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On Hamada's theorem for a certain of the operators with double characteristics

By

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§ 0. Introduction

We consider the non-characteristic Cauchy problem with meromorphic initial data for a linear partial differential operator with holomorphic coefficients in the complex domain.

Y. Hamada, J. Leray and C. Wagschal [1] treated this problem for the operator with constant multiple characteristics. Y. Hamada and G. Nakamura [2], [4] treated this problem for the operator with involutive characteristics of variable multiplicities. In the preceeding paper [5], the author treated this problem for the Tricomi operator $D_t^2 - tD_x^2$ with lower order term whose coefficients depended on only t. In this paper we remove the condition on the lower order term's coefficients and treat the more general operator than the Tricomi operator with arbitrary lower order term.

Our method is to costruct the formal solution which was developed by D. Ludwig in [3] and to verify its convergence by using the majorant functions $\phi_\alpha(z, \zeta, y)$ due to Y. Hamada, which make us be able to remove the conditions on lower order term.

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§ 1. Assumptions and results

Let Ω be a neighbourhood of the origin of C^{n+1}, and $x=(x_0, x_1, \cdots, x_n)$ be a point of Ω. By $\mathrm{L}^k(\Omega)$, we mean the set of all linear partial differential operators of order k whose coefficients are holomorphic in Ω. Let $P(x, \mathrm{D}) \in \mathrm{L}^m(\Omega)$, $Q(x, \mathrm{D}) \in \mathrm{L}^{2m}(\Omega)$ and $R(x, \mathrm{D}) \in \mathrm{L}^{2m-1}(\Omega)$. We shall be studying a linear partial differential operator belonging to $\mathrm{L}^{2m}(\Omega)$:

$$L(x, \mathrm{D}) = P(x, \mathrm{D})^2 - x_0 Q(x, \mathrm{D}) + R(x, \mathrm{D}) .$$

We shall impose on $P(x, \xi)$ and $Q(x, \xi)$ the following conditions, where $\xi = (\xi_0, \xi_1, \cdots, \xi_n)$.

Assumption (A) (i) $P(x, \xi)$ is a homogeneous polynomial in ξ of degree m.

(ii) $P(x, 1, 0, \cdots, 0) = 1$.

(iii) The equation $P(0, \xi_0, 1, 0, \cdots, 0)=0$ has mutually distinct m roots λ_i $(i=1, \cdots, m)$.

Assumption (B) (i) $Q(x, \xi)$ is a homogeneous polynomial in ξ of degree $2m$.

(ii) $Q(0, \lambda_i, 1, 0, \cdots, 0)\neq 0$ $(i=1, \cdots, m)$

Then there exist m characteristic surfaces K_i $(i=1, \cdots, m)$ issuing from $(n-1)$-plane $x_0=x_1=0$. K_i are defined by the eqations $\varphi_i^{\pm}=0$. Here $\varphi_i^{\pm}(x)$ are the solutions of the eikonal equations,

$$\begin{cases} \mathring{L}(x, \varphi_{ix}^{\pm})=0 \\ \varphi_i^{\pm}(0, x')=x_1, \quad \text{and } \varphi_{ix_0}^{\pm}(0)=\lambda_i, \end{cases}$$

where $\mathring{L}(x, \xi)=P(x, \xi)^2-x_0Q(x, \xi)$ and $x'=(x_1, \cdots, x_n)$.

(In §4, we shall study the construction and some properties of $\varphi_i^{\pm}(x)$ precisely.) We write $\boldsymbol{K}=\bigcup_{i=1}^{m} K_i$. In order to describe the results, we need the auxiliary functions X_α and Y_α. For the precise definition of X_α and Y_α, first we introduce the so-called wave forms $k_\alpha(\rho)$:

$$k_\alpha(\rho)=\begin{cases} \dfrac{\rho^\alpha}{\Gamma(\alpha+1)}(\log \rho+\psi(\alpha+1)) \\ \text{especially } |\alpha+1|!(-1)^{\alpha-1}\rho^\alpha \text{ for } \alpha=-1, -2, \cdots, \end{cases}$$

where $\psi(\alpha)$ is *di-Γ* function, namely $\dfrac{\frac{d}{d\alpha}\Gamma(\alpha)}{\Gamma(\alpha)}$, and α is a complex parameter.

Next, we introduce the multi-valued functions $X_\alpha(\theta, \rho)$ and $Y_\alpha(\theta, \rho)$ as the solution of the Cauchy problem for the Tricomi equations:

$$(\partial_\theta^2-\theta\partial_\rho^2)X_\alpha(\theta, \rho)=0$$

with the initial data

$$\begin{cases} X_\alpha(0, \rho)=k_\alpha(\rho) \\ X_{\alpha\theta}(0, \rho)=0, \end{cases}$$

$$(\partial_\theta^2-\theta\partial_\rho^2)Y_\alpha(\theta, \rho)=0$$

with the initial data

$$\begin{cases} Y_\alpha(0, \rho)=0 \\ Y_{\alpha\theta}(0, \rho)=k_\alpha(\rho) \end{cases}$$

We remark that the following explicit representations of X_α and Y_α are known;

$$X_\alpha(\theta, \rho)=\frac{\partial}{\partial\alpha}\left[F\left(\frac{1}{6}, -\alpha, \frac{1}{3}; 1-\frac{\varphi^+}{\varphi^-}\right)\frac{(\varphi^+)^\alpha}{\Gamma(\alpha+1)}\right]$$

$$Y_\alpha(\theta, \rho)=\frac{\partial}{\partial\alpha}\left[F\left(\frac{5}{6}, -\alpha, \frac{5}{3}; 1-\frac{\varphi^+}{\varphi^-}\right)\frac{\theta(\varphi^+)^\alpha}{\Gamma(\alpha+1)}\right],$$

where $\varphi^+=\rho+\frac{2}{3}\theta^{3/2}$ and $\varphi^-=\rho-\frac{2}{3}\theta^{3/2}$ are so-called characteristic coordinates of the Tricomi operator, (see [5]).

Now, we consider the non-characteristic Cauchy problem with singular data,

$$(1,1)\qquad \begin{cases} L(x, \mathrm{D})u(x)=0 \\ \mathrm{D}_0^h u(0, x')=w_h(x') \ (h=0, \cdots, 2m-1), \end{cases}$$

where $w_h(x')$ have poles along $x_0=x_1=0$ and $\mathrm{D}_i=\dfrac{\partial}{\partial x_i}$.

Then our theorem is as follows.

Theorem. *Under the assumptions (A) and (B), for $r>0$ sufficiently small, the Cauchy problem* (1,1) *has a unique holomorphic solution on the universal covering space over $\boldsymbol{D}_r\backslash\boldsymbol{K}$, where $\boldsymbol{D}_r=\{x\in\Omega, |\varphi_i^{\pm}(x)|<r\}$. More precisely, the solution is expressed by*

$$u(x)=\sum_{\beta=1}^{m}\sum_{\alpha=-l-2m+1}^{+\infty} u_{\alpha,\beta}(x)X_{\alpha\rho}(\theta_\beta(x), \rho_\beta(x))+g_{\alpha,\beta}(x)X_{\alpha\theta}(\theta_\beta(x)\rho_\beta(x))$$

$$+v_{\alpha,\beta}(x)Y_{\alpha\rho}(\theta_\beta(x), \rho_\beta(x))+h_{\alpha,\beta}(x)Y_{\alpha\theta}(\theta_\beta(x), \rho_\beta(x)),$$

where l is the highest order of poles of the initial data and $u_{\alpha,\beta}(x)$, $g_{\alpha,\beta}(x)$, $v_{\alpha,\beta}(x)$, $h_{\alpha,\beta}(x)$, $\theta_\beta(x)$ and $\rho_\beta(x)$ are holomorphic in $\boldsymbol{D}_r$, (as for $\theta_\beta(x)$ and $\rho_\beta(x)$ such that $\varphi_\beta^{\pm}(x)=\rho_\beta(x)\pm\frac{2}{3}[\theta_\beta(x)]^{2/3}$, see § 4).

To prove this theorem, first we construct the formal solution of the Cauchy problem (1,1), which is due to D. Ludwig [3] and then confirm the convergence of the formal solution by the method of majorant function. For the construction of the formal solution, we prepare some caluculations and some properties of the auxiliary functions in the next section.

Note: We show the following examples as the simple operators which satisfy Assumptions (A) and (B).

Example 0. (Tricomi operator)

$$\mathrm{D}_0^2-x_0\mathrm{D}_1^2$$

Example 1. $L(x, \mathrm{D})=P(x, \mathrm{D})^2-x_0Q(x, \mathrm{D}')$, $\mathrm{D}'=(\mathrm{D}_1, \cdots, \mathrm{D}_n)$ where $P(x, \xi)$ satisfies Assumption (A) and $Q(x, \xi')$ is homogeneous in ξ' and $Q(0, 1, 0, \cdots, 0)\neq 0$.

Example 2. $L(x, \mathrm{D})=\prod_{h=1}^{g}[P_h(x, \mathrm{D})^2-x_0Q_h(x, \mathrm{D}')]$ where $P_h(x, \mathrm{D})\in L^{m_h}(\Omega)$ are such that $P(x, \xi)=\prod_{h=1}^{g}P_h(x, \xi)$ satisfies Assumption (A), and $Q_h(x, \xi')$ are homogeneous polynomials in ξ' of degree $2m_h$ and $Q_h(0, 1, 0, \cdots, 0)\neq 0$.

§ 2. Preliminary calculation

In the construction of the formal solution of the Cauchy problem (1,1), we need to represent $\partial_\rho^i\partial_\theta^j X_\alpha$ or $\partial_\rho^i\partial_\theta^j Y_\alpha$ in terms of $\partial_\rho^k X_\alpha$, $\partial_\rho^{k-1}\partial_\theta X_\alpha$, and $\partial_\rho^k Y_\alpha$, $\partial_\rho^{k-1}\partial_\theta Y_\alpha$ $(k\leqq i+j)$ respectively. So we employ the following formula.

