m-2} \frac{y_i}{i!} u_i(y') . \]
For \( \alpha = (\alpha_0, \beta) \), \( \alpha_0 \leq m-1 \),
\[ D^\alpha u = T^{m-1} \sum_{\beta=0}^{m-1} \frac{y_0}{(1-\alpha_0)!} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} D^\beta a_i^{(m)} \{ D^\beta' u_{i-1} + T D^\beta (a_i^{(m)} f) \} + \sum_{i=0}^{m-2} \frac{y_i}{(i-\alpha_0)!} D_0^i u_i . \]
Hence from (3.6), (3.11) and Lemma 3.2 we have
\[ |D^\alpha u| \leq \left( \frac{1}{\sqrt{1-T}} \right)^{m-1-\alpha_0} \sum_{\beta=0}^{m-1} \frac{m_1^{(\beta)}}{r_1^{1-1}} C (|\beta| + \mu + \mu + \mu + \mu + \mu + \mu + \mu + \mu + \mu)! \]
\[ + \left( \frac{r_1}{r_1-1} \right)^{y_0} \left( \frac{1}{\sqrt{1-T}} \right) C A^{1+1} \sum_{\mu=0}^{\beta} \frac{|y_0 A|^{ \mu}}{\mu!} (|\beta| + \mu + \mu + \mu)! \]
\[ + \sum_{i=0}^{m-2} \frac{y_i}{(i-\alpha_0)!} A^{1+1} (|\beta| + i + \mu)! \]
\[ \leq \hat{C} A^{m+1} \sum_{\alpha=0}^{m+1} \frac{|y_0 A|^{ \mu}}{\mu!} (|\alpha| + \mu + \mu)! , \]
where \( \hat{C} = m_1^{(\beta_1)} (r_1-1)^{-1} \geq \max (1, m_1^{(\beta_1)} (1-r_1)^{-1}, r_1^{(m_1-1)^{-1}}) \). Next let \( \alpha = (\alpha_0, \alpha') \), \( \alpha_0 \geq m \) and put \( \beta = (\alpha_0 - m + 1, \alpha') \). Then from (3.13) we have
\[ (3.14) \quad D^\alpha u = D^\beta \{ \sum_{\beta_0=0}^{m} \frac{y_0}{(1-\alpha_0)!} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} D^\beta a_i^{(m)} \{ D^\beta' u_{i-1}(y') + \sum_{\beta_0=0}^{m} \frac{y_0}{(1-\alpha_0)!} \begin{pmatrix} \beta \\ \beta' \end{pmatrix} D^\beta a_i^{(m)} D^\beta (T a_i^{(m)} f) \} . \]
By virtue of (3.11) we obtain
\[ |D^\beta T a_i^{(m)} f| \leq \left( \frac{C_1}{r_1-1} \right)^{m+1} \sum_{\mu=0}^{m} \frac{|y_0 A|^{ \mu}}{\mu!} (|\beta| + m + \mu + \mu + \mu)! , \quad \text{if } \beta_0^2 \neq 0 \]
\[ \leq \left( \frac{C_1}{r_1-1} \right)^{m+1} \sum_{\mu=0}^{m} \frac{|y_0 A|^{ \mu}}{\mu!} (|\beta| + m + \mu + \mu + \mu + \mu)! , \quad \text{if } \beta_0^2 = 0 . \]
Hence we have from (3.14) and (3.7)
\[ |D^a u| \leq C \sum_{p=0}^{m} \frac{m!}{\mu!} |y_0 A|^\mu (|\alpha| + m + \mu + p)!^* \]

Thus we complete the proof of Proposition 3.5. We can prove the following theorem analogously

**Proposition 3.6.** We assume that the initial data \( u_h = 0, h=0, 1, \ldots, m-1 \) in (3.4) and \( f(y) \) satisfied for any \( \alpha \) and \( y \in K \)

\[ |D^a f(y)| \leq CA^{\alpha + m} \sum_{\mu=(k_0-m-a_0)}^{m+1} \frac{y_0 A^\mu}{\mu!} (|\alpha| + m + \mu - p_0)!^* , \]

where \( m_0, k_0 \) and \( p \) are non-negative integers. Then the solution \( u(y) \) of (3.4) is estimated by

\[ |D^a u(y)| \leq CCA^a \sum_{\mu=(k_0-m-a_0)}^{m+1} \frac{y_0 A^\mu}{\mu!} (|\alpha| + m + \mu - p_0)!^* , \quad y \in K , \]

where \( C=m\gamma(\gamma_1-1)^{-}\gamma_1 A(2^{\gamma_1}C_0 A_0)^{-1}>1 \) and \( h! = 0, \text{if}, h<0 \).

Now we shall prove Theorem 2.1 and Theorem 2.2. We return to the equations (2.15), and (3.5) if we replace the variables \( (z_0 - y_0, z', \xi') \) by \( y \) and \( K = K_0 \times S_{\xi'}^{-1} \), where \( K_0 \) is a compact set in \( R^{n+1} \) and \( S_{\xi'}^{-1} \) is a unit sphere in \( R^n \). Here we note that \( v_0^{(i)} \) and the coefficients of \( A^{(i)} \) and \( B^{(i)} \) are homogeneous degree zero in \( A \). Then we can prove that there are positive constants \( C \) and \( A \) such that

\[ (3.16)_{k,j} \quad |D^a U_{p,j}^{(i)}| \leq C(C_0 \hat{C}^{j+k+1} A^{(i)+j-k\kappa(\xi')-s_{p}^{(i)}} \times \sum_{\mu} \frac{y_0 A^\mu}{\mu!} [|\alpha| + j\kappa(\xi') - k\kappa(\xi') + \mu - s_{p}^{(i)}]!^* , \]

where the summation in \( \mu \) is from \([k\kappa(\xi')q(\xi') + s_{p}^{(i)} - a_0]_+ \) to \([km(\xi') + k\kappa(\xi')q(\xi') + m_{0}(\xi') - a_0]_+ + s_{p}^{(i)} \) .

We shall prove (3.16)_{k,j} by induction on \( k \) and \( j \). At first we estimate \( U_{p,j}^{(i)} \) by induction on \( j \). We recall the definition of \( U_{p,j}^{(i)} \), that is,

\[ U_{p,j}^{(i)} = U_{p,j}^{(i)}|_{x=(\xi, \hat{x})} = \begin{cases} v_0^{(i)}x_{\xi}^{(i)}|_{x=(\xi, \hat{x})} , & j \geq n_{p}^{(i)} \\ 0 , & j < n_{p}^{(i)} \end{cases} \]
where \( h_p^{(l)} = \max_{\lambda} n_{\lambda}^{(l)} - n_{\lambda}^{(l)} \) and \( v_p^{(l)} \) is the solution of (2.7) and (2.9). Put

\[
V_j^{(l)} = v_j^{(l)}|_{x=(z_0, \tilde{x}', \tilde{t}' \choose 0, 0)}.
\]

Then we shall prove

\[
(3.17)_{j} \quad |D^\alpha V_j^{(l)}| \leq C(C_0 C)^{j+1} A^{\alpha+\mu} \sum_{\rho=0}^{m} \frac{|y_\rho A|^\mu}{\mu!} (|\alpha| + j + \mu)!.
\]

from which it follows that \( U_{p}^{(j)} \) satisfies (3.16) if we choose \( s_p^{(j)} \leq h_p^{(j)} \) for each \( p \) and \( l \). From (2.7) we have

\[
(3.18) \quad A_{m}^{(j)}(y, D_0) V_j^{(l)} = F_j^{(l)},
\]

where \( F_j^{(l)} = \sum_{\rho=0}^{m} A_{m}^{(j)}(y, D_0) V_j^{(l)} \), and we replaced \( (z_0, y_0, z', \xi') \) to \( y \).

