J. Math. Kyoto Univ. (JMKYAZ)
22-2 (1982) 223-242

Spectral representation for Schrödinger operators with magnetic vector potentials

By

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(Received Jan. 22, 1981)

1. Introduction

The present paper is concerned with the Schrödinger operator

$$(1.1)\qquad L=-\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}+ib_j\right)^2+V \quad \text{in} \quad \boldsymbol{R}^n,$$

where b_j and V denote the multiplication operators by real-valued functions $b_j(x)$ and $V(x)$. $b(x)=(b_1(x),\dots, b_n(x))$ is the magnetic vector potential (thus rot b represents the magnetic field when $n=3$) and $V(x)$ is the electric scalar potential. In their classical work [5], Ikebe and Kato have proved the essential self-adjointness of L on $C_0^\infty(\boldsymbol{R}^n)$ in the Hilbert space $\mathscr{H}=L_2(\boldsymbol{R}^n)$ for an arbitrary $b(x)$ which is continuously differentiable and for $V(x)$ in an appropriate class of functions. Recently, the condition on $b(x)$ has been improved considerably by Leinfelder-Simader [9]. In the present paper, for simplicity, we impose the differentiability condition on $b(x)$ and somewhat strong conditions on the local behavior of $V(x)$ as well as on its decay rate at infinity, which guarantee the uniqueness of the self-adjoint realization of L in $\mathscr{H}$, and we shall denote it by H. Moreover, the magnetic field is assumed to tend to zero at infinity (for the study of the Schrödinger operators with constant magnetic fields, see Avron-Herbst-Simon [2]). We assume the following conditions throughout the paper:

(V) $V(x)$ is a real-valued measurable function and there exist positive constants C_0, δ such that $|V(x)|\leqq C_0(1+|x|)^{-1-\delta}$ for all $x\in\boldsymbol{R}^n$.

(b) $b_j(x)$ $(j=1,\dots, n)$ are real-valued C^2 functions and there exist positive constants C_0, δ such that $B_{jk}=\dfrac{\partial b_k}{\partial x_j}-\dfrac{\partial b_j}{\partial x_k}$ $(j, k=1,\dots, n)$ satisfies $|B_{jk}(x)|\leqq C_0(1+|x|)^{-3/2-\delta}$ for all $x\in\boldsymbol{R}^n$.

Logically, the constants C_0 and δ in (V) and (b) are different. But we may and do assume that they are identical. Moreover, we can take δ so that $0<\delta<1$,

$\delta \neq \frac{1}{2}$. It is known that these assumptions imply the following properties of H (Ikebe-Saitō [6], Kuroda [8]):

(i) The essential spectrum of H is $[0, \infty)$,

(ii) $E((0, \infty))H$ is an absolutely continuous operator, where E denotes the spectral measure associated with H.

Our purpose in the present paper is to show that H admits a spectral representation, which needs a stronger assumption on $b(x)$, i.e., the following:

(b') In addition to (b), $\left|\frac{\partial B_{jk}}{\partial x_j}(x)\right| \leqq C_0(1+|x|)^{-2-\delta}$ for $j, k=1,\dots, n$ and for all $x \in \boldsymbol{R}^n$.

Namely, we shall establish the existence of an unitary operator $\mathscr{F}$ from the subspace $\mathscr{H}_{ac}$ of absolute continuity for H onto $\hat{\mathscr{H}}=L_2((0, \infty); L_2(\Omega))$ (Ω denotes the unit sphere in $\boldsymbol{R}^n$), which diagonalizes H (Theorem 4.2).

Let us make a brief sketch of some well-known results about the spectral representation in the case $n=3$ and $b=0$, i.e., for $H=-\Delta+V$ (see Ikebe [3], [4]). Let k be a non zero vector in $\boldsymbol{R}^3$. A generalized eigenfunction $\psi_k(x)$ for $-\Delta+V$, which behaves asymptotically like the plane wave $e^{ik\cdot x} \equiv \phi_k(x)$, is obtained by solving the Lippmann-Schwinger equation

$$\psi_k(x)=\phi_k(x)-\frac{1}{4\pi}\int_{\boldsymbol{R}^3}\frac{e^{i|k||x-y|}}{|x-y|}V(y)\psi_k(y)dy. \tag{1.2}$$

The spectral representation for $-\Delta+V$ can be obtained in terms of generalized Fourier transforms

$$\mathscr{F}f(k)=(2\pi)^{-3/2}\int_{\boldsymbol{R}^3}\overline{\psi_k(x)}f(x)dx.$$

However, this procedure works only for V which decays faster than $|x|^{-2-\delta}$ for some positive δ. Agmon [1] has used a version of the Lippmann-Schwinger equation to construct the generalized eigenfunctions and has obtained the spectral representation in the case of short-range V (i.e. $V(x)=O(|x|^{-1-\delta})$). On the other hand, in the case of long-range V which satisfies $V(x)=O(|x|^{-1/2-\delta})$, Ikebe [4] has obtained the spectral representation by considering the following limit, instead of using the generalized eigenfunctions explicitly:

$$\operatorname*{s-lim}_{m\to\infty} r_m^{(n-1)/2}e^{-iK(r_m\cdot,\lambda)}R(\lambda+i0)f(r_m\cdot) \tag{1.3}$$