(F,1) *Let $U(\theta, \rho)$ satisfy the Tricomi equation, namely $(\partial_\theta^2-\theta\partial_\rho^2)U=0$. Then we have*

$$\begin{cases} \partial_\theta^{2r}U=\theta^r\partial_\rho^{2r}U+r(r-1)\theta^{r-2}\partial_\rho^{2r-2}\partial_\theta U+\cdots, \\ \partial_\theta^{2r+1}U=\theta^r\partial_\rho^{2r}\partial_\theta U+r^2\theta^{r-1}\partial_\rho^{2r}U+\cdots. \end{cases}$$

Let $K(x, \xi)$ be the homogeneous polynomial of degree l in $\xi=(\xi_0, \xi_1, \cdots, \xi_n)$. We shall write $K^{(i)}(x, \xi)=\frac{\partial}{\partial\xi_i}K(x, \xi)$, $K^{(i,j)}(x, \xi)=\frac{\partial^2}{\partial\xi_i\partial\xi_j}K(x, \xi)$, $K_{(i)}(x, \xi)=\mathrm{D}_iK(x, \xi)$ and $K^{(\alpha)}(x, \xi)=\mathrm{D}_\xi^\alpha K(x, \xi)$, where $\alpha=(\alpha_0, \alpha_1, \cdots, \alpha_n)\in(N\cup\{0\})^{n+1}$. We define $K_i(x, \xi, \eta)$ by

$$K(x, r\xi+s\eta)=\sum_{i=0}^{l}K_i(x, r\xi, s\eta)=\sum_{i=0}^{l}r^is^{l-i}K_i(x, \xi, \eta)$$

where $\eta=(\eta_0, \eta_1, \cdots, \eta_n)$ and $r, s\in C^1$. We shall use $\partial_i=\theta_{x_i}\partial_\theta+\rho_{x_i}\partial_\rho$ $(i=0, \cdots n)$, $\partial=(\partial_0, \cdots, \partial_n)$ and $\mathrm{D}_i\partial_j=\theta_{x_ix_j}\partial_\theta+\rho_{x_ix_j}\partial_\rho$.

We shall sometimes use the following Leibniz formula

(F,2) $$K(x, \mathrm{D})[u(x)v(x)]=\sum_{|\alpha|\leq l}\frac{1}{\alpha!}\mathrm{D}^\alpha u\cdot K^{(\alpha)}(x, \mathrm{D})v$$

and the following formula by chain rule

(F,3) $$\mathrm{D}^\alpha U(\theta(x), \rho(x))=\partial^\alpha U+\frac{1}{2}\sum_{i,j=0}^{n}(\partial^\alpha)^{(i,j)}(\mathrm{D}_i\partial_j)U+\cdots.$$

From (F,2) and (F,3), we have

(F,4) $$K(x, \mathrm{D})[u(x)U(\theta(x), \rho(x))]=u\cdot K(x, \partial)U+u\cdot\frac{1}{2}K^{(i,j)}(x, \partial)(\mathrm{D}_i\partial_j)U$$
$$+\mathrm{D}_iu\cdot K^{(i)}(x, \partial)U+(\text{lower order term}).$$

(repeated index i, j will be always summed from 0 to n).

From the definition of $K_i(x, \xi, \eta)$, we have

(F,5) $$K(x, \partial)=K(x, \theta_x\partial_\theta+\rho_x\partial_\rho)=\sum_{i=0}^{l}K_i(x, \theta_x, \rho_x)\partial_\theta^i\partial_\rho^{l-i}.$$

(F,1) and (F,5) lead us to the following formula (F,6) and (F,7)

(F,6) *Let $U(\theta, \rho)$ satisfy the Tricomi equation.*

(i)

$$K(x, \partial)U_\rho(\theta(x), \rho(x))=\left[\sum_{i=0}^{[l/2]}K_{2i}(x, \theta_x, \rho_x)\theta^i\right]\partial_\rho^{l+1}U$$
$$+\left[\sum_{i=0}^{[(l-1)/2]}K_{2i+1}(x, \theta_x, \rho_x)\theta^i\right]\partial_\rho^l\partial_\theta U+\left[\sum_{i=0}^{[l/2]}i^2K_{2i+1}(x, \theta_x, \rho_x)\theta^{i-1}\right]\partial_\rho^l U$$
$$+\left[\sum_{i=0}^{[l/2]}i(i-1)K_{2i}(x, \theta_x, \rho_x)\theta^{i-2}\right]\partial_\rho^{l+1}\partial_\theta U+\cdots$$

$$={}^1K(x,\theta_x,\rho_x)\partial_\rho^{l+1}U+{}^2K(x,\theta_x,\rho_x)\partial_\rho^l\partial_\theta U+{}^3K(x,\theta_x,\rho_x)\partial_\rho^l U$$

$$+{}^4K(x,\theta_x,\rho_x)\partial_\rho^{l-1}\partial_\theta U+\cdots.$$

(ii)

$$K(x,\partial)U_\theta(\theta(x),\rho(x))=\left[\sum_{i=0}^{[(l-1)/2]}K_{2i+1}(x,\theta_x,\rho_x)\theta^{i+1}\right]\partial_\rho^{l+1}U$$

$$+\left[\sum_{i=0}^{[l/2]}K_{2i}(x,\theta_x,\rho_x)\theta^i\right]\partial_\rho^l\partial_\theta U+\left[\sum_{i=0}^{[l/2]}K_{2i}(x,\theta_x,\rho_x)i^2\theta^{i-1}\right]\partial_\rho^l U$$

$$+\left[\sum_{i=0}^{[(l-1)/2]}K_{2i+1}(x,\theta_x,\rho_x)i(i+1)\theta^{i-1}\right]\partial_\rho^{l-1}U+\cdots$$

$$={}^1K'(x,\theta_x,\rho_x)\partial_\rho^{l+1}U+{}^2K'(x,\theta_x,\rho_x)\partial_\rho^l\partial^\theta U+{}^3K'(x,\theta_x,\rho_x)\partial_\rho^l U$$

$$+{}^4K'(x,\theta_x,\rho_x)\partial_\rho^{l-1}\partial_\theta U+\cdots.$$

We immediately see the relation ${}^1K'={}^2K\cdot\theta$ and ${}^2K'={}^1K$. So we have (F,7).

(F,7)
$$\begin{cases}\text{(i)} & K(x,\partial)\partial_\theta^2U={}^1K'\partial_\rho^{l+1}\partial_\theta U+\theta({}^2K')\partial_\rho^{l+2}U+\cdots.\\ \text{(ii)} & K(x,\partial)\partial_\rho\partial_\theta U={}^1K'\partial_\rho^{l+2}U+{}^2K'\partial_\rho^{l+1}\partial_\theta U+\cdots.\\ \text{(iii)} & K(x,\partial)\partial_\rho^2U={}^1K\partial_\rho^{l+2}U+{}^2K\partial_\rho^{l+1}\partial_\theta U+\cdots.\end{cases}$$

To examine ${}^hK(x,\theta_x,\rho_x)$ and ${}^hK'(x,\theta_x,\rho_x)$ $(h=1, 2, 3)$, note the relations $K_0(x,\xi,\eta)=K(x,\eta)$, $K_1(x,\xi,\eta)=K^{(i)}(x,\eta)\xi_i$, and $K_2(x,\xi,\eta)=\frac{1}{2}K^{(i,j)}(x,\eta)\xi_i\xi_j$ and then we have

$$\begin{cases}\text{(i)} & {}^1K(x,\theta_x,\rho_x)=K(x,\rho_x)+\theta\cdot{}_1\tilde{K}(x,\theta_x,\rho_x),\\ \text{(ii)} & {}^2K(x,\theta_x,\rho_x)=K^{(i)}(x,\rho_x)\theta_{x_i}+\theta\cdot{}_2\tilde{K}(x,\theta_x,\rho_x),\\ \text{(iii)} & {}^1K'(x,\theta_x,\rho_x)={}^2K(x,\theta_x,\rho_x)\cdot\theta,\\ \text{(iv)} & {}^2K'(x,\theta_x,\rho_x)={}^1K(x,\theta_x,\rho_x),\\ \text{(v)} & {}^3K'(x,\theta_x,\rho_x)=\frac{1}{2}K^{(i,j)}(x,\rho_x)\theta_{x_i}\theta_{x_j}+\theta\cdot{}_3\tilde{K}(x,\theta_x,\rho_x).\end{cases}$$

Next we shall study the composition of two operators of the form $K(x,\partial)$.

(F,8) *Let $M(x,\partial)$ and $N(x,\partial)$ be linear differential operators in ∂ of order m and n respectively. We get*

(i) $$M(x,\partial)N(x,\partial)\partial_\rho U=({}^1M\cdot{}^1N+{}^1M'\cdot{}^2N)\partial_\rho^{m+n+1}U$$

$$+({}^2M\cdot{}^1N+{}^2M'\cdot{}^2N)\partial_\rho^{m+n}\partial_\theta U+({}^3M\cdot{}^1N$$

$$+{}^3M'\cdot{}^2N+{}^1M\cdot{}^3N+{}^1M'\cdot{}^4N)\partial_\rho^{m+n}U$$

$$+({}^4M\cdot{}^1N+{}^4M'\cdot{}^2N+{}^2M\cdot{}^3N+{}^2M'\cdot{}^4N)\partial_\rho^{m+n-1}\partial_\theta U+\cdots.$$

(ii) $$M(x,\partial)N(x,\partial)\partial_\theta U=({}^1M\cdot{}^1N'+{}^1M'\cdot{}^2N')\partial_\rho^{m+n+1}U$$

$$+({}^2M\cdot{}^1N'+{}^2M\cdot{}^1N')\partial_\rho^{m+n}\partial_\theta U$$

$$+({}^3M\cdot{}^1N'+{}^3M'\cdot{}^2N'+{}^1M\cdot{}^3N'+{}^1M'\cdot{}^4N')\partial_\rho^{m+n}U$$
$$+({}^4M\cdot{}^1N'+{}^4M'\cdot{}^2N'+{}^2M\cdot{}^3N'+{}^2M'\cdot{}^4N')\partial_\rho^{m+n-1}\partial_\theta U+\cdots.$$

where ${}^hM={}^hM(x,\ \theta_x,\ \rho_x)$, ${}^hN={}^hN(x,\ \theta_x,\ \rho_x)$ $(h=1,\ 2,\ 3,\ 4)$ *and so on.*

Now, making use of (F,1) (F,8), we calculate

$$L(x,\ \mathrm{D})[u(x)U_\rho(\theta(x),\ \rho(x))+g(x)U_\theta(\theta(x),\ \rho(x))]$$

where $U(\theta,\ \rho)$ satisfies the Tricomi equation.