We note that \( A_{m}^{(j)}(y, D_0) \) is a differential operator of order \( m^{(j)} + \mu \) and in particular

\[
A_{m}^{(j)}(y, D_0) = \sum_{\rho=0}^{m} a^{(j)}_{\rho}(y) D_0^\rho, \quad (a^{(j)}_{m} \neq 0).
\]

We may assume that the coefficients of \( A_{m}^{(j)}(y, D_0) \) satisfy (3.5), if we choose \( C_0, \rho \) suitably. Hence by virtue of Proposition 3.5 with \( p=0 \), \( f=0 \), \( m_0=0 \) and \( m=m^{(j)} \), we can obtain (3.17) if we assume that (3.17) are valid. Then in (3.18) we have

\[
F_j^{(l)} = \sum_{\rho=0}^{m} a^{(j)}_{\rho}(y) D_0^\rho V_j^{(l)}.
\]

Hence the hypothesis of induction and (3.5) imply

\[
(3.19) \quad |D^\alpha F_j^{(l)}| \leq \sum |\alpha| |D^\alpha a^{(j)}_{\rho}||D^\beta a^\rho V_j^{(l)}|
\]

\[
\leq C(C_0 C)^{j} A^{m^{(j)} + \mu} \sum_{\rho=0}^{m} \frac{|y_\rho A|^\mu}{\mu!} (|\alpha| + m^{(j)} + j + \mu)!.
\]

where

\[
C_1 = m(n^m-1)\gamma_0(n-1)^{-1}(\gamma_0-1)^{-1}, \quad \gamma_0 = A/A_0 > 1.
\]

As the initial condition we have from (2.9)
\[
D^\mu V^{(l)} = m^{-1} \sum_{h=1}^{m-1} C^h \sum_{\rho \neq \mu} \sum_{\rho = m}^{m-1} M^h_{\rho, \mu} V^{(l)}_{\rho - \mu + m(t) - m(t)} + \sum_{l' = 1}^d \sum_{\rho \neq \mu}^{m-1} m^{-1} m(t) - m(t)
\]

where \( M^h_{\rho, \mu} = N^h_{\rho, \mu} \) \( x = (\rho, \rho') \). If we assume that \( m^{(1)} \geq m^{(2)} \geq \cdots \geq m^{(d)} \), we have

\[
V^{(l)}_{\rho - \mu + m(t) - m(t)} |_{\rho = 0} = \sum_{h=1}^{m-1} C^h \sum_{\rho \neq \mu} \sum_{\rho = m}^{m-1} M^h_{\rho, \mu} V^{(l)}_{\rho - \mu + m(t) - m(t)} |_{\rho = 0}.
\]

Hence by the assumption of induction we obtain

\[
|D^\mu C^{(l)}| \leq CA^{(\alpha)} |\alpha|^!.
\]

By induction of \( \mu \) we obtain

\[
|D^\mu V^{(l)}| |_{\rho = 0} \leq C(C_0 C_1)(C_0 \hat{C})^j A^{(\alpha + j + \mu)} |(\alpha + j + \mu)|!.
\]

for \( \mu = 0, 1, \ldots, m^{(1)} - 1 \). Moreover by induction on \( l \) we have

\[
|D^\mu V^{(l)}| |_{\rho = 0} \leq C(C_0 C_1)(C_0 \hat{C})^j A^{(\alpha + j + \mu)} |(\alpha + j + \mu)|!.
\]

for \( \mu = 0, 1, \ldots, m^{(1)} - 1 \) and \( l = 1, 2, \ldots, d \). Therefore by virute of Proposition 3.5 we obtain (3.17), if we choose \( \hat{C} \) and \( A \) such that \( \hat{C} \geq C_1 m^{(1)} \gamma_1 \gamma_1^{-1} \), \( \gamma_1 = A(2^{\gamma_1} + A_0)^{-1} > 1 \).

Next we shall prove (3.16) \( k, j, k \geq 1 \) by induction in \( k \) and \( j \). We assume (3.17) \( k-1, j, j_0 \), \( j \geq 1 \). From (2.15) \( k, j \) we have

\[
A^{(l)} U^{(l)} = G^{(l)}_{k, j},
\]

where

\[
G^{(l)}_{k, j} = \sum_{q=1}^N B^{(l)}_{k, j} U^{(l)}_{q, j_0} = \sum_{q=1}^N \sum_{\rho = 0}^{d} B^{(l)}_{k, j} U^{(l)}_{q, j_0} D^{(l)}_{k, j} U^{(l)}_{q, j_0}.
\]

Noting that \( d^{(l)}_{q, \rho} \leq m^{(1)} - \kappa^{(1)} q^{(1)} + s^{(1)} q^{(1)} - s^{(1)} q^{(1)} \) and that the coefficients of \( B^{(l)}_{k, j} \) satisfy (3.5), we have

\[
|D^\mu G^{(l)}_{k, j}| \leq |\sum_{\alpha} \alpha^! D^{\alpha} B^{(l)}_{k, j} | |D^{\alpha} D^{(l)}_{k, j} | |D^{\alpha} U^{(l)}_{k, j} |
\]

\[
\leq C |\sum_{\alpha} \alpha^! C_0 (C_0 \hat{C})^k A^{(\alpha + 1)} |(\alpha + 1)! A^{(\alpha + 1)} d^{(l)}_{q, \rho} - (k-1) \kappa^{(1)} q^{(1)} + \mu - s^{(1)} q^{(1)} |\gamma_0 A \mu |^{!}
\]

\[
\times [\alpha^! + d^{(l)}_{q, \rho} - t - (k-1) \kappa^{(1)} q^{(1)} + \mu - s^{(1)} q^{(1)}]!,
\]

\[
\leq C \sum_{\alpha} \sum_{\rho = 1} \gamma_0 \gamma_0^{-1} (C_0 \hat{C})^k A^{(\alpha + 1)} m^{(1)} - \kappa^{(1)} q^{(1)} - s^{(1)} q^{(1)} - 1 |\gamma_0 A \mu |^{!}
\]