(strong limit in $L_2(\Omega)$), where $\{r_m\}$ is a sequence tending to infinity as $m\to\infty$, $r_m\cdot$ stands for $r_m\omega(\omega\in\Omega)$, $K(x, \lambda)$ is a real-valued function which behaves like $\lambda|x|$ at infinity and $R(\lambda+i0)$ denotes the boundary value of the resolvent of H on the upper side of the positive real axis (see Theorem 2.3 for details).

The spectral representation for Schrödinger operators with long-range potentials have been investigated by several authors since [4] (e.g. Isozaki [7], Saitō [10]). But it seems that, except for the case of constant magnetic field, the spectral representation for H with magnetic vector potentials has not been studied yet.

In the present paper, $K(x, \lambda)$ is of the form $\lambda|x|-A(x)$, where $A(x)$ is a certain function depending only on $b(x)$, which will be constructed in § 2. This function $A(x)$ has been utilized in Kuroda [8] and, as noticed there, is closely related to the gauge transformation, which changes the magnetic potential b into $b-\operatorname{grad} A$, but does not change the magnetic field. Our assumption (b) implies that $A(x)$ can be chosen so that $|b(x)-\operatorname{grad} A(x)|\leqq|x|^{-1/2-\delta}$ for some positive δ.

§ 2 is a preliminary section including the construction of $A(x)$ and the limiting absorption theorem. In § 3, we study the asymptotic behavior of $R(\lambda+i0)f$, that is, the existence of the limit (1.3) for *any* sequence $\{r_m\}$ tending to infinity. For this purpose, we need further the following assumption:

(V') In addition to (V), $V(x)$ satisfies $|V(x)|\leqq C_0(1+|x|)^{-3/2-\delta}$ for all $x\in \boldsymbol{R}^n$.

Theorem 3.9 asserts that the limit (1.3) exists for $f\in L_{2,1}$ (i.e. $(1+|x|)f(x)\in L_2(\boldsymbol{R}^n)$) without taking subsequences if the assumptions (V') and (b) are fulfilled.

It must be noted that, for obtaining our final result, the spectral representation theorem, it suffices to show that the limit (1.3) exists for certain specified sequences $\{r_m\}$. This is, in fact, what we are going to do in § 4 under the assumptions (V) and (b').

2. Preliminaries.

Throughout the paper we use the following notations:

$$B_r=\{x\in \boldsymbol{R}^n|\ |x|<r\},$$

$$E_r=\{x\in \boldsymbol{R}^n|\ |x|>r\},$$

$$S_r=\{x\in \boldsymbol{R}^n|\ |x|=r\},\quad (r>0).$$

For $\alpha\in\boldsymbol{R}$ and a domain $G\subset\boldsymbol{R}^n$, let $L_{2,\alpha}(G)$ denote the Hilbert space of all measurable functions over G such that

$$\|u\|_{\alpha,G}^2=\int_G(1+|x|)^{2\alpha}|u(x)|^2dx<\infty.$$

The L_2 inner product over G will be denoted by

$$(u, v)_G=\int_G u(x)\overline{v(x)}dx,$$

which makes sense if $u\in L_{2,\alpha}(G)$ and $v\in L_{2,\beta}(G)$ with $\alpha+\beta\geqq 0$. When u and v are vector-valued, we also write

$$(u, v)_G=\sum_{j=1}^n (u_j, v_j)_G,$$

$$\|u\|_{\alpha,G}^2=\sum_{j=1}^n \|u_j\|_{\alpha,G}^2.$$

If $\alpha=0$ or if $G=\boldsymbol{R}^n$, the subscript α or G will be omitted.

$H_{2,loc}$ is the set of all locally L_2 functions on $\boldsymbol{R}^n$ with locally L_2 distribution derivatives up to the second order.

Let $\partial_j=\partial/\partial x_j$ $(j=1,\dots,n)$, $\mathrm{grad}\, u=(\partial_1 u,\dots,\partial_n u)$, $r=|x|$, $\tilde{x}=x/r=(\tilde{x}_1,\dots,\tilde{x}_n)$, $\langle x, y\rangle=\sum_{j=1}^n x_j y_j$ and $\partial_r u=\langle\tilde{x}, \mathrm{grad}\, u\rangle$. Then, for $z\in\boldsymbol{C}\backslash\boldsymbol{R}^+$ and $\lambda\in\boldsymbol{R}^+$($\boldsymbol{R}^+$ is the set of all positive real numbers), several differential operators are defined as follows:

$$D_j=\partial_j+ib_j,$$

$$Du=(D_1u,\dots, D_nu)=\mathrm{grad}\, u+ibu,$$

$$D_ru=\langle\tilde{x}, Du\rangle=\partial_ru+i\langle\tilde{x}, b\rangle u,$$

$$D_Tu=Du-\tilde{x}D_ru,$$

$$\mathscr{D}(z)u=Du+\left(\frac{n-1}{2r}-i\sqrt{z}\right)\tilde{x}u,$$

$$\mathscr{D}_r(z)u=\langle\tilde{x}, \mathscr{D}(z)u\rangle,$$

$$\mathscr{D}_\pm u=\mathscr{D}(\lambda\pm i0)u=Du+\frac{n-1}{2r}\tilde{x}u\mp i\sqrt{\lambda}\tilde{x}u,$$

$$\mathscr{D}_{\pm,r}u=\langle\tilde{x}, \mathscr{D}_\pm u\rangle=D_ru+\frac{n-1}{2r}u\mp i\sqrt{\lambda}u,$$

where $\sqrt{z}$ is the square root of z such that $\mathscr{I}m\sqrt{z}>0$ ($\mathscr{I}m$ = the imaginary part).

Proposition 2.1. *Suppose the assumption (b) is satisfied. Define*

$$A(x)=\sum_{j=1}^n\int_0^{|x|}b_j(s\tilde{x})\tilde{x}_j ds \quad (x\in\boldsymbol{R}^n).$$

Then A is a C^2 function and the following assertions hold:
(1) There exists a constant C such that $|D(e^{-iA})|=|b-\mathrm{grad}\, A|\leqq C(1+|x|)^{-1/2-\delta}$.
(2) If the assumption (b′) is satisfied, then (1) holds and, in addition, there exists a constant C such that $|L_0(e^{-iA})|\leqq C(1+|x|)^{-1-\delta}$, *where* $L_0=-\sum_{j=1}^n D_j^2=-\sum_{j=1}^n(\partial_j-ib_j)^2$.

Proof. Since $A(x)=\sum_{k=1}^n x_k\int_0^1 b_k(tx)dt$, $A(x)$ is a C^2 function and, differentiating this by x_j and integrating by parts, we obtain

$$\begin{aligned}\partial_jA(x)&=\int_0^1 b_j(tx)dt+\sum_{k=1}^n x_k\int_0^1(\partial_kb_j(tx)+B_{jk}(tx))tdt\\&=\int_0^1 b_j(tx)dt+\int_0^1 t\frac{d}{dt}(b_j(tx))dt+\int_0^1\Phi_j(tx)dt\\&=b_j(x)+\int_0^1\Phi_j(tx)\,dt,\end{aligned}$$

where $\Phi_j(x)=\sum_{k=1}^n x_kB_{jk}(x)$. From the assumption (b) [resp. (b′)], we obtain the following estimate:

$$|\Phi_j(x)|\leqq C(1+|x|)^{-1/2-\delta}$$

$$[|\partial_j\Phi_j(x)|=|\sum_k x_k(\partial_j B_{jk})(x)|\leqq C(1+|x|)^{-1-\delta}].$$

(Here we have used $B_{jj}=0$.) Hence we have

$$|\partial_j A-b_j|\leqq\int_0^1|\Phi_j(tx)|dt$$

$$\leqq C\int_0^1(1+t|x|)^{-1/2-\delta}dt$$

$$=C\frac{1}{|x|}\int_0^{|x|}(1+s)^{-1/2-\delta}ds\leqq C'(1+|x|)^{-1/2-\delta}$$

$$[|\partial_j(\partial_j A-b_j)|=|\int_0^1(\partial_j\Phi_j(tx))tdt|\leqq C'(1+|x|)^{-1-\delta}].$$

Consequently, we obtain the required inequalities by noting that

$$D(e^{-iA})=i(b-\text{grad }A)e^{-iA},$$

$$L_0(e^{-iA})=\sum_j\{(\partial_j A-b_j)^2-i\partial_j(\partial_j A-b_j)\}e^{-iA}.$$

Proposition 2.2. *Suppose that the assumption (b) is satisfied. Let λ be a positive number, ϕ a smooth function on Ω (the unit sphere in $\boldsymbol{R}^n$) and $\rho_0(r)$ a smooth function such that $\rho_0(r)=1$ ($r\geqq 1$) and $\rho_0(r)=0$ ($r\leqq 1/2$). For $x\in\boldsymbol{R}^n\left(r=|x|\text{ and }\omega=\frac{x}{r}\right)$, let $v_\phi(x,\lambda)$ be defined by*

$$v_\phi(x,\lambda)=C(\lambda)^{-1}r^{-(n-1)/2}e^{i(\sqrt{\lambda}r-A(r\omega))}\phi(\omega)\rho_0(r), \tag{2.1}$$

where A is as in Proposition 2.1 and $C(\lambda)=\pi^{-1/2}\lambda^{1/4}$. Then the following assertions hold with a constant C which can be taken uniformly bounded when λ varies in a compact set in $\boldsymbol{R}^+$:

(1) $$|v_\phi(r\omega)|\leqq C|\phi(\omega)|r^{-(n-1)/2},$$

$$|\mathscr{D}_+v_\phi(r\omega)|\leqq C(r^{-1/2-\delta}|\phi(\omega)|+r^{-1}|\text{grad}_\Omega\phi|)r^{-(n-1)/2},$$

where grad_Ω denotes the gradient on Ω: $\text{grad}_\Omega=r(\text{grad}-\tilde{x}\partial_r)$. Hence $v_\phi\in L_{2,-(1+\varepsilon)/2}$, $\mathscr{D}_+v_\phi\in L_2(E_1)$ for an arbitrary positive ε.
(2) *If the assumption (b′) is satisfied, then (1) holds and in addition*

$$|(L_0-\lambda)v_\phi(r\omega)|\leqq C(|\phi|+|\text{grad}_\Omega\phi|+|\Lambda\phi|)r^{-\frac{n-1}{2}-1-\delta},$$

where Λ is the Laplace-Beltrami operator on Ω. Hence $(L_0-\lambda)v_\phi\in L_{2,(1+\varepsilon)/2}$ for sufficiently small ε ($0<\varepsilon<\delta/2$).

Proof. First note that, by simple calculation,

$$\mathscr{D}_+(r^{-(n-1)/2}e^{i\sqrt{\lambda}r}\{\cdot\})=r^{-(n-1)/2}e^{i\sqrt{\lambda}r}D\{\cdot\}, \tag{2.2}$$

$$D(e^{-iA})=i(b-\text{grad }A)e^{-iA},$$

and, since A is so constructed that $\partial_r A = \langle \tilde{x}, b \rangle$,

$$(2.3) \qquad \langle \tilde{x}, D(e^{-iA}) \rangle = D_r(e^{-iA}) = 0.$$

The estimate for v_ϕ in (1) is immediate from (2.1). We have for $r \geqq 1$, using (2.2),

$$(2.4) \qquad C(\lambda)\mathscr{D}_+ v_\phi = r^{-(n-1)/2} e^{i\sqrt{\lambda}r} D(e^{-iA}\phi)$$

$$= r^{-(n-1)/2} e^{i\sqrt{\lambda}r} \left\{ D(e^{-iA})\phi + \frac{e^{-iA}}{r} \operatorname{grad}_\Omega \phi \right\}.$$

Hence, the estimate for $\mathscr{D}_+ v_\phi$ in (1) follows from (2.4) combined with (1) of Proposition 2.1.

Let $v_{0,\phi}$ denote the function $r^{-(n-1)/2} e^{i\sqrt{\lambda}r} \phi(\omega)$. Then we have by direct computation

$$\operatorname{grad} v_{0,\phi} = \left(-\frac{n-1}{2r} + i\sqrt{\lambda} \right) \tilde{x} v_{0,\phi} + r^{-(n-1)/2} e^{i\sqrt{\lambda}r} \frac{1}{r} \operatorname{grad}_\Omega \phi,$$

$$(\Delta + \lambda) v_{0,\phi} = -\frac{(n-1)(n-3)}{4r^2} v_{0,\phi} - r^{-(n-1)/2} e^{i\sqrt{\lambda}r} \frac{\Lambda \phi}{r^2}.$$

Hence, by noting $v_\phi = C(\lambda)^{-1} e^{-iA} v_{0,\phi}$ and by the use of (2.3), we obtain

$$(2.5) \qquad C(\lambda)(L_0 - \lambda) v_\phi = v_{0,\phi} L_0(e^{-iA}) - 2\langle \operatorname{grad} v_{0,\phi}, D(e^{-iA}) \rangle$$

$$- (\Delta v_{0,\phi} + \lambda v_{0,\phi}) e^{-iA}$$

$$= r^{-(n-1)/2} e^{i\sqrt{\lambda}r} \left[\left\{ e^{-iA} \frac{(n-1)(n-3)}{r^2} + L_0(e^{-iA}) \right\} \phi - \right.$$

$$\left. - \frac{2i}{r} \langle \operatorname{grad}_\Omega \phi, D(e^{-iA}) \rangle + \frac{\Lambda \phi}{r^2} e^{-iA} \right].$$

The required estimate in (2) follows from (2.5) combined with Proposition 2.1.

The following theorem which has been established in Ikebe-Saitō [6] is fundamental to this paper and will be stated without proof. In what follows ε will denote a positive constant smaller than $\delta/2$.