$$L(x,\ \mathrm{D})[u(x)U_\rho+g(x)U_\theta]$$
$$=u\cdot\mathring{L}(x,\ \partial)U_\rho+g\cdot\mathring{L}(x,\ \partial)U_\theta+u\cdot[P^{(i)}(x,\ \partial)P^{(j)}(x,\ \partial)$$
$$+P(x,\ \partial)P^{(i,j)}(x,\ \partial)-\frac{1}{2}x_0Q^{(i,j)}(x,\ \partial)](\mathrm{D}_i\partial_j)U_\rho$$
$$+g\cdot[P^{(i)}(x,\ \partial)P^{(j)}(x,\ \partial)+P(x,\ \partial)P^{(i,j)}(x,\ \partial)$$
$$+\frac{1}{2}x_0Q^{(i,j)}(x,\ \partial)](\mathrm{D}_i\partial_j)U_\theta+\mathrm{D}_iu\cdot[2P(x,\ \partial)P^{(i)}(x,\ \partial)$$
$$-x_0Q^{(i)}(x,\ \partial)]U_\rho+\mathrm{D}_ig\cdot[2P(x,\ \partial)P^{(i)}(x,\ \partial)-x_0Q^{(i)}(x,\ \partial)]U_\theta$$
$$+u\cdot\mathring{R}(x,\ \partial)U_\rho+g\cdot\mathring{R}(x,\ \partial)U_\theta+(\text{lower order term}),$$

where $\mathring{R}(x,\ \xi)=P^{(i)}(x\ \text{`}\xi)P_{(i)}(x,\ \xi)+R_{2m-1}(x,\ \xi)$. We have

(F,9) $$L(x,\ \mathrm{D})[u(x)U_\rho+g(x)U_\theta]$$
$$=({}^1\mathring{L}\cdot u+\theta\cdot{}^2\mathring{L}\cdot g)\partial_\rho^{2m+1}U+({}^2\mathring{L}\cdot u+{}^1\mathring{L}\cdot g)\partial_\rho^{2m}\partial_\theta U$$
$$+[\theta\cdot\mathcal{L}g+\mathcal{M}u+(\theta\cdot\rho_{x_ix_j}\cdot\mathcal{P}_{ij}^1+\theta\cdot\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^2+{}^3\mathring{L}'+\theta\cdot{}^2\mathring{R})g]$$
$$+(\rho_{x_ix_j}\cdot P_{ij}^2+\theta\cdot\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^1+{}^3\mathring{L}+{}^1\mathring{R})u]\partial_\rho^{2m}U$$
$$+[\mathcal{L}u+\mathcal{M}g+(\rho_{x_ix_j}\mathcal{P}_{ij}^1+\theta_{x_ix}\ \cdot\mathcal{P}_{ij}^2+{}^4\mathring{L}+{}^2\mathring{R})u$$
$$+(\rho_{x_ix_j}\cdot\mathcal{P}_{ij}^2+\theta\cdot\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^1+{}^4\mathring{L}'+{}^1\mathring{R})g]\partial_\rho^{2m-1}\partial_\theta U$$
$$+(\text{lower order term}),$$

where

$$\mathcal{L}=\mathcal{L}(x,\ \theta_x,\ \rho_x,\ \mathrm{D})=[2({}^1P(x,\ \theta_x,\ \rho_x)\cdot{}^2P^{(i)}(x,\ \theta_x,\ \rho_x)$$
$$+{}^2P(x,\ \theta_x,\ \rho_x)\cdot{}^1P^{(i)}(x,\ \theta_x,\ \rho_x))-x_0\cdot{}^2Q^{(i)}(x,\ \theta_x,\ \rho_x)]\mathrm{D}_i,$$
$$\mathcal{M}=\mathcal{M}(x,\ \theta_x,\ \rho_x,\ \mathrm{D})=[2({}^1P(x,\ \theta_x,\ \rho_x)\cdot{}^1P^{(i)}(x,\ \theta_x,\ \rho_x)$$
$$+\theta\cdot{}^2P^{(i)}(x,\ \theta_x,\ \rho_x)\cdot{}^2P(x,\ \theta_x,\ \rho_x))-x_0\cdot{}^1Q^{(i)}(x,\ \theta_x,\ \rho_x)]\mathrm{D}_i,$$
$$\mathcal{P}_{ij}^1=[{}^1P^{(i)}\cdot{}^2P^{(j)}+{}^2P^{(i)}\cdot{}^1P^{(j)}+{}^1P\cdot{}^2P^{(i,j)}+{}^2P\cdot{}^1P^{(i,j)}$$
$$-\frac{1}{2}x_0\cdot{}^2Q^{(i,j)}](x,\ \theta_x,\ \rho_x),$$
$$\mathcal{P}_{ij}^2=[{}^1P^{(i)}\cdot{}^1P^{(i,j)}+{}^1P\cdot{}^1P^{(i,j)}+{}^2P^{(i)}\cdot{}^2P^{(j)}+{}^2P\cdot{}^2P^{(i\ j)}$$

$$-\frac{1}{2}x_0\cdot{}^1Q^{(i,j)}](x,\ \theta_x,\ \rho_x).$$

For the proof of the convergence of the formal solution of the Cauchy problem (1,1) we need the closer information on the lower order term in (F,9). From (F,3) and (F,5), we have

(F,10) (i)

$$L(x,\ \mathrm{D})u\cdot U_\rho$$

$$=\Big(\sum_{|\alpha|=0}^{2m}\frac{1}{\alpha!}\mathrm{D}^\alpha u\,L^{(\alpha)}(x,\ \mathrm{D})U_\rho=\sum_{|\alpha|=0}^{2m}\frac{1}{\alpha!}\mathrm{D}^\alpha u$$

$$\cdot\Big[\sum_{\mu=0}^{2m-|\alpha|}{}^1\mathring{L}_\mu^{(\alpha)}\partial_\rho^\mu U+\sum_{\mu=0}^{2m-1-|\alpha|}{}^2\mathring{L}_\mu^{(\alpha)}\cdot\partial_\rho^\mu\partial_\theta U_\rho+\cdots\Big]$$

$$=\sum_{\nu=0}^{2m}\Big[\sum_{|\alpha|=0}^{2m-\nu}{}^1\mathring{L}_\nu^{(\alpha)}\cdot\frac{1}{\alpha!}\mathrm{D}^\alpha u+\cdots\Big]\partial_\rho^\nu U_\rho$$

$$+\sum_{\nu=0}^{2m-1}\Big[\sum_{|\alpha|=0}^{2m-1-\nu}{}^2\mathring{L}_\nu^{(\alpha)}\cdot\frac{1}{\alpha!}\mathrm{D}^\alpha u+\cdots\Big]\partial_\rho^\nu\partial_\theta U_\rho\Big)$$

$$=\sum_{\nu=0}^{2m}{}^1L_\nu[u]\partial_\rho^\nu U+\sum_{\nu=1}^{2m}{}^2L_\nu[u]\partial_\rho^\nu\partial_\theta U\,,$$

where

$${}^1L_\nu={}^1L_\nu(x,\ \theta,\ \rho,\ \mathrm{D})\in\mathrm{L}^{2m-\nu}(\Omega)$$

$${}^2L_\nu={}^2L_\nu(x,\ \theta,\ \rho,\ \mathrm{D})\in\mathrm{L}^{2m-\nu}(\Omega)$$

and we have

(ii)
$$L(x,\ \mathrm{D})g\cdot U_\theta=\sum_{\nu=1}^{2m}{}^3L_\nu[g]\partial_\rho^\nu U_\rho+\sum_{\nu=0}^{2m}{}^4L_\nu[g]\partial_\rho^\nu\partial_\theta U\,,$$

where

$${}^3L_\nu={}^3L_\nu(x,\ \theta,\ \rho,\ \mathrm{D})\in\mathrm{L}^{2m-\nu}(\Omega)$$

$${}^4L_\nu={}^4L_\nu(x,\ \theta,\ \rho,\ \mathrm{D})\in\mathrm{L}^{2m-\nu}(\Omega)$$

We see also the following relations from the above.

(F,11)
$$\begin{cases}{}^1\mathring{L}_\nu\equiv\displaystyle\sum_{|\alpha|=2m-\nu}{}^1\mathring{L}^{(\alpha)}(x,\ \theta_x,\ \rho_x)\frac{\mathrm{D}^\alpha}{\alpha!}\quad(\mathrm{mod.}\ \theta)\\ {}^2\mathring{L}_\nu\equiv\displaystyle\sum_{|\alpha|=2m-\nu}{}^2\mathring{L}^{(\alpha)}(x,\ \theta_x,\ \rho_x)\frac{\mathrm{D}^\alpha}{\alpha!}\quad(\mathrm{mod.}\ \theta)\\ {}^3\mathring{L}_\nu\equiv\theta\cdot\Big[\displaystyle\sum_{|\alpha|=2m-\nu}\mathring{L}^{(\alpha)}(x,\ \theta_x,\ \rho_x)\frac{\mathrm{D}^\alpha}{\alpha!}\Big]\quad(\mathrm{mod.}\ \theta^2)\\ {}^4\mathring{L}_\nu\equiv\displaystyle\sum_{|\alpha|=2m-\nu}{}^1\mathring{L}^{(\alpha)}(x,\ \theta_x,\ \rho_x)\frac{\mathrm{D}^\alpha}{\alpha!}\quad(\mathrm{mod.}\ \theta)\,,\end{cases}$$

where ${}^h\mathring{L}_\nu$ are the principal part of ${}^hL_\nu$ respectively ($h=1,\ 2,\ 3,\ 4$).

From (F,9) and (F,10) we have

$$(F,12)\qquad \begin{cases} {}^{1}L_{2m}(x,\ \theta,\ \rho,\ \mathrm{D})={}^{1}\mathring{L}(x,\ \theta_{x},\ \rho_{x}) \\ {}^{2}L_{2m}(x,\ \theta,\ \rho,\ \mathrm{D})={}^{2}\mathring{L}(x,\ \theta_{x},\ \rho_{x}) \\ {}^{3}L_{2m}(x,\ \theta,\ \rho,\ \mathrm{D})={}^{2}\mathring{L}(x,\ \theta_{x},\ \rho_{x})\cdot\theta \\ {}^{4}L_{2m}(x,\ \theta,\ \rho,\ \mathrm{D})={}^{1}\mathring{L}(x,\ \theta_{x},\ \rho_{x}) \end{cases}$$

$$(F,13)\qquad \begin{cases} {}^{1}L_{2m-1}(x,\ \theta,\ \rho,\ \mathrm{D})=\mathcal{M}+(\rho_{x_ix_i}\cdot\mathcal{P}_{ij}^{2}+\theta\cdot\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^{1}+{}^{3}\mathring{L}+{}^{1}\mathring{R}) \\ {}^{2}L_{2m-1}(x,\ \theta,\ \rho,\ \mathrm{D})=\mathcal{L}+(\rho_{x_ix_j}\cdot\mathcal{P}_{ij}^{1}+\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^{2}+{}^{4}\mathring{L}+{}^{2}\mathring{R}) \\ {}^{3}L_{2m-1}(x,\ \theta,\ \rho,\ \mathrm{D})=\theta\cdot[\mathcal{L}+(\rho_{x_ix_j}\cdot\mathcal{P}_{ij}^{1}+\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^{2}+{}^{2}\mathring{R})]+{}^{3}L' \\ {}^{4}L_{2m-1}(x,\ \theta,\ \rho,\ \mathrm{D})=\mathcal{M}+(\rho_{x_ix_j}\cdot\mathcal{P}_{ij}^{2}+\theta\cdot\theta_{x_ix_j}\cdot\mathcal{P}_{ij}^{1}+{}^{4}\mathring{L}'+{}^{1}\mathring{R}). \end{cases}$$

In § 4, we shall treat (F,11) for the determination of the phase function and study (F,13) for the research of the transport equations.