\[
\times [\alpha^! + m^{(1)} - k \kappa^{(1)} q^{(1)} - t - \mu - s^{(1)} q^{(1)}]!.
\]
where the summation in \( \mu \) is from \([(k-1)\kappa^{(1)}q^{(1)}} + s^{(1)} + (d_E^{(1)} - t + \alpha_0) \) to \([(k-1)\kappa^{(1)}q^{(1)} + [m^{(1)} - \alpha_0 - d_E^{(1)} + t] \) + \( s^{(1)} \). \( \gamma_0 = A/A_0 > 1 \) and \( h \neq 0, \) if \( h < 0. \) It follows from the assumption (1.17) and (1.14)

\[
(3.20) \quad s^{(1)} - \kappa^{(1)}q^{(1)} + [m^{(1)} - \alpha_0 - d_E^{(1)} + t] + t
\]

\[
\leq s^{(1)} - \alpha_0 + s^{(1)}(\text{if } m^{(1)} - \alpha_0 - d_E^{(1)} + t > 0)
\]

\[
\leq s^{(1)} - \kappa^{(1)}q^{(1)} + 1 - t \leq s^{(1)}(\text{if } m^{(1)} - \alpha_0 - d_E^{(1)} + t \leq 0).
\]

Hence replacing \( \mu \) to \( \mu + t, \) we obtain

\[
|D^\alpha G^{p,q}_{k,j} | \leq CC_0 \frac{\gamma_0}{\gamma_k - 1} (C_0 \hat{C})^A A_{\alpha' + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p} \times m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p
\]

\[
\leq CC_0 ( \gamma_0 ) \frac{\gamma_0}{\gamma_k - 1} (C_0 \hat{C})^A A_{\alpha + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p}
\]

\[
\times \frac{y_0 A^\mu |y_0|^t}{(\mu + t)!} [\alpha | + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p + \mu] ! ,
\]

where \( C_0 = \gamma_0/M(1 + \tilde{\alpha}^m) / \gamma_k - 1. \) As the initial condition of \( U^{(1)}_{k,j} \) it follows from (2.18) at that

\[
D_{k,j} U^{(1)}_{k,j} |_{y_0 = 0}, \quad h = 0, 1, \cdots, m^{(1)} - 1.
\]

Therefore we can apply Proposition 3.6 to \( U^{(1)}_{k,j} \), putting \( k_0 = \kappa^{(1)}q^{(1)} + s^{(1)}_p, \) \( m = m^{(1)}, \) \( m_0 = 0 \) and \( p_0 = \kappa^{(1)}q^{(1)} + s^{(1)}_p. \) Thus we obtain (3.16) at if we choose \( \hat{C} \) and \( A \) such that

\[
\hat{C} \geq m^{(1)} \gamma_i (1 - \gamma_1)^{-j} C_0,
\]

\[
\gamma_i = A(2^{+s} C_0 A_0)^{-1} > 1 ,
\]

\[
\gamma_0 = A/A_0 > 1.
\]

Next we assume that (3.16) at are valid for \( i = 0, 1, \cdots, j - 1. \) Then we shall estimate \( F^{p,q}_{k,j} \) and \( G^{p,q}_{k,j} \) in (2.15)at. We have by (2.61) and (2.17)

\[
|D^\alpha F^{p,q}_{k,j} | \leq \sum_{\alpha', \mu} \sum_{\alpha' + m^{(1)} - \kappa^{(1)}q^{(1)} + s^{(1)}_p} |D^\alpha A_{m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p} | \times \frac{y_0 A^\mu |y_0|^t}{(\mu + t)!} [\alpha | + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p + \mu] ! ,
\]

\[
\leq CC_0 ( \gamma_0 ) \frac{\gamma_0}{\gamma_k - 1} (C_0 \hat{C})^A A_{\alpha + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p}
\]

\[
\times \frac{y_0 A^\mu |y_0|^t}{(\mu + t)!} [\alpha | + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p + \mu] ! ,
\]

\[
|D^\alpha G^{p,q}_{k,j} | \leq CC_0 ( \gamma_0 ) \frac{\gamma_0}{\gamma_k - 1} (C_0 \hat{C})^A A_{\alpha + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p}
\]

\[
\times \frac{y_0 A^\mu |y_0|^t}{(\mu + t)!} [\alpha | + m^{(1)} - \kappa^{(1)}q^{(1)} - s^{(1)}_p + \mu] ! .
\]
where the summation in $\mu$ is from $[k\kappa^{(i)}q^{(l)}+s^{(i)}]-m^{(i)}-\alpha_0]$ to $[jm^{(i)}+k\kappa^{(i)}q^{(l)}+s^{(i)}]$. Next we shall estimate $G_{\beta,i}$. From (2.16) and (2.17) we have

$$G_{\beta,i}^{(l)} = \sum_{q=1}^{N} B_{\beta,q}^{(l)} U_{\beta,q-1,j}^{(l)} + \sum_{q=1}^{N} \sum_{t=1}^{\infty} B_{\beta,q}^{(l)} U_{\beta,q-t,j}^{(l)},$$

$$B_{\beta,q}^{(l)} = \sum_{\beta=1}^{\beta(i)} B_{\beta,q}^{(l)} \eta(y) D^{\gamma \delta},$$

where

$$(3.21)$$

$$d_{\beta,q}^{(l)} \leq m^{(i)} - \kappa^{(i)}(q^{(l)} - i) + s^{(i)} - s^{(i)}$$

and in particular we note that $B_{\beta,q}^{(l)}$ are differential operators only in $y_0$. Hence

$$|D^{\alpha}(B_{\beta,q}^{(l)} U_{\beta,q-1,j})| \leq \sum_{\alpha'=\alpha} \left( \sum_{\alpha'=\alpha} \right) |D^{\alpha'} B_{\beta,q}^{(l)}| |D^{\alpha''} U_{\beta,q-1,j}|$$

$$\leq \sum_{\alpha} |C_0 A_{\alpha}^{(l')}| |\alpha'| |C(C_0 C)_{\alpha}^{(l')} A|^{\alpha'} + m^{(i)} - \kappa^{(i)}(j-i) - (k - 1)C_{\beta}^{(l)} q^{(l)} - s^{(i)} + \mu \right) !,$$

where the summation in $\mu$ is from $[(j-i)m^{(i)}+(k-1)\kappa^{(i)}q^{(l)}+s^{(i)}]-\beta_0 - \alpha_0]$ to $[jm^{(i)}+k\kappa^{(i)}q^{(l)}+s^{(i)}+m^{(i)}]$. Hence

$$|D^{\alpha}(B_{\beta,q}^{(l)} U_{\beta,q-1,j})| \leq C(C_0 C)_{\alpha}^{(l')} A|^{\alpha'} + m^{(i)} - \kappa^{(i)}(j-i) - (k - 1)C_{\beta}^{(l)} q^{(l)} - s^{(i)} + \mu \right) !,$$