Theorem 2.3 (*Limiting absorption principle*). *Let the assumptions* (V) *and* (b) *be satisfied. Then the following assertions hold*:
(1) *Let K be a bounded domain in $\boldsymbol{C}\backslash\boldsymbol{R}$ such that $\bar{K}$, the closure of K in $\boldsymbol{C}$, does not intersect $(-\infty, 0]$, and let $f \in L_{2,(1+\varepsilon)/2}$. Then, if we denote the resolvent of H by $R(z)$ ($z \in K$), $u \equiv R(z) f \in L_2 \subset L_{2,-(1+\varepsilon)/2}$ satisfies the inequalities*

$$\|u\|_{-(1+\varepsilon)/2} \leqq C \|f\|_{(1+\varepsilon)/2},$$

$$\|\mathscr{D}(z) u\|_{(-1+\varepsilon)/2, E_1} \leqq C \|f\|_{(1+\varepsilon)/2},$$

where C is a domain constant independent of f.
(2) *$R(z) f$ is continuous in $L_{2,-(1+\varepsilon)/2}$ with respect to $z \in \boldsymbol{C}\backslash\boldsymbol{R}$ and $f \in L_{2,(1+\varepsilon)/2}$, and for any $\lambda > 0$, the limit*

$$R(\lambda \pm i0)f = \underset{\substack{z\to\lambda \\ \pm \mathscr{I}_m z>0}}{\text{s-lim}}\ R(z)f$$

exists in $L_{2,-(1+\varepsilon)/2}$ *in such a way that* $R(z)f$ *can be extended to a continuous map from* $\boldsymbol{C}^{\pm} \cup \boldsymbol{R}^{+}$ $(\boldsymbol{C}^{\pm}=\{z\in\boldsymbol{C} \mid \pm\mathscr{I}_m z>0\})$ *to* $L_{2,-(1+\varepsilon)/2}$. *The inequlaities in* (1) *are satisfied with* $u=R(\lambda\pm i0)f$ *and* $\mathscr{D}(z)$ *replaced by* $\mathscr{D}(\lambda\pm i0)$ *when* $\lambda\in\bar{K}$.

(3) *Given* $\lambda>0$ *and* $f\in L_{2,(1+\varepsilon)/2}$, $R(\lambda\pm i0)f$ *in* (2) *solves the following problem uniquely*:

$$\text{(2.6)}\qquad \begin{cases} Lu-\lambda u=f, \quad u\in H_{2,loc}\cap L_{2,-(1+\varepsilon)/2}, \\ \mathscr{D}(\lambda\pm i0)u\in L_{2,(-1+\varepsilon)/2}(E_1). \end{cases}$$

(4) *For* $f, g\in L_{2,(1+\varepsilon)/2}$ *and any Borel set* B *in* $\boldsymbol{R}^{+}$, *we have*

$$\begin{aligned}(E(B)f, g) &= \frac{1}{2\pi i}\int_B (R(\lambda+i0)f-R(\lambda-i0)f, g)d\lambda \\ &= \frac{1}{2\pi i}\int_B \{(R(\lambda+i0)f, g)-(f, R(\lambda+i0)g)\}d\lambda,\end{aligned}$$

where E *is the spectral measure associated with* H. *The part of* H *in* $E((0,\infty))$ *is absolutely continuous.*

Remark. When $\mathscr{D}_+u\in L_{2,(-1+\varepsilon)/2}[\mathscr{D}_-u\in L_{2,(-1+\varepsilon)/2}]$, u is said to satisfy the outgoing [resp. incoming] radiation condition, and $R(\lambda+i0)f$ $[R(\lambda-i0)f]$ is called the outgoing [resp. incoming] solution for the equation $Lu-\lambda u=f$. For example, if the assumptions (V) and (b') are fulfilled and if g_ϕ denotes $(L-\lambda)v_\phi$, where v_ϕ is a function as in Proposition 2.2, v_ϕ is the outgoing solution for $(L-\lambda)v_\phi=g_\phi$ (thus $R(\lambda+i0)g_\phi=v_\phi$), since $v_\phi\in L_{2,-(1+\varepsilon)/2}$, $\mathscr{D}_+v_\phi\in L_2(E_1)$ and $g_\phi=(L_0-\lambda)v_\phi+Vv_\phi\in L_{2,(1+\varepsilon)/2}$ as noticed in Proposition 2.2.

3. Asymptotic behavior of outgoing solutions

As has been seen in (4) of Theorem 2.3, the following quantity is important in investigating the spectral representation for H:

$$\text{(3.1)}\qquad (R(\lambda+i0)f-R(\lambda-i0)f, f) \quad \text{for} \quad f\in L_{2,(1+\varepsilon)/2}.$$

We shall utilize the following Green's formula for computing (3.1):

$$\text{(3.2)}\qquad \int_{B_r}(u\bar{f}-f\bar{u})dx=\int_{S_r}[(\mathscr{D}_{+,r}u)\bar{u}-u(\overline{\mathscr{D}_{+,r}u})]dS+2i\sqrt{\lambda}\int_{S_r}|u|^2dS,$$

where $u=R(\lambda+i0)f$. The left-hand side of (3.2) converges to (3.1) as $r\to\infty$. As remarked after Theorem 2.3, v_ϕ in Proposition 2.2 is the outgoing solution for $(L-\lambda)v_\phi=g_\phi$ under the assumptions (V) and (b'), and, by letting $f=g_\phi$ and $r\to\infty$ in (3.2), we have

$$\text{(3.3)}\qquad \frac{1}{2\pi i}(R(\lambda+i0)g_\phi-R(\lambda-i0)g_\phi, g_\phi)=\lim_{r\to\infty}\frac{\sqrt{\lambda}}{\pi}\int_{S_r}|v_\phi|^2dS=\|\phi\|^2_{L_2(\Omega)},$$

where $\| \ \|_{L_2(\Omega)}$ is the norm of $L_2(\Omega)$ (=the L_2 space over the unit sphere Ω). In the next lemma and the succeeding propositions, we are going to prove that $R(\lambda+i0)f$ behaves like v_ϕ near infinity and the analogue of (3.3) holds for f satisfying an appropriate condition under the assumptions (V) and (b).