§ 3. Construction of the formal solution of the Cauchy problem

Employing the formulae obtained in the preceeding section, we shall construct the formal solution of the Cauchy problem (1,1). Taking account of the principle of the superposition, we have only to treat the following Cauchy problem with the special data;

$$(3,1)\qquad \begin{cases} L(x,\ \mathrm{D})u(x)=0 \\ \mathrm{D}_0^h u(0,\ x')=w_h(x'')\cdot k_{-l}(x_1)\quad (h=0,\ \cdots,\ 2m-1), \end{cases}$$

where $x''=(x_2,\ \cdots,\ x_n)$ and $w_h(x'')$ are holomorphic functions in x'' in the neighbourhood of $0\in C^{n-1}$.

We seek the formal solution in the following form

$$(3,2)\qquad u(x)=\sum_{\beta=1}^{m}\sum_{\alpha=-l-2m+1}^{+\infty}[u_{\alpha,\beta}(x)X_{\alpha\rho}(\theta_\beta(x),\ \rho_\beta(x))$$
$$+g_{\alpha,\beta}(x)X_{\alpha\theta}(\theta_\beta(x),\ \rho_\beta(x))+v_{\alpha,\beta}(x)Y_{\alpha\rho}(\theta_\beta(x),\ \rho_\beta(x))$$
$$+h_{\alpha,\beta}(x)Y_{\alpha\theta}(\theta_\beta(x),\ \rho_\beta(x)).$$

Making use of the relations $X_{\alpha\rho}=X_{\alpha-1}$ and $Y_{\alpha\rho}=Y_{\alpha-1}$, we determine the coefficients $u_{\alpha,\beta}$, $g_{\alpha,\beta}$, $v_{\alpha,\beta}$, $h_{\alpha,\beta}$ and the auxiliary phase functions θ_β, ρ_β. Substituting the formal solution (3,2) in $Lu=0$, we obtain

$$Lu=\sum_{\beta=1}^{m}\sum_{\alpha}\Big[\sum_{\nu=1}^{2m}{}^{1}L_{\nu,\beta}u_{\alpha+\nu,\beta}+{}^{3}L_{\nu,\beta}g_{\alpha+\nu,\beta}\Big]X_{\alpha\rho}(\theta_\beta(x),\ \rho_\beta(x))$$
$$+\Big[\sum_{\nu=0}^{2m}{}^{4}L_{\nu,\beta}g_{\alpha+\nu,\beta}+{}^{2}L_{\nu,\beta}u_{\alpha+\nu,\beta}\Big]X_{\alpha\theta}(\theta_\beta(x),\ \rho_\beta(x))$$
$$+\Big[\sum_{\nu=0}^{2m}{}^{1}L_{\nu,\beta}v_{\alpha+\nu,\beta}+{}^{3}L_{\nu,\beta}h_{\alpha+\nu,\beta}\Big]Y_{\alpha\rho}(\theta_\beta(x),\ \rho_\beta(x))$$

$$+\left[\sum_{\nu=0}^{2m}{}^4L_{\nu,\beta}h_{\alpha+\nu,\beta}+{}^2L_{\nu,\beta}v_{\alpha+\nu,\beta}\right]Y_{\alpha\theta}(\theta_\beta(x),\ \rho_\beta(x))=0$$

where ${}^hL_{\nu,\beta}={}^hL_\nu(x,\ \theta_\beta,\ \rho_\beta,\ D)\in L^{2m-\nu}(\Omega)$ $(h=1, 2, 3, 4)$ and especially ${}^3L_{0,\beta}={}^2L_{0,\beta}=0$.

Setting the coefficients of $X_{\alpha\rho}$, $X_{\alpha\theta}$, $Y_{\alpha\rho}$ and $Y_{\alpha\theta}$ equal to zero, we have the recursion formulae for $u_{\alpha,\rho}$, $g_{\alpha,\rho}$, $v_{\alpha,\beta}$ and $h_{\alpha,\rho}$;

$$\begin{cases} \sum_{\nu=0}^{2m}[{}^1L_{\nu,\beta}u_{\alpha+\nu,\beta}+{}^3L_{\nu,\beta}g_{\alpha+\nu,\beta}]=0 \\ \sum_{\nu=0}^{2m}[{}^4L_{\nu,\beta}g_{\alpha+\nu,\beta}+{}^2L_{\nu,\beta}u_{\alpha+\nu,\beta}]=0 \\ \sum_{\nu=0}^{2m}[{}^1L_{\nu,\beta}v_{\alpha+\nu,\beta}+{}^3L_{\nu,\beta}h_{\alpha+\nu,\beta}]=0 \\ \sum_{\nu=0}^{2m}[{}^4L_{\nu,\beta}h_{\alpha+\nu,\beta}+{}^2L_{\nu,\beta}v_{\alpha+\nu,\beta}]=0 \end{cases}$$

Here we set

$$(3,3) \qquad {}^1L_{2m,\beta}={}^2L_{2m,\beta}={}^3L_{2m,\beta}={}^4L_{2m,\beta}=0\,,$$

From these non-linear partial differential equations of first order in θ_β and ρ_β, we determine θ_β and ρ_β. We shall study these equations in the next section. Thus we have reached the system of the transport equations.

$$(3,4)\qquad \begin{cases} \text{(i)}\begin{cases} {}^2L_{2m-1,\beta}u_{\alpha+2m-1,\beta}=-\sum_{\nu=0}^{2m-1}{}^4L_{\nu,\beta}g_{\alpha+\nu,\beta} \\ \qquad\qquad -\sum_{\nu=0}^{2m-2}{}^2L_{\nu,\beta}u_{\alpha+\nu,\beta} \\ {}^3L_{2m-1,\beta}g_{\alpha+2m-1,\beta}=-\sum_{\nu=0}^{2m-1}{}^1L_{\nu,\beta}u_{\alpha+\nu,\beta} \\ \qquad\qquad -\sum_{\nu=0}^{2m-2}{}^3L_{\nu,\beta}g_{\alpha+\nu,\beta} \end{cases} \\ \text{(ii)}\begin{cases} {}^2L_{2m-1,\beta}v_{\alpha+2m-1,\beta}=-\sum_{\nu=0}^{2m-1}{}^4L_{\nu,\beta}h_{\alpha+\nu,\beta} \\ \qquad\qquad -\sum_{\nu=0}^{2m-2}{}^2L_{\nu,\beta}v_{\alpha+\nu,\beta} \\ {}^3L_{2m-1,\beta}h_{\alpha+2m-1,\beta}=-\sum_{\nu=0}^{2m-1}{}^1L_{\nu,\beta}v_{\alpha+\nu,\beta} \\ \qquad\qquad -\sum_{\nu=0}^{2m-2}{}^3L_{\nu,\beta}h_{\alpha+\nu,\beta} \end{cases} \end{cases}$$

On the other hand, from the initial data by calculating $D_0^h u(x)|_{x_0=0}$ in the similar way we have

$$\sum_{\beta=1}^{m} [\rho_{\beta x_0}(0, x')]^h \cdot (u_{\alpha+1,\beta}+h_{\alpha,\beta})+h\cdot[\rho_{\beta x_0}(0, x')]^{h-1}\sigma_\beta(0, x')\cdot v_{\alpha,\beta}$$

$$+\sum_{\beta=1}^{m}\Big[\sum_{k=0}^{h-1} M_{k,\beta}^h(u_{\alpha+k+1-h,\beta}+h_{\alpha+k-h,\beta})+M'^h_{k,\beta}g_{\alpha+k+1-h,\beta}$$

$$+N_{k,\beta}^h v_{\alpha+k-h,\beta}+M''^h_{k,\beta}u_{\alpha+k+1-h,\beta}+N'^h_{k,\beta}h_{\alpha+k-h}\Big]\Big|_{x_0=0}$$

$$=\begin{cases} w_h(x'') & \text{for } \alpha=-l+2m+h \\ 0 & \text{otherwise} \end{cases}$$

where $M_{k,\beta}^h$, $N_{k,\beta}^h$ are linear ordinary differential operator in D_0 of order $h-k$ and $M'^h_{k,\beta}$, $M''^h_{k,\beta}$, $N'^h_{k,\beta}$ are linear ordinary differential operators in D_0 of order $h-k-1$. These operators are determined only by the coefficients of $L(x, D)$, θ_β, ρ_β and have the holomorphic coefficients in x'.

Note that the determinant of the following $2m\times 2m$ matrix

$$\begin{vmatrix} 1, & 0 & \cdots & 1, & 0 \\ \gamma_1, & 1 & & \gamma_m, & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma_1^{2m-1}, & (2m-1)\gamma_1^{2m-2} & \cdots & \gamma_m^{2m-1}, & (2m-1)\gamma_m^{2m-2} \end{vmatrix}$$

$$(\gamma_h=\rho_{h x_0}(0, x'))$$

does not vanish if γ_h $(h=1, \cdots, m)$ are mutually distinct, (in the next §4 we shall see γ_h are mutually distinct). Then we have the following lemma.