where $\mu = [k\kappa^{(i)}q^{(l)}+s^{(i)}]$. Next for $i \geq 1$, we have

$$|D^{\alpha}(B_{\beta,q}^{(l)} U_{\beta,q-1,j})| \leq \sum_{\alpha'} \left( \sum_{\alpha'} |D^{\alpha'} B_{\beta,q}^{(l)}| |D^{\alpha''} U_{\beta,q-1,j}|$$

$$\leq \sum |C_0 A_{\alpha}^{(l')}| |\alpha'| |C(C_0 C)_{\alpha}^{(l')} A|^{\alpha'} + m^{(i)} - \kappa^{(i)}(j-i) - (k - 1)C_{\beta}^{(l)} q^{(l)} - s^{(i)} + \mu \right) !,$$

where the summation in $\mu$ is from $[(j-i)m^{(i)}+(k-1)\kappa^{(i)}q^{(l)}+s^{(i)}]-\beta_0 - \alpha_0]$ to $[jm^{(i)}+k\kappa^{(i)}q^{(l)}+s^{(i)}+m^{(i)}]$. Noting that $[(j-i)m^{(i)}+(k-1)\kappa^{(i)}q^{(l)}+s^{(i)}]-\beta_0 - \alpha_0]$ and $[jm^{(i)}+k\kappa^{(i)}q^{(l)}+s^{(i)}+m^{(i)}]$, we have

$$|D^{\alpha}(B_{\beta,q}^{(l)} U_{\beta,q-1,j})| \leq [jm^{(i)}+k\kappa^{(i)}q^{(l)}+s^{(i)}],$$
we obtain
\[
|D^aG_k^{(i)}| \leq \sum_{q=1}^{N} (|D^a(B_{q,i}^{(i)}U_k^{(i),j})| + \sum_{j=1}^{m} |D^a(B_{q,j}^{(i)}U_k^{(i),j-1})|)
\]
\[
\leq C(C_0C_0^\gamma)A^{i+1}A_{q1}^{(i)+j-\kappa_r^{(i)}q^{(i)}-s_p^{(i)}}
\]
\[
\times \sum_{r} \frac{|y_0|A^{\mu}}{\mu!} \left[ |\alpha| + m^{(i)} + j\kappa_r^{(i)}q^{(i)} - s_p^{(i)} + \mu \right]^r
\]
where \(\mu = [k\kappa^{(i)}q^{(i)} + s_p^{(i)} - m^{(i)} - \alpha_0]_4, \ldots, [jm^{(i)} + k\kappa^{(i)}q^{(i)} + s_p^{(i)}]\) and \(C_0 = \gamma_0 Nm(1 + \delta^m)(n_m - 1)(n - 1)^{-1}(r_0 - 1)^{-1}\).

As the initial condition we have by (2.18),
\[
|D^aU_k^{(i)}|_{y_0=0} = 0, \quad h = 0, 1, \ldots, m^{(i)} - 1.
\]
Thus we can apply Proposition 3.6 to \(U_k^{(i)}\) and we obtain (3.16) if we choose \(C\) and \(A\) such that \(C = 2m_0^{(i)}(r_1 - 1)^{-1} \max C_i, r_1 = A(2^{2m_0}(m_0 + 1)) > 1\) and \(r_0 = AA_0^{-1} > 1\).

\section*{4. Construction of Fundamental solution}

In the previous sections we have sought for the asymptotic solutions for (2.1). In the present section we shall construct the fundamental solution by use of the solutions of (2.2) and (2.3). In the expression (2.11) we denote in brief \(\xi' = m^{(i)} + kq^{(i)} - j\) by \(w_k^{(i)}(x, y_0, \xi')\) which is homogeneous degree \(m^{(i)} + kq^{(i)} - j\) in \(\xi'\), where \(m^{(i)} = -m + m^{(i)} + \bar{n}_k^{(i)}\). Hence we have from Theorem 2.2,
\[
|D^aG_k^{(i)}(x, y_0, \xi')| = C(C_0C_0^\gamma)A^{i+1}A_{q1}^{(i)+j-\kappa_r^{(i)}q^{(i)}-s_p^{(i)}}
\]
\[
\times \sum_{r} \frac{|y_0|A^{\mu}}{\mu!} \left[ |\alpha| + m^{(i)} + j\kappa_r^{(i)}q^{(i)} - s_p^{(i)} + \mu \right]^r
\]
which converges uniformly by (4.1), and put
\[
r^{(i)}(x, y_0, \xi') = e^{-\psi^{(i)}}a(x, D)e^{\psi^{(i)}}\omega^{(i)} - \sum_{q=1}^{N} e^{-\psi^{(i)}b_q^{(i)}(x, D)e^{\psi^{(i)}}\omega^{(i)}}.
\]
It is then clear that \( w^{(1)} \) and \( r^{(1)} \) satisfy (1.20) and (1.21) respectively, according to (4.1) and (4.2). Here our aim is to prove the existence of the solution for the integral equation (1.19).

**Lemma 4.1** (c.f. [5]). There exists a positive constant \( C \) such that for any positive number \( \rho \)
\[
\inf_{j \in \mathbb{N}} \frac{j^4}{\rho^j} \leq C \sqrt{\rho + 2} e^{1 - \rho}.
\]

Noting that \( \sqrt{\rho + 2} \leq 2e^{\rho/2} \), we have from the above lemma
\[
\inf_{M \in \mathbb{N}} (A_0^{-1} |\xi'|)^{-M} M! e^{(t_1 + m(s-1))} \leq C e^{-\frac{1}{2} (\xi(t_1 + m(s-1)))}.\]

Hence by (1.21) we have the estimate
\[
|D_2 D_{y_1}^3 r^{(t)}| \leq CA_1^{(a+\beta)} \exp \left\{ |x_0 - y_0| A_2 |\xi'|^{1+\kappa(t)} - \frac{1}{2} \left( |\xi'| A_1 \right)^{(t_1 + m(s-1)-1)} \right\} \times |\alpha + \beta| |\xi'|^{m-1+\alpha} \exp |\xi'|^{-1}.
\]

We define
\[
R^{(1)}(x_0, y_0) u(x') = \int \int e^{i\phi^{(1)}(x, y, \xi')} r^{(1)}(x, y, \xi') u(y') d\xi' dy',
\]
which is an operator from \( \gamma_s(R^n) \) to \( \gamma_s(R^n) \), where \( \phi^{(1)} \) is a phase function. Moreover we can see more precisely,

**Proposition 4.2.** Assume that \( u \in \gamma_s(R^n) \) satisfies
\[
|D_2^2 u(x')| \leq CA_1^{(a+\beta)} |\alpha| |\xi'|^{1+\kappa(t)} \quad \text{for } x' \in R^n,
\]
When \( s < \kappa^{(1)} \), there exist positive constants \( \hat{C}, \hat{\gamma} \) and \( \hat{A} \) such that
\[
|D_2^2 R^{(1)}[x_0, y_0] u(x')| \leq \hat{C} e^{(|x_0 - y_0| A)^{a_1}} |\alpha| |\xi'|^{m-1+\alpha} \exp |\xi'|^{-1}.
\]
for \( x' \in R^n, |x_0 - y_0| \leq A > \hat{A} \). When \( s = \kappa^{(1)} \), there exist \( \hat{C}, \hat{A} \) and \( \hat{\gamma} \) such that
\[
|D_2^2 R^{(1)}[x_0, y_0] u(x')| \leq \hat{C} e^{(|x_0 - y_0| A)^{a_1}} |\alpha| |\xi'|^{m-1+\alpha} \exp |\xi'|^{-1}.
\]
for \( x' \in R^n, |x_0 - y_0| \leq \min \{|\delta, \delta A^{-1+\gamma(t)}\} \) and \( A > \hat{A} \).