Definition 3.1. Let the operator $\mathscr{F}(\lambda, r)$: $L_{2,(1+\varepsilon)/2}\to L_2(\Omega)$ be defined by

$$\mathscr{F}(\lambda, r)f(\omega)=C(\lambda)r^{(n-1)/2}e^{-i\sqrt{\lambda}r+iA(r\omega)}R(\lambda+i0)f(r\omega),$$

where $\omega\in\Omega$, $f\in L_{2,(1+\varepsilon)/2}$ and $C(\lambda)=\pi^{-1/2}\lambda^{1/4}$.

Lemma 3.2. *Suppose that the assumptions* (V) *and* (b) *are satisfied*, $f\in L_{2,(1+\varepsilon)/2}$ *and* $\mathscr{D}(\lambda+i0)R(\lambda+i0)f\in L_2(E_1)$. *Then there exists the following strong limit in* $L_2(\Omega)$:

$$\mathscr{F}(\lambda; f)=\lim_{r\to\infty}\mathscr{F}(\lambda, r)f.$$

For the proof of this lemma, we need some formulae and propositions. To begin with, we consider:

$$\Gamma_{\pm,r}(u, v)\equiv\int_{S_r}(\mathscr{D}_{\pm,r}u)\bar{v}dS, \tag{3.4}$$

where $u, v\in H_{2,loc}$. First, we obtain

$$\begin{aligned}\frac{d}{dr}\Gamma_{\pm,r}(u, v) &= \frac{d}{dr}(r^{(n-1)/2}e^{\pm i\sqrt{\lambda}r+iA}(\mathscr{D}_{\pm,r}u),\ r^{(n-1)/2}e^{\pm i\sqrt{\lambda}r+iA}v)_{L_2(\Omega)}\\ &=\int_{S_r}(\mathscr{D}_{\mp,r}\mathscr{D}_{\pm,r}u)\bar{v}dS+\int_{S_r}(\mathscr{D}_{\pm,r}u)(\overline{\mathscr{D}_{\mp,r}v})dS,\end{aligned} \tag{3.5}$$

where we have used

$$\partial_r(r^{(n-1)/2}e^{\pm i\sqrt{\lambda}r+iA}u)=r^{(n-1)/2}e^{\pm i\sqrt{\lambda}r+iA}\mathscr{D}_{\mp,r}u.$$

Moreover, we have

$$\mathscr{D}_{\mp,r}\mathscr{D}_{\pm,r}=D_r^2+\frac{n-1}{r}D_r+\frac{(n-1)(n-3)}{4r^2}+\lambda, \tag{3.6}$$

$$\begin{aligned}\int_{S_r}\Big(L_0u+D_r^2u+\frac{n-1}{r}D_ru\Big)\bar{v}dS &=\int_{S_r}\langle Du, \overline{Dv}\rangle dS-\int_{S_r}D_ru\overline{D_rv}dS\\ &=\int_{S_r}\langle D_Tu, \overline{D_Tv}\rangle dS,\end{aligned} \tag{3.7}$$

where $L_0=-\sum_j D_j^2$: (3.6) is obtained by straightforward calculation, and (3.7) is obtained by differentiating Green's formula

$$(L_0 u, v)_{B_r} = (Du, Dv)_{B_r} - \int_{S_r} (D_r u)\bar{v}\, dS$$

with respect to r and by noting that $D = D_T + \tilde{x} D_r$ is an orthogonal sum decomposition. By the use of (3.5), (3.6) and (3.7) we obtain

$$\frac{d}{dr}\Gamma_{\pm,r}(u, v) = \int_{S_r} \langle D_T u, \overline{D_T v}\rangle dS + \int_{S_r} (\tilde{V} u - (L-\lambda)u)\bar{v}\, dS \tag{3.8}$$
$$+ \int_{S_r} (\mathscr{D}_{\pm,r} u)(\overline{\mathscr{D}_{\mp,r} v})\, dS,$$

where $\tilde{V} = V + \frac{(n-1)(n-3)}{4r^2}$. Further, taking the upper side of the double sign of (3.8) and using the relation $\mathscr{D}_{-,r} = \mathscr{D}_{+,r} + 2\sqrt{\lambda} i$ and the orthogonal sum decomposition $\mathscr{D}_+ = D_T + \tilde{x}\mathscr{D}_{+,r}$, we have

$$\left(\frac{d}{dr} + 2\sqrt{\lambda} i\right)\Gamma_{+,r}(u, v) = \int_{S_r} \langle \mathscr{D}_+ u, \overline{\mathscr{D}_+ v}\rangle dS + \int_{S_r} (\tilde{V} u - (L-\lambda)u)\bar{v}\, dS. \tag{3.9}$$

