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A field extension with certain finiteness condition on multiplicative group extension

By

Masayoshi NAGATA

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In a previous paper [1], the writer proved the following result. Let K be a proper subfied of a field L and consider the multiplicative groups K^*, L^* of these fields. If every element of L^*/K^* is of finite order, then either L is purely inseparable over K or L is algebraic over a finite field.

In the present note, we consider similar case with stronger conditions, namely the following two cases: (1) the case where the index $[L^*: K^*]$ is finite and (2) the case where orders of elements of L^*/K^* are bounded. As for the first case, we prove that L and K are finite fields and $\sharp(K)<[L^*: K^*]$. As for the latter case, we prove that either L is purely inseparable over K or L is a finite field. Namely, we can state our main results as follows.

As before, let K be a proper subfield of a field L and consider multiplicative groups K^*, L^* of these fields. Then:

Theorem 1. *If the group index $[L^*: K^*]$ is finite, then L is a finite field and $\sharp(K)<[L^*: K^*]$.*

Theorem 2. *If the orders of elements of L^*/K^* are bounded, then either* (1) *L is purely inseparable extension of K such that, for a power q of the characteristic, it holds that $L^q \subseteq K$, or* (2) *L is a finite field.*

Proof of Theorem 1. Let x be an element of L which is not in K. If K is not a finite field, then there are two elements b, c of K such that $(x+b)K^*=(x+c)K^*$, $b\neq c$. Setting $y=x+b$, $a=c-b$, we have $yK^*=(y+a)K^*$, and $(y+a)y^{-1}\in K^* \subseteq K$. Then $ay^{-1}\in K$. Since $a\neq 0$, we have $y^{-1}\in K$, and $y=x+b\in K$. This implies that $x\in K$, a contradiction. Thus K is a finite field. Since the index $[L^*: K^*]$ is finite, L is finitely generated over K and therefore L is a finite field. Set $q=\sharp(K)$ and $n=[L: K]$, then $\sharp(L^*)=q^n-1$ and $\sharp(K)=q-1$, hence $[L^*: K^*]=1+q+\cdots+q^{n-1}>q$. Q.E.D.

Proof of Theorem 2. In view of the theorem in [1], we see that either L is purely inseparable over K or algebraic over a finite field. In the former case, we have (1) quite easily. So we assume that L is algebraic over a finite field. First, we assume that K is not a finite field. Take an element x of L outside of K and consider the minimal polynomial for x over K. Choose a sequence of subfields K_i of K such that (i) coefficients of the minimal polynomial are in K_1, (ii) $K_1 \subset K_2 \cdots$ and (iii) the union of all the K_i is K. Let b_i be a generator of the multiplicative group of $K_i(x)$. Then the order of the class of b_i modulo K^* is $1+q_i+\cdots q_i^{n-1}$ with $q_i=\sharp(K_i)$ and $n=[K(x)\colon K]$. This contradicts the boundeness. Thus K is a finite field. The boundeness implies that L is a finite algebraic extension of K, and L is also a finite field. Q. E. D.

DEPARTMENT OF MATHEMATICS,
KYOTO UNIVERSITY

Reference

[1] M. Nagata, A type of integral extensions, J. Math. Soc. Japan, **20** (1968), 266–267.

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About a week ago, the writer received a letter from Dr. E. D. Davis of New York State University and was shown how to prove a similar result on the case where L^*/K^* is of finite rank.

The result can be formulated as follows:

Theorem 3. (*Davis*) *Under the notation as above,*

1) *If L^*/K^* is finitely generated, then L is a finite field.*

2) *If L^*/K^* modulo torsion is finitely generated, then L^*/K^* is a torsion group (namely, it is the the case we dealt in* [1]).

Proof. In each of the cases, we see easily that L is algebraic over K. If K is not algebraic over any finite field and if L is not purely inseparable over K, then, in view of the lemma below, the main result of our article [2] below shows that L^*/K^* modulo torsion is not finitely generated. Thus we consider the case where K is algebraic over a finite field and L^*/K^* is finitely generated. In this case, L^*/K^* is a finite gorup, and it is the case of our Theorem 1. Q. E. D.

Lemma. *With L and K as above, assume that there are n distinct rank one valuations of K, each of which has at least two distinct prolongations in L. Then the rank of L^*/K^* modulo torsion is at least n.*

The proof is easy by the approximation theorem of valuations.

[2] M. Nagata, T. Nakayama and T. Tsuzuku, An existence lemma in valuation theory, Nagoya Math. J., **6** (1957), 59–61.