**Proof.** The phase function \( \phi^{(1)} \) can be decomposed as
\[
\phi^{(1)}(x, y, \xi') = \langle \theta^{(1)}(s, y, \xi') + x' - y', \xi' \rangle,
\]
where \( \theta^{(1)} = (\theta_1^{(1)}, \ldots, \theta_n^{(1)}) \) satisfies
\[
|D_2^2 D_{y_1}^3 \theta_i^{(1)}| \leq \gamma_1 |x_0 - y_0| A_1^{(a+\beta)} |\alpha + \beta| |\xi'|^{-1-\delta_i}.
\]
Leray-Volevich's systems and Gevrey class

Then we can write

$$R^p(\alpha_0, \gamma_0)u(x') = \int e^{-i\langle x', \xi \rangle} u(\theta^{(1)} x, \gamma_0)dy'd\xi.$$ 

Hence

$$D_\xi^p R^p(\alpha_0, \gamma_0)u(x') = \sum_{\alpha' \in a_{\alpha_0}} \int e^{-i\langle x', \xi \rangle} D_\xi^p R^p(\alpha_0)D_\xi^{\alpha'}(u(\theta^{(1)} + x' + \gamma'))dy'd\xi',$$

Noting that

$$D_\xi^p(u(\theta^{(1)} + x' + \gamma')) = (X_{1}^\xi \cdots X_{n}^\xi u)(\theta^{(1)} + x' + \gamma'),$$

where $X_i = \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \theta_j^{(1)} \frac{\partial}{\partial y_j} + \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_1}$, we have

$$D_\xi^p R^p(\alpha_0, \gamma_0)u(x') = \sum_{\alpha' \in a_{\alpha_0}} \int e^{i\phi^{(1)}} D_\xi^p R^p(\alpha_0)X_{1}^\xi \cdots X_{n}^\xi u)(\gamma')dy'd\xi'.$$

For $x' \in \mathbb{R}^n$ fixed, there exist a compact set $K$ in $\mathbb{R}^n$ such that for $y' \in K$

$$\phi^{(1)} = \theta^{(1)} + x' - y' \neq 0.$$ 

Hence we can find a first a first order differential operator $L^{(1)}(x', y', D_{y'})$ such that

$$L^{(1)}(e^{i\phi^{(1)}}) = (1 + |x' - y'|^{1/2}) e^{i\phi^{(1)}}, \quad \text{for} \quad y' \in K,$$

that is,

$$L^{(1)} = (1 + |x' - y'|^{1/2}) \left\{ \sum_{j=1}^{n} |\phi_j^{(1)}|^{-1} \sum_{j=1}^{n} \phi_j^{(1)} \frac{\partial}{\partial \xi_j} \right\}.$$

Then the coefficients $a_j^{(1)}$ of $L^{(1)}$ satisfy

$$|D_\xi^p a_j^{(1)}| \leq C_1 A_j^{1/2} |\theta|^{1/2}, \quad \text{for} \quad y' \in K,$$

if we choose $C_1$ and $A_j$ suitably. Let $\Psi(y')$ be a function of $\gamma_0(\mathbb{R}^n)(s=\kappa^{(1)})$ if $s = \kappa^{(1)}$ and $s < s < \kappa^{(1)}$, if $s < \kappa^{(1)}$ such that $\Psi = 1$ on $K$, supp $\Psi$ compact and

$$|D_\xi^p \Psi(y')| \leq C_2 A_j^{1/2} |\theta|^{1/2}$$

for $y' \in \mathbb{R}^n$. Then we decompose (4.5)

$$D_\xi^p R^p(\alpha_0, \gamma_0)u(x') = \sum_{\alpha' \in a_{\alpha_0}} \int e^{i\phi^{(1)}} D_\xi^p R^p(\alpha_0)(1 - \Psi)(X_{1}^\xi \cdots X_{n}^\xi u)(y')dy'd\xi'$$

$$+ \sum_{\alpha' \in a_{\alpha_0}} \int e^{i\phi^{(1)}} D_\xi^p R^p(\alpha_0)\Psi(X_{1}^\xi \cdots X_{n}^\xi u)(y')dy'd\xi'.$$
We at first estimate \( I \). By (4.6) have
\[
I_1 = \sum \left( \frac{\alpha}{\alpha'} \right) \int e^{i\xi \cdot (1 - \Psi)(1 + |\xi - y'|^2)^{-\frac{n+1}{2}}} \\
= (L^{(1)})^{n+1}D^{\alpha_1}_y \cdot r^{\alpha_1}(X_1^* \cdots X_n^* u(y') dy dy' dx_0 dx_1 \cdots dx_n.
\]
We can write
\[
(L^{(1)})^{n+1}D^{\alpha_1}_y \cdot r^{\alpha_1}(X_1^* \cdots X_n^* u(y'))
\]
where \( \beta(x, y_0, \xi) \) satisfies (4.7), if we choose \( A \) and \( C \) suitably. Moreover we have
\[
D^{\alpha_1}_y (X_1^* \cdots X_n^* u)(y) = (Y_1 Y_2 \cdots Y_{\alpha_1} u)(y)
\]
where
\[
Y_j = \sum \frac{\xi \cdot X_j \cdot \partial}{\partial y_j} + \frac{\partial}{\partial y_j} + \frac{\partial}{\partial y_j}.
\]
where \( i(j) \in [1, \cdots, n] \) and \( \beta^{(j)}(x, \xi) = (\beta^{(j)} \cdots \beta^{(j)}) \), \( |\beta^{(j)}| \leq n+1 \). Noting that the coefficients of \( Y_j \) satisfy (4.4), we obtain by virtue of the lemma A.1 (It's proof will be given in the appendix.), for \( y' \in \text{supp} (1 - \Psi) \),
\[
|D^{\alpha_1}_y (Y_1 \cdots Y_{\alpha_1} u)(y')| \leq C^\alpha (1 + |\alpha| + |\alpha'|)^{1/fr}
\]
for any \( \epsilon > 0 \), where we used the inequality
\[
(|\alpha| + |\alpha'|) \leq (1 + \epsilon) |\alpha| + |\alpha'| |\alpha| |\alpha'|.
\]
We choose later on \( \epsilon = r_0 |x_0 - y_0| \). We put
\[
F_{\alpha}(\xi') = \int e^{-i<y', \xi'>} (1 - \Psi(y'))(Y_1 \cdots Y_{\alpha_1} u)(y')(1 + |y' - x'|^2)^{-\frac{n+1}{2}} dy'.
\]
Then we have
\[
\xi'^{\alpha} F_{\alpha}(\xi') = \int e^{-i<y', \xi'>} D^{\alpha_1}_y ((1 - \Psi)(Y_1 \cdots Y_{\alpha_1} u)(y')(1 + |y' - x'|^2)^{-\frac{n+1}{2}} dy'.
\]
Hence from (4.8) and (4.9)
\[
|\xi'^{\alpha} F_{\alpha}| \leq C^\alpha (1 + \epsilon)^{\alpha} |\alpha| |\alpha'| |\alpha'| |\alpha'|.
\]
for any \( \alpha \). Therefore from Lemma 4.1 it follows that
\begin{equation}
|F_{a}(\xi')| \leq CC_{\varepsilon}(e^{x_{0}-y_{0}T(1+\varepsilon)^{T}}A)^{\alpha'_{\varepsilon}}|\alpha^{\prime}_{\varepsilon}|^{!} \exp \left\{ -\left( e^{\varepsilon'}|\xi'| \right)^{1/\varepsilon} \right\} .
\end{equation}