Proposition 3.3. *Let* $u \in L_{2,-(1+\varepsilon)/2}$ *and* $\mathscr{D}_+ u \in L_{2,(-1+\eta)/2}(E_1)$. *Then there exists a sequence* $\{r_m\}$ *of positive numbers diverging to infinity as* $m \to \infty$ *such that*

$$r_m^{-\varepsilon} \int_{S_{r_m}} |u|^2 dS \longrightarrow 0 \quad as \quad m \longrightarrow \infty, \tag{3.10}$$

$$r_m^{\eta} \int_{S_{r_m}} |\mathscr{D}_+ u|^2 dS \longrightarrow 0 \quad as \quad m \longrightarrow \infty. \tag{3.11}$$

Proof. It is not difficult to verify that, for an integrable function g over $\boldsymbol{R}^n$, there exists a sequence $\{r_m\}$ diverging to infinity as $m \to \infty$ such that

$$r_m \int_{S_{r_m}} |g(x)| dS \longrightarrow 0.$$

Considering this fact, we have the assertion of the proposition.

Proposition 3.4. *Let the assumption of lemma* 3.2 *be satisfied. Let* $v \in L_{2,-(1+\varepsilon)/2} \cap H_{2,loc}$ *with* $\mathscr{D}_+ v \in L_2(E_1)$. *Then we have*

$$\int_{S_r} (\mathscr{D}_{+,r} u)\bar{v}\, dS \longrightarrow 0 \quad as \quad r \longrightarrow \infty \qquad (u = R(\lambda + i0)f). \tag{3.12}$$

In particular, we have by (3.12) *with* $v = u$, (3.2) *and Theorem* 2.3

$$\lim_{r \to \infty} \|\mathscr{F}(\lambda, r)f\|^2_{L_2(\Omega)} = \frac{1}{2\pi i}(R(\lambda + i0)f - R(\lambda - i0)f, f). \tag{3.13}$$

Proof. Let $F(r)$ be the right-hand side of (3.9). Then, $F(r)$ is integrable over $(1, \infty)$ because by assumption $\mathscr{D}_+ u, \mathscr{D}_+ v \in L_2(E_1)$, $\tilde{V} u$, $(L-\lambda)u = f \in L_{2,(1+\varepsilon)/2}$, $v \in L_{2,-(1+\varepsilon)/2}$. Since we can rewrite (3.9) as

$$\frac{d}{dr}(e^{2i\sqrt{\lambda} r}\Gamma_{+,r}) = e^{2i\sqrt{\lambda} r}F(r),$$

we have

$$e^{2i\sqrt{\lambda}r}\Gamma_{+,r}=-\int_r^\infty e^{2i\sqrt{\lambda}r}F(r)dr. \tag{3.14}$$

Here we have used the existence of a sequence $\{r_m\}$ such that $r_m\to\infty$ and $\Gamma_{+,r_m}\to 0$ as $m\to\infty$, which can be verified by an argument similar to the proof of Proposition 3.3, since $r^{-(1+\varepsilon)/2}(\mathscr{D}_{+,r}u)\bar{v}$ is integrable over E_1 and $\Gamma_{+,r}=\int_{S_r}(\mathscr{D}_{+,r}u)\bar{v}dS$ ((3.4)). (3.12) follows from (3.4) and (3.14).

Proposition 3.5. *Under the assumption of Lemma 3.2, there exists a function $\alpha(r)$ such that $\alpha(r)\downarrow 0$ as $r\to\infty$, and for all r and s satisfying $1\leqq r\leqq s$ and for $\phi\in H_1(\Omega)$ (= the set of L_2 functions over Ω with L_2 distribution derivatives up to the first order), the following inequality holds*:

$$|(w_+(r),\ \phi)_{L_2(\Omega)}|+|(w_-(r)-w_-(s),\ \phi)_{L_2(\Omega)}|$$
$$\leqq\alpha(r)(\|\phi\|_{L_2(\Omega)}+r^{-1/2}\|\mathrm{grad}_\Omega\,\phi\|_{L_2(\Omega)}),$$

where $w_\pm(r)\in L_2(\Omega)$ are defined by

$$w_\pm(r)(\,\cdot\,)=C(\lambda)r^{(n-1)/2}e^{-i\sqrt{\lambda}r+iA(r\cdot)}\mathscr{D}_{\pm,r}u(r\cdot)$$
$$(u=R(\lambda+i0)f,\quad C(\lambda)=\pi^{-1/2}\lambda^{1/4}).$$