Lemma 3.1. *$u_{\alpha+1,\beta}+h_{\alpha,\beta}|_{x_0=0}$ and $v_{\alpha,\beta}|_{x_0=0}$ are represented as the linear combination of $H^1_{\mu\beta}(x', D_0)(u_{\alpha+1-\mu,\beta}+h_{\alpha-\mu,\beta})(0, x')$, $H^2_{\mu-1\beta}(x', D_0)g_{\alpha+1-\mu,\beta}(0, x')$, $H^3_{\mu,\beta}(x', D_0)v_{\alpha-\mu,\beta}(0, x')$, $H^4_{\mu-1,\beta}(x', D_0)h_{\alpha-\mu,\beta}(0, x')$ and $H^5_{\mu-1,\beta}(x', D_0)u_{\alpha+1-\mu,\beta}(0, x')$ where $H^q_{\nu,\beta}(x', D_0)$ $(q=1, \cdots, 5)$ are the ordinary differential operators in D_0 of order ν and are determined only by $L(x, D)$ and θ_β, ρ_β. Hence we have the Cauchy problem for the first order system (3,4) with the initial data;*

(3,5) (i)

$$u_{\alpha+2m-1,\beta}(0, x')+h_{\alpha-2m-2,\beta}(0, x')$$

$$=\Big[\sum_{\mu=1}^{2m-1}\sum_{\gamma=1}^{m} d^1_{\mu,\gamma}(x')H^1_{\mu,\gamma}(x', D_0)(u_{\alpha+2m-1-\mu,\gamma}+h_{\alpha+2m-2-\mu,\gamma})(x)$$

$$+d^2_{\mu,\gamma}(x')H^2_{\mu-1,\gamma}(x', D_0)g_{\alpha+2m-1-\mu,\gamma}(x)$$

$$+d^3_{\mu,\gamma}(x')H^3_{\mu,\gamma}(x', D_0)v_{\alpha+2m-2-\mu,\gamma}(x)+d^4_{\mu,\gamma}(x')H^4_{\mu-1,\gamma}(x', D_0)h_{\alpha+2m-2-\mu,\gamma}(x)$$

$$+d^5_{\mu,\gamma}(x')H^5_{\mu-1,\gamma}(x', D_0)u_{\alpha+2m-1-\mu,\gamma}(x)\Big]\Big|_{x_0=0},$$

(ii) $v_{\alpha+2m-1,\mu}(0, x')$

$$=\Big[\sum_{\mu=1}^{2m-1}\sum_{\gamma=1}^{m} e_{\mu,\gamma}^{1}(x')H_{\mu,\gamma}^{1}(x', \mathrm{D}_0)(u_{\alpha+2m-1-\mu,\gamma}+h_{\alpha+2m-2-\mu,\gamma})(x)$$

$$+e_{\mu,\gamma}^{2}(x')H_{\mu-1,\gamma}^{2}(x', \mathrm{D}_0)g_{\alpha+2m-1-\mu,\gamma}(x)$$

$$+e_{\mu,\gamma}^{3}(x')H_{\mu,\gamma}^{3}(x', \mathrm{D}_0)v_{\alpha+2m-2-\mu,\gamma}(x)$$

$$+e_{\mu,\gamma}^{4}(x')H_{\mu-1,\gamma}^{4}(x', \mathrm{D}_0)h_{\alpha+2m-2-\mu,\gamma}(x)$$

$$+e_{\mu,\gamma}^{5}(x')H_{\mu-1,\gamma}^{5}(x', \mathrm{D}_0)u_{\alpha+2m-1-\mu,\gamma}(x)\Big]\Big|_{x_0=0},$$

where $d_{\mu,\gamma}^{\nu}(x')$ *and* $e_{\mu,\gamma}^{\nu}(x')$ *are holomorphic functions in* x' *in a common neighbourhood of* $0\in C^n$.

We first remark these problems take the same form

$$\begin{cases} [(2x_0\mathrm{D}_0+1)+(x_0^2\cdot\alpha_i(x)\mathrm{D}_i+x_0\gamma_1(x))]g+(x_0\beta_i(x)\mathrm{D}_i+\delta_1(x))u=S(x) \\ (2\mathrm{D}_0+x_0\alpha_i(x)\mathrm{D}_i+\gamma_2(x))u+(x_0\beta_i(x)\mathrm{D}_i+\delta_2(x))g=T(x) \end{cases}$$

$$u(0, x')=u_0(x')$$

where $\alpha_i(x)$, $\beta_i(x)$, $\gamma_i(x)$, $\delta_i(x)$, $S(x)$ and $T(x)$ are holomorphic in a neighbourhood of $0\in C^{n+1}$, (for the proof see § 5).

For this Cauchy problem, there exist unique holomorphic solutions $u(x)$ and $g(x)$. We can easily prove this fact using the method of indetermined coefficients due to Fuchs.

To be precise, coefficients $u_{\alpha,\beta}$, $g_{\alpha,\beta}$, $v_{\alpha,\beta}$, $h_{\alpha,\beta}$ are determined in the following way:

First, we suppose all $u_{\gamma,\beta}$, $g_{\gamma,\beta}$, $v_{\gamma,\beta}$, $h_{\gamma,\beta}$ ($\gamma\leqq\alpha+2m-2$, $\beta=1, \cdots, m$) are determined and then the right hand side of (3,4) and (3,5) (i) are known. From the remark described above, we determine $u_{\alpha+2m-1}$, and $g_{\alpha+2m-1}$, by solving the Cauchy problem (3,4) (i) with the initial data (3,5) (i). Then $v_{\alpha+2m-1}(0, x')$ are given by (3,5) (ii). From the remark described above, we determine $v_{\alpha+2m-1}$, and $h_{\alpha+2m-1}$, by solving the Cauchy problem (3,4) (ii) with the initial data (3,5) (ii). Thus we can determine holomorphic coefficients $u_{\alpha,\beta}$, $g_{\alpha,\beta}$, $v_{\alpha,\beta}$, $h_{\alpha,\beta}$ inductively. We shall prove that these coefficients have a common existence domain and suitable estimates in § 6.

§ 4. Construction of the phases $\varphi_\beta^{\pm}(x)$ and $\theta_\beta(x)$, $\rho_\beta(x)$

First we solve the eikonal equations (3,3) ${}^1L_{2m,\beta}={}^2L_{2m,\beta}={}^3L_{2m,\beta}={}^4L_{2m,\beta}=0$ and study the phases $\varphi_\beta^{\pm}(x)$ and the auxiliary phases $\theta_\beta(x)$, $\rho_\beta(x)$. According to (F,11) and (F,5) we see the eikonal equations (3,3) are equal to the following equations (4,1) and (4,2)

(4,1) $${}^1\mathring{L}(x, \theta_{\beta x}, \rho_{\beta x})=\sum_{\nu=0}^{m}\mathring{L}_{2\nu}(x, \theta_{\beta x}, \rho_{\beta x})\theta^{\nu}=0$$

(4,2) $$ {}^2\mathring{L}(x, \theta_{\beta x}, \rho_{\beta x}) = \sum_{\nu=0}^{m-1} \mathring{L}_{2\nu+1}(x, \theta_{\beta x}, \rho_{\beta x})\theta^{\nu} = 0 . $$

Multiplying (4,2) by $\pm\theta_\beta^{1/2}$ and taking account of the definition of $\mathring{L}_h(x, r\xi, \eta) = \mathring{L}_h(x, \xi, \eta)r^h$ $(h=0, \cdots, 2m)$, we have

$$ {}^1\mathring{L}(x, \theta_{\beta x}, \rho_{\beta x}) = \sum_{\nu=0}^{m} \mathring{L}_{2\nu}(x, \pm\theta_\beta^{1/2}\cdot\theta_{\beta x}, \rho_{\beta x}) = 0 $$

$$ {}^2\mathring{L}(x, \theta_{\beta x}, \rho_{\beta x}) = \sum_{\nu=0}^{m-1} \mathring{L}_{2\nu+1}(x, \pm\theta_\beta^{1/2}\cdot\theta_{\beta x}, \rho_{\beta x}) = 0 $$

And then adding these two equations and noting the relations

$$ \mathring{L}(x, \xi+\eta) = \sum_{\nu=0}^{2m} \mathring{L}_\nu(x, \xi, \eta) \quad \text{and} \quad \frac{2}{3}(\theta_\beta^{3/2})_x = \theta_\beta^{1/2}\cdot\theta_{\beta x} , $$

we obtain the ordinary eikonal equations

$$ \mathring{L}\Big(x, \Big(\rho_\beta \pm \frac{2}{3}\theta_\beta^{3/2}\Big)_x\Big) = 0 $$

or

$$ \mathring{L}(x, , \varphi_{\beta x}^{\pm}) = 0 \quad \text{where } \varphi_\beta^{\pm} = \rho_\beta \pm \frac{2}{3}\theta_\beta^{3/2} . $$

Solving the Cauchy problem $\mathring{L}(x, \varphi_{\beta x}^{\pm}) = 0$ with the initial data $\varphi_\beta^{\pm}(0, x') = x_1$, we get the phases $\varphi_\beta^{\pm}$ and the auxiliary phases θ_β, ρ_β. For this aim, we prove the next proposition.

Proposition 4.1. *There exist the solutions $\varphi_\nu^{\pm}(x)$ $(\nu=1, \cdots, m)$ of the Cauchy problem*

$$ \begin{cases} \mathring{L}(x, \varphi_{\nu x}^{\pm}) = 0 \\ \varphi_\nu^{\pm}(0, x') = x_1 \quad \text{and} \quad \varphi_{\nu x}^{\pm}(0) = (\lambda_\nu, 1, 0, \cdots, 0) \end{cases} $$

Precisely speaking $\varphi_\nu^{\pm}(x)$ are expressed in the form

$$ \varphi_\nu^{\pm}(x) = \rho_\nu(x) \pm \frac{2}{3}(\theta_\nu(x))^{3/2} . $$

$\theta_\nu(x)$ and $\rho_\nu(x)$ are holomorphic functions in a neighbourhood of $0 \in C^{n+1}$ and satisfy the equations (4,1) *and* (4,2). *Moreover $\theta_\nu(x)$ are represented as follows*

$$ \theta_\nu(x) = x_0 \cdot \sigma_\nu(x) $$

where $\sigma_\nu(x)$ are holomorphic and do not vanish in a neighbourhood of $0 \in C^{n+1}$. As for $\rho_\nu(x)$, $\rho_\nu(0, x') = \varphi_\nu^{\pm}(0, x')$ hold.