\[
\leq CC_{\varepsilon}(e^{x_{0}-y_{0}T(1+\varepsilon)^{T}}A)^{\alpha'_{\varepsilon}}|\alpha^{\prime}_{\varepsilon}|^{!} \exp \left\{ -\left( \frac{|x_{0}-y_{0}|T}{(1+\varepsilon)^{T}|\xi'| \varepsilon} \right)^{1/\varepsilon} \right\}
\]
where we put \( \varepsilon = \gamma_{o} |x_{0}-y_{0}| \) and \( \hat{\varepsilon} = \gamma_{o} \delta \). We choose \( \gamma_{o} \) such that
\[
|x_{o}-y_{o}|A_{2}|\xi|^{1/k^{(t)}} - \frac{|x_{0}-y_{0}|\gamma_{0}}{(1+|x_{0}-y_{0}|\gamma_{0})A^{2/k}} |\xi'|^{1/\hat{\varepsilon}} \lesssim 0 \quad (|\xi'| \to \infty).
\]
When \( s < k^{(t)} \), the above is valid. When \( s = k^{(t)} \), for \( |x_{o}-y_{o}|A_{2}^{1/k^{(t)}} \leq 1/2 \), we may put \( \gamma_{o} = 2A^{1/k^{(t)}} \). Then we can estimate from (4.3) and (4.11)
\[
|I_{1}| \leq \sum \left( \alpha_{\varepsilon}^{\prime} \right)^{1} \sum_{\beta \cdot \gamma \in \mathbb{Z}^{2n}} |a_{\beta \cdot \gamma}| |D_{\varepsilon}^{\beta_{\varepsilon}}D_{\varepsilon}^{\gamma_{\varepsilon}}P_{\varepsilon}^{(t)}| |F_{a_{\varepsilon}}| d\xi'
\leq CC_{\varepsilon}(e^{x_{0}-y_{0}T(1+\varepsilon)^{T}}A)^{\alpha(|\alpha^{\prime}_{\varepsilon}|^{!})},
\]
for \( |x_{o}-y_{o}| \leq \delta \), \( (|x_{o}-y_{o}| \leq \min (\delta, (2A_{2}A)^{-1}) \), if \( s = k^{(t)} \). For \( I_{2} \), we put
\[
G_{a_{\varepsilon}} = e^{-i(x_{o} \cdot \varepsilon' + X^{\prime}_{X_{0}} \cdot t)}X_{a_{\varepsilon}}(y')dy'.
\]
Then noting that \( \text{supp} \mathcal{V} \) is compact, we can prove analogously that \( G_{a_{\varepsilon}} \) satisfies (4.10) and therefore \( I_{2} \) also satisfies (4.4).

Now we shall construct a solution \( F_{p}^{0}[x_{0}, y_{o}] \) of the integral equation (1.19). We define inductively
\[
F_{p}^{0}[x_{0}, y_{o}] = R_{p}^{0}[x_{0}, y_{o}] = \sum_{i=1}^{d} R_{p}^{i}(x_{0}, y_{o}) ,
\]
\[
F_{p}^{j}[x_{0}, y_{o}] = \int_{y_{0}}^{x_{0}} R_{p}^{0}[x_{0}, t] F_{p}^{j-1}[t, y_{o}] dt , \quad j = 1, 2, \cdots .
\]
Then we have

**Proposition 4.3.** Let \( \psi(x') \) be in \( \gamma_{s}(R_{n}) \) and satisfies
\begin{equation}
|D^{\alpha} \psi(x')| \leq C A^{\alpha_{0}}|\alpha^{\prime}|^{!}, \quad x' \in R_{n}.
\end{equation}
Then when \( s < \inf_{i} k^{(t)} \), there exists positive constants \( \hat{C}, \hat{C}_{o}, \hat{p} \) and \( \hat{A} \) independent of \( j \) and \( \alpha \) such that
\begin{equation}
|D^{\alpha} F_{p}^{j}[x_{0}, y_{o}] \psi(x')| \leq \hat{C} \sum_{j=1}^{d} \left( \frac{|x_{0}-y_{o}|}{\hat{A}^{1/k^{(t)}}} \right)^{j} (e^{\hat{p}|x_{0}-y_{o}|A})^{\alpha_{0}}|\alpha^{\prime}|^{!},
\end{equation}
for \( |x_{0}-y_{o}| \leq \delta \) and \( A > \hat{A} \). When \( s = \inf_{i} k^{(t)} = \hat{p} \), moreover there exists \( \hat{\delta} > 0 \) such that \( F_{p}^{j}[x_{0}, y_{o}] \) satisfies (4.1) for \( |x_{0}-y_{o}| \leq \min (\delta, \hat{\delta}A^{-1/x}) \) and \( A > \hat{p} \).

**Proof.** We shall prove (4.12) \( j \) by induction. For \( j = 0 \) it follows from Proposition 4.2 that (4.12) \( j = 0 \) is valid. Assume that (4.12) \( j = 1 \) is valid. Then again
applying Proposition 4.2 to \( u = F_{t}^{y}[t, y_{o}]v \), we obtain
\[
| R^{p}[x_{o}, t] F_{t}^{y}[t, y_{o}]v | \leq \mathcal{C}C \frac{(e^{(t-x_{0})} e^{(t-y_{0})} A)^{|\alpha|}}{(j-1)!} | \alpha | !^{s}
\]
which we integrate in \( t \), we have (4.12).

We define
\[
F^{p}[x_{o}, y_{o}] = \sum_{j=0}^{\infty} F^{p}[x_{o}, y_{o}]
\]
which is evidently a solution of (1.19) and satisfies
\[
| D_{p}^{s} F^{p}[x_{o}, y_{o}]v(x') | \leq \mathcal{C}C (e^{(t-x_{0})} e^{(t-y_{0})} A)^{|\alpha|} | \alpha | !^{s}.
\]

Thus we have prove that the fundamental solution \( K^{p}[x_{o}, y_{o}] \) is given by (1.18) and satisfies
\[
| D_{p}^{s} K^{p}[x_{o}, y_{o}]v(x') | \leq \mathcal{C}C (e^{(t-x_{0})} e^{(t-y_{0})} A)^{|\alpha|} | \alpha | !^{s}.
\]
for \( | x_{o} - y_{o} | \leq \delta \) and \( A > \hat{A} \) \((| x_{o} - y_{o} | \leq \inf (\delta, \hat{A} A^{1/2}) \) if \( s = \hat{k} \) where \( v \) satisfies (4.11).

Appendix

Here we shall prove the estimate (4.9), which is sharper one than Lemma 3.3. Let \( X_{i} = \sum_{l=1}^{n} a_{ij}(x) \frac{\partial}{\partial y_{j}} + \frac{\partial}{\partial y_{j}} + \frac{\partial}{\partial x_{j}} \) \((j = 1, 2, \ldots, n) \) be first order differential operators. We assume that
\[
| D_{p}^{s} a_{ij}(x) | \leq \varepsilon_{0} A_{0} \left| \alpha \right| !^{s}, \quad x \in X, \quad i = j = 1, \ldots, n,
\]
where \( X \) is a compact set in \( \mathbb{R}^{n} \) and that \( u(y) \) satisfies
\[
| D_{p}^{s} u(y) | \leq C A \left| \alpha \right| !^{s}, \quad y \in X_{1}.
\]
Then we have

**Lemma A.1.** Assume that \( A > A_{0} \). Then there exist positive constants \( \hat{C} \) and \( \gamma \) such that
\[
| D_{p}^{s} X_{j_{1}} X_{j_{2}} \ldots X_{j_{k}} u(y) | \leq \hat{C} C \left( 1 + \gamma \varepsilon_{0} A \right) A_{0} \left| \alpha \right| k \left| \alpha \right| !^{s},
\]
for \( p = 1, 2, \ldots, \) and \( x \in X, \) \( y \in X_{1}, \) where \( (j_{1}, \ldots, j_{k}) \subset \{1, \ldots, n\} \).

**Proof.** We assume that
\[
| D^{(s)} X_{j_{1}} \ldots X_{j_{k}} u(y) | \leq C \sum_{l=1}^{s} C_{i_{1}, \ldots, i_{s}} | i_{1} \ldots i_{s} | A \left| \alpha \right| + j \left| \alpha \right| !^{s},
\]
where \( b_{i_{1}, \ldots, i_{s}}^{(s)} \) and \( d_{i_{1}, \ldots, i_{s}}^{(s)} \) are positive constants. We shall prove that there exists a positive constant \( \gamma \) such that for any positive integer \( k \)
We shall at first derive the recursive relation of $C_{i_1,i_j}^{(k)}$. We have
\[
|(D_x^2 D_y^2 X_{j_1})(X_{j_2} \cdots X_{j_k})u)|
\leq \left| D_x^2 D_y^2 \left( \sum a_{j_1i}(x) \frac{\partial}{\partial y_{j_1}} + \frac{\partial}{\partial x_{j_1}} \right) X_{j_2} \cdots X_{j_k} u \right|
\leq \sum \left( \frac{\beta}{\beta'} \right) |D_x^2 a_{j_1i}| \left| D_y^2 D_y^2 \frac{\partial}{\partial y_{j_1}} X_{j_2} \cdots X_{j_k} u \right|
+ \left| D_x^2 D_x^2 \frac{\partial}{\partial x_{j_1}} X_{j_2} \cdots X_{j_k} u \right|
\leq C \sum \left( \frac{\beta}{\beta'} \right) n \varepsilon_0 A_{i_1}^{i_1+1} |\beta'| \left| \varepsilon_0 \sum_{j=1}^{k-1} C_{i_1^{(k-1)},j}^{(k-1)} A_{i_0}^{i_0+1+j}(|\alpha| + 1 + j)! \right|
+ C \sum_{j=1}^{k-1} C_{i_1^{(k-1)},j}^{(k-1)} A_{i_0}^{i_0+1+j}(|\alpha| + j)!\varepsilon_0

Hence we obtain for $k \geq 2$
\[
\begin{align*}
C_{i_1^{(k-1)},0}^{(k)} & = C_{i_1^{(k-1)},1}^{(k-1)} \\
C_{i_1^{(k-1)},j}^{(k)} & = C_{i_1^{(k-1)},j-1}^{(k-1)} + n \varepsilon_0 \varepsilon_0 \sum_{j_1=0}^{k-2} \left( \frac{\beta}{\beta'} \right) A_{i_1^{(k-1)}}^{i_1^{(k-1)}+1+j} C_{i_1^{(k-1)},j-1}^{(k-1)}, \quad j=2, \ldots, k-1, \\
C_{i_1^{(k-1)},k}^{(k)} & = C_{i_1^{(k-1)},k-1}^{(k-1)} + n \varepsilon_0 \varepsilon_0 \sum_{j_1=0}^{k-2} \left( \frac{\beta}{\beta'} \right) A_{i_1^{(k-1)}}^{i_1^{(k-1)}+1+j} C_{i_1^{(k-1)},k-1}^{(k-1)}.
\end{align*}
\]
From (A.4) we have
\[
\sum_{i=0}^{k-1} b_{i,1}^{(k)} A_{i}^{i+1+j}(|\beta| + i)! + d_{i}^{(k)} = \sum_{i=0}^{k-1} b_{i,1}^{(k-1)} A_{i}^{i+1+j}(|\beta| + 1 + i)! + d_{i}^{(k-1)}
= \sum_{i=1}^{k-1} b_{i,1}^{(k-1)} A_{i}^{i+1+j}(|\beta| + i)! + d_{i}^{(k-1)}.
\]
Hence we obtain for $k \geq 2$, 
\[ b^{(k)}_{i, k} = 0 \]
\[ b^{(k)}_{i, i} = b^{(k-1)}_{i, i-1}, \quad i = 1, \ldots, k-1, \]
\[ d^{(k)}_{i} = d^{(k-1)}_{i}. \]

For $k = 1$ noting that
\[
|D_2 D_2 X_{j_1} u(y)| \leq \sum |D_2 x_{j_1}(x) \frac{\partial}{\partial y_j} D_2 u(y)| + |D_2 D_2 \frac{\partial}{\partial y_j} u(y)| 
\leq \left\{ \begin{array}{ll}
\varepsilon_0 C A_0^{|\beta|} |\beta|! A^{|\alpha|+i}(|\alpha|+1)!^s, & (|\beta| \neq 0) \\
C(n \varepsilon_0 + 1) A_0^{|\alpha|+1}(|\alpha|+1)!^s, & (|\beta| = 0) 
\end{array} \right.
\]
we have
\[
b^{(1)}_{i, 0} = \gamma \varepsilon_0 \\
d^{(1)}_{i} = 1.
\]

where $\gamma = n A_1/(A_1 - A_0)$. Hence we have
\[
b^{(1)}_{i, i} = 0, \quad i = 0, \ldots, k-2, 
\]
\[
(A.7)_1 \\
b^{(1)}_{i, k-1} = 1, \\
d^{(1)}_{i} = 1.
\]