Proof. Considering the change of the variables $x_0 = t^2$, $x' = x'$, we get

$$ 0 = \mathring{L}(x, \varphi_x) = \mathring{L}\Big(t^2, x' ; \frac{1}{2t}\varphi_t, \varphi_{x'}\Big) $$

$$=\left[P\left(t^2, x'; \frac{1}{2t}\varphi_t, \varphi_{x'}\right)\right]^2 - t^2\cdot Q\left(t^2, x'; \frac{1}{2t}\varphi_t, \varphi_{x'}\right)$$

So we have the new Cauchy problem

$$(4,3)\qquad \begin{cases} P\left(t^2, x'; \frac{1}{2t}\varphi_t, \varphi_{x'}\right) = \pm t\cdot\left[Q\left(t^2, x'; \frac{1}{2t}\varphi_t, \varphi_{x'}\right)\right]^{1/2} \\ \varphi[t, x']_{t=0} = x_1 \end{cases}$$

Taking account of Assumption (A,iii) and (B,ii), implicit function theorem and Cauchy-Kowalevskaya theorem guarantee that this Cauchy problem has $2m$ solutions $\varphi_\nu^\pm[t, x']$ which are holomorphic in $[t, x']$. We have the relations $\varphi_\nu^+[-t, x'] = \varphi_\nu^-[t, x']$ and $\varphi_\nu^-[-t, x'] = \varphi_\nu^+[t, x']$ by the change of the variables from t to $-t$. Setting $\rho_\nu[t, x'] = \frac{1}{2}(\varphi_\nu^+[t, x'] + \varphi_\nu^-[t, x'])$, we see $\rho_\nu[-t, x'] = \rho_\nu[t, x']$ from the above relations. Thus $\rho_\nu[t, x']$ are even functions in t and so $\rho_\nu[t, x']$ are holomorphic in $x = (x_0, x')$. We write $\rho_\nu[t, x'] = \rho_\nu(x)$. Setting $\theta_\nu[t, x'] = \left(\frac{3}{4}(\varphi_\nu^+[t, x'] - \varphi_\nu^-[t, x'])\right)^{2/3}$, we see $\theta_\nu[-t, x'] = \theta_\nu[t, x']$, too. Hence $\theta_\nu[t, x']$ are holomorphic in x, too. We write $\theta_\nu[t, x'] = \theta_\nu(x)$. Therefore $\varphi_\nu^\pm[t, x'] = \rho_\nu(x) \pm \frac{2}{3}(\theta_\nu(x))^{3/2} = \varphi_\nu^\pm(x)$. On the other hand differentiating the equation (4,3), we have the relations $\varphi_\nu^+[0, x'] = \varphi_\nu^-[0, x'] = x_1$, $\varphi_{\nu t}^+[0, x'] = \varphi_{\nu t}^-[0, x'] = 0$, $\varphi_{\nu tt}^+[0, x'] = \varphi_{\nu tt}^-[0, x']$, and $\varphi_{\nu ttt}^\pm[0, 0] = \pm 2(Q(0, \lambda_\nu, 1, 0, \cdots 0))^{1/2} \times \prod_{\mu\neq\nu}(\lambda_\mu - \lambda_\nu) \neq 0$ from Assumptions (A iii) and (B ii). So $\theta_\nu(x)$ can be expressed in the form

$$\theta_\nu(x) = x_0\sigma_\nu(x)\ \ (\sigma_\nu(0)\neq 0).$$

§ 5. Properties of the operators ${}^hL_{\nu,\beta}$

As for the transport operator ${}^3L_{2m-1,\beta}$ namely the most important operator in ${}^hL_{\nu,\beta}$, we have the following proposition

Proposition 5.1. *${}^3L_{2m-1,\beta}$ are expressed in the following form*

$$^3L_{2m-1,\beta} = \mathring{L}^{(0,i)}(0, x', \rho_{\beta x}(0, x'))\cdot x_0(\sigma_\beta(0, x'))^2 \mathrm{D}_i$$

$$+\frac{1}{2}\mathring{L}^{(0,0)}(0, x_{\prime}, \rho_{\beta x}(0, x'))(\sigma_\beta(0, x'))^2 + x_0^2\cdot\mathcal{L}_{\beta,i}(x)\mathrm{D}_i + x_0 c_\beta(x)$$

where $\mathcal{L}_{\beta,i}(x)$ and $c_\beta(x)$ are holomorphic in a neighbourhood of $0\in C^{n+1}$.

Proof. From the definition of ${}^3L_{2m-1,\beta}$, we have

$$^3L_{2m-1,\beta} = \theta_\beta\mathcal{L}_\beta + \theta_\beta[\rho_{\beta_{x_i x_j}}\cdot\mathcal{P}^1_{ij,\beta} + \theta_{\beta_{x_i x_j}}\cdot\mathcal{P}^2_{ij,\beta}$$

$$+\mathring{R}^{(i)}(x, \rho_{\beta_x})\theta_{\beta_{x_i}} + \theta_\beta\cdot{}_3\tilde{\mathring{R}}(x, \theta_{\beta_x}, \rho_{\beta_x})] + {}^3L_\beta,$$

where

$$\mathcal{L}_{\beta}=[2({}^{1}P(x, \theta_{\beta_x}, \rho_{\beta_x})\cdot{}^{2}P^{(i)}(x, \theta_{\beta_x}, \rho_{\beta_x})+{}^{2}P(x, \theta_{\beta_x}, \rho_{\beta_x})$$
$$\cdot{}^{1}P^{(i)}(x, \theta_{\beta_x}, \rho_{\beta_x})-x_0\cdot{}^{2}Q^{(i)}(x, \theta_{\beta_x}, \rho_{\beta_x})]\mathrm{D}_i$$

and

$${}^{3}\mathring{L}_{\beta}=\frac{1}{2}\mathring{L}^{(i,j)}(x, \rho_{\beta_x})\theta_{\beta_{x_i}}\theta_{\beta_{x_j}}+\theta_{\beta}\tilde{\mathring{L}}(x, \theta_{\beta_x}, \rho_{\beta_x}).$$

On the other hand, from the definition of ${}^{h}K(x, \xi, \eta)$ $(h=1, 2)$ and $\theta_{\beta}(x)=x_0\cdot\sigma_{\beta}(x)$, we see $\mathcal{L}_{\beta}|_{x_0=0}=\mathring{L}^{(0,i)}(0, x', \rho_{\beta_x}(0, x'))\sigma_{\beta}(0, x')\mathrm{D}_i$ and

$${}^{3}\mathring{L}_{\beta}|_{x_0=0}=\frac{1}{2}\mathring{L}^{(0,0)}(0, x', \rho_{\beta_x}(0, x'))(\sigma_{\beta}(0, x'))^2.$$

Thus we proved the Proposition 5.1.

For the method of the majorant function, we introduce the new variables $y_{\beta}=(y_{0,\beta}, y_{1,\beta}, \cdots, y_{n,\beta})$ $(\beta=1, \cdots, m)$ as follows.

We set
$$\begin{cases} y_{0,\beta}=x_0 \\ y_{\alpha,\beta}=\psi_{\alpha,\beta}(x) \quad (\alpha=1, \cdots, n) \end{cases}$$

where $\psi_{\alpha,\beta}(x)$ are the solutions of the Cauchy problem

$$P^{(i)}(0, x', \rho_{\beta_x}(0, x'))\mathrm{D}_i\psi_{\alpha,\beta}(x)=0$$

with the initial data $\psi_{\alpha,\beta}(0, x')=x_{\alpha}$.

Considering the transformation of the variables x into y_{β}, we have the following proposition about the operator ${}^{h}L_{\nu,\beta}$ which shall be employed in the estimates of the coefficients of the formal solution in the next section.

Proposition 5.2. (i) *Using the new variables, we represent*

$${}^{3}L_{2m-1,\beta}=a_{\beta}(y_{\beta})\left(y_{0,\beta}\ \mathrm{D}_{y_{0,\beta}}+\frac{1}{2}\right)+y_{0,\beta}^2 a_{\beta i}(y_{\beta})\mathrm{D}_{y_{0,\beta}}+y_{0,\beta}c_{\beta}(y_{\beta}),$$

where

$$a_{\beta}(y_{\beta})=\mathring{L}^{(0,0)}(0, x', \rho_{\beta_x}(0, x'))(\sigma_{\beta}(0, x'))^2\neq 0,$$

and $a_{\beta}(y_{\beta})$, $c_{\beta}(y_{\beta})$ *are holomorphic functions in a neighbourhood of* $0\in C^{n+1}_{y_{\beta}}$.
(ii) *In* ${}^{h}\mathring{L}_{\nu,\beta}$ $(h=1, 2, 4)$, *the terms* $\mathrm{D}^{2m-\nu-1}_{y_{1,\beta}}\mathrm{D}_{y_{\mu,\beta}}$ $(\mu=0, \cdots, n)$ *always have the factor* $y_{0,\beta}$.
(iii) *In* ${}^{3}\mathring{L}_{\nu,\beta}$, *the terms* $\mathrm{D}^{2m-\nu-1}_{y_{1,\beta}}\mathrm{D}_{y_{\mu,\beta}}$ $(\mu=0, \cdots n)$ *always have the factor* $y_{0,\beta}^2$.

Proof. (i) Note the relations

$$\mathrm{D}_0=\mathrm{D}_{y_{0,\beta}}+\sum_{\alpha=1}^{n}\psi_{\alpha,\beta_{x_0}}\cdot\mathrm{D}_{y_{\alpha,\beta}}, \qquad \mathrm{D}_{\gamma}=\sum_{\alpha=1}^{n}\psi_{\alpha,\beta_{x_\gamma}}\cdot\mathrm{D}_{y_{\alpha,\beta}}$$

$(\gamma=1, \cdots, n)$ and $P^{(i)}(0, x', \rho_{\beta_x}(0, x')\cdot\mathrm{D}_i=P^{(0)}(0, x', \rho_{\beta_x}(0, x'))\cdot\mathrm{D}_{y_{0,\beta}}$ and then we obtain (i).