For $j \geq 2$, we have from (A.5)
\[
C^{(k)}_{j} \equiv \sum_{i=0}^{k-1-j} b^{(k-1-j)}_{j, i} A_i^{\beta + i} (|\beta| + 1 + i)!^s + d^{(k-1)}_{j}
\]
\[
+ \sum_{i=0}^{k-1-j+1} b^{(k-1)}_{j, i} A_i^{\beta + i} (|\beta| + i)!^s + d^{(k-1)}_{j-1}
\]
\[
+ \gamma \varepsilon_0 \sum_{\beta \neq 0} (|\beta| + 1 + i)!^s \left\{ \sum_{i=0}^{k-1-j} b^{(k-1)}_{j, i} A_i^{\beta + i} (|\beta| + i)!^s + d^{(k-1)}_{j-1} \right\}
\]
\[
\leq \sum_{i=0}^{k-1-j} b^{(k-1)}_{j, i} A_i^{\beta + i} (|\beta| + i)!^s + d^{(k-1)}_{j}
\]
\[
+ \gamma \varepsilon_0 \sum_{i=0}^{k-1-j} b^{(k-1)}_{j, i} A_i^{\beta + i} (|\beta| + i)!^s + d^{(k-1)}_{j-1}
\]
\[
+ \gamma \varepsilon_0 \sum_{i=0}^{k-1-j} b^{(k-1)}_{j, i} A_i^{\beta + i} (|\beta| + i)!^s
\]
\[
+ \gamma \varepsilon_0 d^{(k-1)}_{j-1} A_i^{\beta + i} (|\beta| + i)!^s,
\]
where we used Lemma 3.1 and $\gamma = n A_1/(A_1 - A_0)$. Hence we obtain
Leray-Volevich's systems and Gevrey class

\( b^{(k)}_{\beta} = \gamma_\varepsilon_0 d^{(k-1)} + b^{(k-1)}_{\beta} (\gamma_\varepsilon_0 + 1) \)

(A.8)

\( b^{(k)}_{\beta} = b^{(k-1)}_{\beta} + b^{(k-1)}_{\beta} (\gamma_\varepsilon_0 + 1), \quad i = 1, \ldots, k - j, \)

\( d^{(k)} = d^{(k-1)} + d^{(k-1)}_{\beta} \), \quad \beta = 2, \ldots, k - 1. \)

From (A.6) we have

\[
C^{(k)}_{(\beta)} \leq (b^{(k-1)}_{\beta} (\gamma_\varepsilon_0 + 1) + \gamma_\varepsilon_0 d^{(k-1)}_{\beta}) A_{(j')_{\beta}} |\beta| !^i + d^{(k-1)}_{\beta},
\]

from which it follows

\( b^{(k)}_{\beta} = b^{(k-1)}_{\beta} (\gamma_\varepsilon_0 + 1) + \gamma_\varepsilon_0 d^{(k-1)}_{\beta} \)

(A.1)

\( d^{(k)} = d^{(k-1)} = d^{(k)}_{1} = 1. \)

We shall prove by induction on \( k \) that the \( b^{(0)}_{\beta,i} \) and \( d^{(k)}_{\beta} \) given by (A.7) and (A.8) satisfy (A.3). It is evident from (A.7), that (A.3) is valid. Assume that (A.7) is valid for \( p = 1, \ldots, k - 1 \) and (A.8) is valid for \( j \leq k - 1 \). Then by virtue of (A.8) we have for \( j \leq k - 1 \),

\[
d^{(k)} = \frac{(k - 1)!}{j!} + \frac{(k - 1)!}{(j - 1)!} = \frac{k!}{j!} \cdot \frac{j + 1}{k} \leq \frac{k!}{j!},
\]

\[
b^{(k)}_{\beta,i} \leq \gamma_\varepsilon_0 \frac{(k - 1)!}{j!} \gamma_\varepsilon_0 + (1 + \gamma_\varepsilon_0)^j - 1 \frac{(k - 1)!}{(j - 1)!} \gamma_\varepsilon_0 + 1
\]

\[
\leq (1 + \gamma_\varepsilon_0)^j \frac{k!}{j!} \left( \frac{\gamma_\varepsilon_0^j}{(1 + \gamma_\varepsilon_0)^k} + \frac{j}{k} \right)
\]

\[
\leq (1 + \gamma_\varepsilon_0)^j \frac{k!}{j!} \left( \frac{j + 1}{k} \right)
\]

\[
\leq (1 + \gamma_\varepsilon_0)^j \frac{k!}{j!} \cdot \gamma_\varepsilon_0^j,
\]

where we used the inequality \( \gamma_\varepsilon_0^j \leq (1 + \gamma_\varepsilon_0)^j \), and

\[
b^{(k)}_{\beta,i} \leq (1 + \gamma_\varepsilon_0)^j \frac{(k - 1)!}{j!} \gamma_\varepsilon_0 + (1 + \gamma_\varepsilon_0)^j \frac{(k - 1)!}{(j - 1)!} \gamma_\varepsilon_0
\]

\[
= (1 + \gamma_\varepsilon_0)^j \frac{k!}{j!} \frac{j + 1}{k} \leq (1 + \gamma_\varepsilon_0)^j \frac{k!}{j!} \gamma_\varepsilon_0^j.
\]

It is evident from (A.9) that \( d^{(k)}_{\beta} \) satisfies (A.3) and

\( b^{(k)}_{\beta} = (1 + \gamma_\varepsilon_0)^k - 1 \leq (1 + \gamma_\varepsilon_0)^k. \)

Thus we have proved (A.3). Hence we have

\[
|D_{\alpha}^\gamma X_{\beta,j} \cdots X_{\beta,j} u(y)| \leq C \sum_{j=1}^{k} C^{(k)}_{(\beta)} A^{(\alpha + j)} (|\alpha| + j)!^i
\]

\[
\leq C \sum_{j=1}^{k} \left( \sum_{i=1}^{j} (1 + \gamma_\varepsilon_0)^j \frac{k!}{j!} A^i! + \frac{k!}{j!} \right) A^{(\alpha + j)} (|\alpha| + j)!^i
\]

\[
\leq C \sum_{j=1}^{k} \left( \sum_{i=1}^{j} (1 + \gamma_\varepsilon_0)^j A^{(\alpha + j)} (|\alpha| + j)!^i \right) \sum_{i=1}^{k} \left( \frac{k!}{j!} \right) A^{(\alpha + j)} (|\alpha| + j)!^i.
\]
which and Lemma 3.1 imply our lemma, if we choose \( C \) suitably and \( A_1 \), such that \( A > A_1 > A_0 \).

References