(ii) From (F,11), we have ${}^1\mathring{L}_{\nu,\beta}|_{x_0=0}={}^4\mathring{L}_{\nu,\beta}|_{x_0=0}$,

$$ {}^1\mathring{L}_{\nu,\beta}|_{x_0=0}=\sum_{|\alpha|=2m-\nu} {}^1\mathring{L}^{(\alpha)}(0, x', \theta_{\beta_x}(0, x'), \rho_{\beta_x}(0, x'))\frac{\mathrm{D}^\alpha}{\alpha!}, $$

$$ {}^2\mathring{L}_{\nu,\beta}|_{x_0=0}=\sum_{|\alpha|=2m-\nu} {}^2\mathring{L}^{(\alpha)}(0, x', \theta_{\beta_x}(0, x'), \rho_{\beta_x}(0, x'))\frac{\mathrm{D}^\alpha}{\alpha!}. $$

From $\mathring{L}(0, x', \phi_{1,\beta_x}(0, x'))=[P(0, x', \phi_{1,\beta_x}(0, x'))]^2=0$, we have $\mathring{L}^{(\mu)}(0, x', \phi_{1,\beta_x}(0, x'))=0$ ($\mu=0, \cdots, n$), too. We get the Euler's identity

$$ \sum_{|\alpha|=k-1}\frac{(\phi_{1,\beta_x}(0, x'))^\alpha}{\alpha!} L^{(\mu)(\alpha)}(0, x', \phi_{1,\beta_x}(0, x')) $$

$$ =\frac{(2m-1)!}{(k-1)!(2m-k)!} L^{(\mu)}(0, x', \phi_{1,\beta_x}(0, x'))=0. $$

Hence $\mathrm{D}_{y_{1,\beta}}^{2m-\nu-1}\mathrm{D}_{y_{\mu,\beta}}$ ($\mu=0, \cdots, n$) cannot appear in ${}^h\mathring{L}_{\nu,\beta}|_{x_0=0}$ ($h=1, 2, 4$). (iii) In the similar way in the proof of (ii), we can verify (iii).

§ 6. Convergence of the formal solution

As for our formal solution (3,2), we know the coefficients are determined successively by solving the Cauchy problem for the system of transport equations (3,4) with the initial data (3,5) in § 3. Moreover, we investigate transport equations, especially transport operators more precisely in § 5. It remains to verify the convergence and the uniqueness of the formal solution (3,5). However, the uniqueness of the solution follows from the Cauchy-Kowalevskaya theorem. In this section, we are to prove the convergence of the formal solution (3,2) by the method of the majorant function. To do so, we introduce a family of functions $\{\phi_\alpha(z, \zeta, y)\}_{\alpha=0}^\infty$ which play an important role in the proof of the convergence of the formal solutions (3,2),

$$ \phi_\alpha(z, \zeta, y)=\partial_\zeta^\alpha\phi_0(z, \zeta, y)=\sum_{n\geq 0, j\geq 0}\frac{(j+n)!}{j!\,n!}\left(\frac{3}{2}\right)^\alpha $$

$$ \cdot\frac{\Gamma\left(\frac{2}{3}(j+n+1)+\alpha\right)}{\Gamma\left(\frac{2}{3}(j+n+1)\right)}\cdot\frac{y^j\cdot(\rho z)^n}{[R-(3/2)\zeta]^{\alpha+(2/3)(j+n+1)}}. $$

The following proposition can be easily verified.

Proposition 6.1. *$\phi_\alpha(z, \zeta, y)$ have the following properties.*

(1) $(2z\mathrm{D}_z+1)\phi_{\alpha+2m}\gg z\mathrm{D}_z\phi_{\alpha+2m},\ z\mathrm{D}_z^2\phi_{\alpha+2m-1}$.

(2) $(2z\mathrm{D}_z+1)\phi_{\alpha+2m}\gg\dfrac{a}{\rho^\nu R^{(1/3)\nu}}\mathrm{D}_z^\nu\phi_{\alpha+2m+1-\nu}$ ($\nu=3, \cdots, 2m+1$)

where a is a constant.

(3) $D_z^2\phi_{\alpha+2m-1}\gg\frac{\rho^2}{R}zD_z\phi_{\alpha+2m},\ z\phi_{\alpha+2m-1}$.

(4) $D_z^2\phi_{\alpha+2m-1}\gg\frac{1}{(R^{(1/3)}\rho)^{\nu-2}}D_z^{\nu}\phi_{\alpha+2m+1-\nu}\ (\nu=2,\cdots,2m)$.

(5) $\frac{1}{[R'-(2/3)\zeta]\cdot[R''-\rho z-y]}D_z^{\beta}\phi_\alpha\gg\frac{1}{(R'-R)(R'-R^{(2/3)})}D_z^{\beta}\phi_\alpha$

where $R'>R,\ R''>R^{(2/3)}$.

(6) $\phi_{\alpha+1}\gg\frac{1}{\rho R^{(1/3)}}D_z\phi_\alpha$.

Proposition 6.2. *For the Cauchy problem*

$$\begin{cases}(2x_0D_0+1)g+(x_0^2\alpha_iD_i+x_0\gamma_1)g+(x_0\beta_iD_i+\delta_1)u=S(x),\\ 2D_0u+(x_0\alpha_iD_i+\gamma_2)u+(x_0\beta_iD_i+\delta_2)g=T(x)\end{cases}$$

with the initial data $u(0,x')=u_0(x)$.

there exist unique holomorphic solutions $u(x)$ *and* $g(x)$ *in the neighbourhood of* $0\in C^{n+1}$. *Moreover, assuming* $\alpha_i\ll\tilde{\alpha}_i,\ \beta_i\ll\tilde{\beta}_i,\ \gamma_h\ll\tilde{\gamma}_h,\ \delta_h\ll\tilde{\delta}_h,\ S\ll\tilde{S},\ T\ll\tilde{T}$ *and* $u_0\ll\tilde{u}_0$, *it is verified that* $u(x)\ll\tilde{u}(x)$ *and* $g(x)\ll\tilde{g}(x)$ *if* $\tilde{u}(x)$ *and* $\tilde{g}(x)$ *satisfy the following majorant relations,*

$$\begin{cases}(2x_0D_0+1)\tilde{g}\gg(x_0^2\tilde{\alpha}_iD_i+x_0\tilde{\gamma}_1)\tilde{g}+(x_0\tilde{\beta}_iD_i+\tilde{\delta}_1)\tilde{u}\\ 2D_2\tilde{u}\gg(x_0\tilde{\alpha}_iD_i+\tilde{\gamma}_2)u+(x_0\tilde{\beta}_iD_i+\tilde{\delta}_2)\tilde{g}\\ \tilde{u}(0,x')\gg\tilde{u}_0,\end{cases}$$

(*for the proof see* [5]).

From these propositions, we obtain the following proposition.

Proposition 6.3. *There exist positive constants* A, B, C, D, E, F, R *and* K *such that*

$$u_{\alpha,\beta}(y_\beta)\ll AK^{\tilde{\alpha}}D_z\phi_{\tilde{\alpha}}\Big(y_{0,\beta},\ y_{1,\beta},\ \sum_{\nu=2}^{n}y_{\nu,\beta}\Big)$$

$$g_{\alpha,\beta}(y_\beta)\ll BK^{\tilde{\alpha}}\phi_{\tilde{\alpha}+1}\Big(y_{0,\beta},\ y_{1,\beta},\ \sum_{\nu=2}^{n}y_{\nu,\beta}\Big)$$

$$v_{\alpha,\beta}(y_\beta)\ll CK^{\tilde{\alpha}+1}D_z\phi_{\tilde{\alpha}+1}\Big(y_{0,\beta},\ y_{1,\beta},\ \sum_{\nu=2}^{n}y_{\nu,\beta}\Big)$$

$$h_{\alpha,\beta}(y_\beta)\ll DK^{\tilde{\alpha}+1}\phi_{\tilde{\alpha}+2}\Big(y_{0,\beta}y_{1,\beta},\ \sum_{\nu=2}^{n}y_{\nu,\beta}\Big)$$

$$(u_{\alpha,\beta}+h_{\alpha-1,\beta})(0,x')\ll EK^{\tilde{\alpha}}D_z\phi_{\tilde{\alpha}}\Big(0,\ x_1,\ \sum_{\nu=2}^{n}x_\nu\Big)$$

$$u_{\alpha,\beta}(0, x')\ll FK^{\tilde{\alpha}}D_z\phi_{\tilde{\alpha}}\Big(0, x_1, \sum_{\nu=2}^n x_\nu\Big) \quad (\tilde{\alpha}=\alpha+1+2m)$$

where these constants are independent of α *and depend only on* $L(x, D)$ *and* $\theta_\beta(x)$, $\rho_\beta(x)$.

Therefore we know $u_{\alpha,\beta}$, $g_{\alpha,\beta}$, $v_{\alpha,\beta}$, $h_{\alpha,\beta}$ have a common existence domain that is a neighbourhood of $0\in C^{n+1}$ and the estimates $|u_{\alpha,\beta}|$, $|g_{\alpha,\beta}|$, $|v_{\alpha,\beta}|$, $|h_{\alpha,\beta}|<C\alpha!\gamma^\alpha$ in this domain, where C, γ are positive constants independent of α. On the other hand, we have the estimates $|X_\alpha|$, $|Y_\alpha|<C_{\mathcal{K}}\dfrac{1}{(\alpha-1)!}r^{\alpha-1}$ on any compact set $\mathcal{K}$ in $\boldsymbol{D}_r\backslash\boldsymbol{K}$, where $\alpha>0$ and $C_{\mathcal{K}}$ is a constant independent of α and depends only on the compact set $\mathcal{K}$, (for the proof see [5]). Thus choosing r such that $r<\gamma^{-1}$, we prove the convergence of the formal solution of the Cauchy problem (1,1) on $\boldsymbol{D}_r\backslash\boldsymbol{K}$. We remark $u(x)$ does not ramify on $x_0=0$ $x_1\neq 0$ because of the Cauchy-Kowalevskaya theorem.

Proof of Proposition 6.3. We prove this proposition by induction of α. Assume that these estimates are valid for $\alpha=-1+2m-1, \cdots, \alpha+2m-2$. Let coefficients of differential operators is (3.4) and (3.5) $\ll M(R'-(2/3)y_{1,\beta})^{-1}(R''-\rho y_{0,\beta}-\sum_{\nu=2}^n y_{\nu,\beta})^{-1}$. First we have the following estimates about the initial data from (3,5 i)

$$(u_{\alpha+2m-1,\beta}+h_{\alpha+2m-2,\beta})(0, x')\ll MD_0^\mu(u_{\alpha+2m-1-\mu,\gamma}+h_{\alpha+2m-2-\mu,\gamma})(0, x')$$
$$+MK^{\alpha+2m-2}(\phi_{\tilde{\alpha}+2m-1}+D_z\phi_{\tilde{\alpha}+2m-1}+\phi_{\tilde{\alpha}+2m-1})\Big(0, x_1, \sum_{\nu=2}^n x_\nu\Big).$$

As for the estimates of $D_0^\mu(u_{\alpha+2m-1-\mu,\beta}+h_{\alpha+2m-2-\mu,\beta})$, we make use of the fact that $D_0^\mu=S_\beta(y_\beta, D_{y_\beta})D_{y_{0,\beta}}+T_\beta(y_\beta, D'_{y_\beta})$, where S_β is a partial difierential operator in D_{y_β} of order $\mu-1$ and T_β is a partial differential operator in $D'_{y_\beta}=(D_{y_{1,\beta}}, \cdots, D_{y_{n,\beta}})$ of order μ. So we get

$$D_0^\mu(u_{\alpha+2m-1-\mu,\beta}+h_{\alpha+2m-2-\mu,\beta})(y_\beta)|_{y_{0,\beta}=0}$$
$$=S_\beta(y_\beta, D'_{y_\beta})D_{y_{0,\beta}}(u_{\alpha+2m-1-\mu,\beta}+h_{\alpha+2m-2-\mu,\beta}(y_\beta)|_{y_{0,\beta}=0}$$
$$+T_\beta(0, y'_\beta, D'_{y_\beta})(u_{\alpha+2m-1-\mu,\beta}+h_{\alpha+2m-2-\mu,\beta})(0, y'_\beta)$$

To the former part of the right hand side of this identity we apply the estimates $u_{\alpha,\beta}(y_\beta)\ll AK^{\tilde{\alpha}}D_z\phi_{\tilde{\alpha}}$, $h_{\alpha,\beta}(y_\beta)\ll DK^{\tilde{\alpha}+1}\phi_{\tilde{\alpha}+2}$. To latter part of the right hand side of this identity, we apply the estimates $(u_{\alpha,\beta}+h_{\alpha-1,\beta})(0, x')\ll EK^{\tilde{\alpha}}D_z\phi_{\tilde{\alpha}}$. Then, we get the estimates from (3,5 i):

$$(u_{\alpha+2m-1,\beta}+h_{\alpha+2m-2,\beta})(0, x')\ll 4MK^{\tilde{\alpha}+2m-1}D_z\phi_{\tilde{\alpha}+2m-1}\Big(0, x_1, \sum_{\nu=2}^n x_\nu\Big)$$
$$\ll EK^{\tilde{\alpha}+2m-1}D_z\phi_{\tilde{\alpha}+2m-1}\Big(0, x_1, \sum_{\nu=2}^n x_\nu\Big)$$

(in this section we use M as a suitable positive constant).

As for the estimates $h_{\alpha+2m-2}(0, x')$, we restrict the equation

$$^{3}L_{2m-1,\beta}h_{\alpha+2m-2,\beta}=-\sum_{\nu=0}^{2m-1}{}^{1}L_{\nu,\beta}v_{\alpha+\nu-1,\beta}-\sum_{\nu=0}^{2m-2}{}^{3}L_{\nu,\beta}h_{\alpha+\nu-1,\beta}$$

on $x_0=0$, and estimating the right hand side of this restricted equation by assumption of the induction, we get the estimates

$$h_{\alpha+2m-2,\beta}(0, x')\ll MK^{\tilde{\alpha}+2m-1}\mathrm{D}_z\phi_{\tilde{\alpha}+2m-1}\Big(0,\ x_1,\ \sum_{\nu=2}^{n}x_\nu\Big).$$

From this estimates and the relation $(u_{\alpha+2m-1,\beta}+h_{\alpha+2m-2,\beta})-h_{\alpha+2m-2,\beta}=u_{\alpha+2m-1,\beta}$, we get the estimates

$$u_{\alpha+2m-1,\beta}(0, x')\ll FK^{\tilde{\alpha}+2m-1}\mathrm{D}_z\phi_{\tilde{\alpha}+2m-1}\Big(0,\ x_1,\ \sum_{\nu=2}^{n}x_\nu\Big).$$

Next, taking account of the properties of $^{h}L_{\nu,\beta}$ obtained in Proposition 5,2 (ii), (iii) and using Proposition 6,1 with the assumptions of the induction, we get the following estimates

$$\begin{cases} {}^{4}L_{\nu,\beta}g_{\alpha+\nu,\beta}\ll BMK^{\tilde{\alpha}+\nu}\Big(x_0\phi_{\tilde{\alpha}+2m+1}+x_0\mathrm{D}_z\phi_{\tilde{\alpha}+2m}+\sum_{\mu=2}^{2m-\nu}\mathrm{D}_z^{\mu}\phi_{\tilde{\alpha}+2m-\mu} \\ \qquad\ll BMK^{\tilde{\alpha}+\nu}\Big(\rho^{-3}+\rho^{-2}+\sum_{\mu=2}^{2m-\nu}(\rho R^{1/3})\Big)\mathrm{D}_z^2\phi_{\tilde{\alpha}+2m-1} \\ \qquad\text{for } \nu=0,\cdots,2m-2 \\ {}^{4}L_{2m-1,\beta}g_{\alpha+2m-1,\beta}\ll BMK^{\tilde{\alpha}+2m-1}\mathrm{D}_z^2\phi_{\tilde{\alpha}+2m-1}, \end{cases}$$

$$\begin{cases} {}^{2}L_{\nu,\beta}u_{\alpha+\nu,\beta}\ll AMK^{\tilde{\alpha}+\nu}\Big(x_0\mathrm{D}_z\phi_{\tilde{\alpha}+2m}+\sum_{\mu=1}^{2m}\mathrm{D}_z^{\mu}\phi_{\tilde{\alpha}+2m-\mu} \\ \qquad\ll AMK^{\tilde{\alpha}+\nu}\Big(\rho^{-2}+\sum_{\mu=1}^{2m}(\rho R^{1/3})^{-2}\Big)\mathrm{D}_z^2\phi_{\tilde{\alpha}+2m-1} \\ \qquad\text{for } \nu=0,\cdots,2m-2, \end{cases}$$

$$\begin{cases} {}^{1}L_{\nu,\beta}u_{\alpha+\nu,\beta}\ll AMK^{\tilde{\alpha}+\nu}\Big(x_0\mathrm{D}_z\phi_{\tilde{\alpha}+2m}+x_0\mathrm{D}_z^2\phi_{\tilde{\alpha}+2m-1}+\sum_{\mu=2}^{2m-\nu}\mathrm{D}_z^{\mu+1}\phi_{\tilde{\alpha}+2m-\mu}\Big) \\ \qquad\text{for } \nu=0,\cdots,2m-2, \\ {}^{1}L_{2m-1,\beta}u_{\alpha+2m-1,\beta}\ll AMK^{\tilde{\alpha}+2m-1}(1+\rho^{-1})(2\mathrm{D}_z+1)\phi_{\tilde{\alpha}+2m}, \end{cases}$$

$$\begin{cases} {}^{4}L_{\nu,\beta}g_{\alpha+\nu,\beta}\ll BMK^{\tilde{\alpha}+\nu}\Big(x_0^2\phi_{\tilde{\alpha}+2m}+\sum_{\mu=1}^{2m-\nu}x_0\mathrm{D}_z^{\mu}\phi_{\tilde{\alpha}+2m-\mu}\Big) \\ \qquad\ll BMK^{\tilde{\alpha}+\nu}\Big(\rho^{-1}R^{-(1/3)}+\sum_{\mu=1}^{2m-\nu}(\rho R^{1/3})^{\mu-1}\Big)(2\mathrm{D}_z+1)\phi_{\alpha+2m} \\ \qquad\text{for } \nu=0,\cdots,2m-2. \end{cases}$$

If the following majorant relations are proved, from Proposition 6,2 we see that our statement is valid for $\alpha+2m-1$

$$\begin{cases} AK D_z^2 \phi_{\tilde{\alpha}+2m-1} \gg [2mM(A+B)(\rho^{-3}+\rho^{-2}+4mb^{2m-2}) \\ \qquad +BMK(\rho^{-3}+\rho^{-2})] D_z^2 \phi_{\tilde{\alpha}+2m-1} \\ BK(2D_z+1)\phi_{\tilde{\alpha}+2m} \gg [2mM(A+B)(1+\rho^{-1}+4mb^{2m-2} \\ \qquad +AMK(1+\rho^{-1})](2D_z+1)\phi_{\tilde{\alpha}+2m} \\ CK D_z^2 \phi_{\tilde{\alpha}+2m} \gg [2mM(C+D)(\rho^{-3}+\rho^{-2}+4mb^{2m-2})+DMK(\rho^{-3}+\rho^{-2})] D_z^2 \phi_{\tilde{\alpha}+2m} \\ DK(2D_z+1)\phi_{\tilde{\alpha}+2m+1} \gg [2mM(C+D)(1+\rho^{-1}+4mb_2{}^{m-2}) \\ \qquad +CMK(1+\rho^{-1})](2D_z+1)\phi_{\tilde{\alpha}+2m+1} \\ CD_z\phi_{\tilde{\alpha}+2m}|_{x_0=0} \gg EMD_z\phi_{\tilde{\alpha}+2m}|_{x_0=0} \\ AD_z\phi_{\tilde{\alpha}+2m-1}|_{x_0=0} \gg MFD_z\phi_{\tilde{\alpha}+2m-1}|_{x_0=0} \\ \text{where } b=\rho R^{1/3}. \end{cases}$$

These majorant relations are reduced to the following systems of inequalities,

$$\begin{cases} AK>2mM(A+B)b^{2m-2}+BM\rho^{-2} \\ BK>2mM(A+B)b^{2m-2}+AMK \\ CK>2mM(C+D)b^{2m-2}+DM\rho^{-2} \\ DK>2mM(C+D)b^{2m-2}+CMK \\ A>FM \\ CK>EM \end{cases}$$

On the other hand, for ρ and K sufficiently large, and R sufficiently small, we can choose positive constants $A, B, C, D, E, F, K, R, \rho$ such that the system of these inequalities are valid. Thus we prove Proposition 6,3.

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References

[1] Y. Hamada, J. Leray and C. Wagshal, Systems d'equations aux derivées partielles a caractéristiques multiples; problème de Cauchy ramifié, hyperbolicité partielles, J. Math. pure et appl., **55** (1976), p. 297 à 352.

[2] Y. Hamada and G. Nakamura, On the singularities of the solutions of the Cauchy problem for the operator with non-uniform multiple characteristics, Ann. Scoula Norm. Sup. Pisa, 1977, 725-755.

[3] D. Ludwig, Uniform asymptotic expansions at acaustic, Comm. Pure Appl. Math, **19** (1966), 215-250.

[4] G. Nakamura, The singularities of solutions of the Cauchy problems for systems whose characteristic roots are non-uniform, Publ. RIMS. Kyoto Univ., **13** (1977), 255-275.

[5] J. Urabe, On the theorem of Hamada for a linear second order equation with variable multiplicities, J. Math. Kyoto Univ., **19** (1979), 153-169.