Proof. Let $\phi\in C^\infty(\Omega)$, v_ϕ be as in Proposition 2.2 and $F(r)$ be the right-hand side of (3.9) with v replaced by v_ϕ, i.e.,

$$F(r)=\int_{S_r}\langle\mathscr{D}_+u,\ \mathscr{D}_+v_\phi\rangle dS+\int_{S_r}(\tilde{V}u-f)\overline{v_\phi}dS. \tag{3.15}$$

Then, since $v_\phi\in L_{2,-(1+\varepsilon)/2}\cap H_{2,loc}$, $\mathscr{D}_+v_\phi\in L_2(E_1)$ as noted in Proposition 2.2, the argument in the proof of Proposition 3.4 is applicable to the case $v=v_\phi$. That is, $F(r)$ is integrable over $(1,\infty)$, and we have

$$e^{2i\sqrt{\lambda}r}\Gamma_{+,r}(u,\ v_\phi)=-\int_r^\infty e^{2i\sqrt{\lambda}r}F(r)dr. \tag{3.16}$$

Moreover, taking the lower side of the double sign of (3.8) and replacing v by v_ϕ, we have

$$\frac{d}{dr}\Gamma_{-,r}(u,\ v_\phi)=\int_{S_r}\langle D_Tu,\ D_Tv_\phi\rangle dS+\int_{S_r}(\mathscr{D}_{-,r}u)(\overline{\mathscr{D}_{+,r}v_\phi})dS \tag{3.17}$$
$$+\int_{S_r}(\tilde{V}u-f)\overline{v_\phi}dS.$$

The right-hand side of (3.17) coincides with $F(r)$ because we have $\mathscr{D}_{+,r}v_\phi=0$ by straightforward calculation and the orthogonal sum decomposition $\mathscr{D}_+=D_T+\tilde{x}\mathscr{D}_{+,r}$. Hence, we obtain by integrating (3.17),

$$\Gamma_{-,s}(u,\ v_\phi)-\Gamma_{-,r}(u,\ v_\phi)=\int_r^s F(r)dr\qquad(1\leqq r\leqq s). \tag{3.18}$$

On the other hand, (3.4) and the definition of $w_{\pm}(r)$ and v_ϕ leads to

$$(3.19) \qquad (w_{\pm}(r),\ \phi)_{L_2(\Omega)} = C(\lambda)^2 \Gamma_{\pm,r}(u,\ v_\phi),$$

from which, in view of (3.16) and (3.18), the following estimate can be obtained for $1 \leqq r \leqq s$:

$$\begin{aligned}
&|(w_+(r),\ \phi)_{L_2(\Omega)}| + |(w_-(r) - w_-(s),\ \phi)_{L_2(\Omega)}| \\
&\leqq C\int_r^\infty |F(r)|dr \\
&\leqq C\int_r^\infty dr\int_{S_r} |\langle \mathscr{D}_+u,\ \overline{\mathscr{D}_+v_\phi}\rangle| dS + C\int_r^\infty dr\int_{S_r} |\tilde{V}u\overline{v_\phi}|dS + C\int_r^\infty dr\int_{S_r} |f\overline{v_\phi}|dS \\
&\equiv \qquad I_1 \qquad + \qquad I_2 \qquad + \qquad I_3,
\end{aligned}$$

where C is a constant depending only on λ. According to Schwarz' inequality and Proposition 2.2 (1), we have

$$\begin{aligned}
I_1 &\leqq C\int_r^\infty r^{(n-1)/2}\|\mathscr{D}_+u\|_{L_2(\Omega)}(r^{-1/2-\delta}\|\phi\|_{L_2(\Omega)} + r^{-1}\|\mathrm{grad}_\Omega\,\phi\|_{L_2(\Omega)})dr \\
&\leqq C\sqrt{\int_r^\infty r^{n-1}\|\mathscr{D}_+u\|^2_{L_2(\Omega)}dr} \times \\
&\qquad \times\left(\sqrt{\int_r^\infty r^{-1-2\delta}dr}\,\|\phi\|_{L_2(\Omega)} + \sqrt{\int_r^\infty r^{-2}dr}\,\|\mathrm{grad}_\Omega\,\phi\|_{L_2(\Omega)}\right) \\
&= C\|\mathscr{D}_+u\|_{E_r}\left(\frac{r^{-\delta}}{\sqrt{2\delta}}\|\phi\|_{L_2(\Omega)} + r^{-1/2}\|\mathrm{grad}_\Omega\,\phi\|_{L_2(\Omega)}\right), \\
I_2 &\leqq C\int_r^\infty r^{(n-1)/2}r^{-1-\delta}\|u\|_{L_2(\Omega)}dr\|\phi\|_{L_2(\Omega)} \\
&\leqq Cr^{-\varepsilon/2}\|u\|_{-(1+\varepsilon)/2}\|\phi\|_{L_2(\Omega)}, \\
I_3 &\leqq C\int_r^\infty r^{(n-1)/2}\|f\|_{L_2(\Omega)}dr\|\phi\|_{L_2(\Omega)} \\
&\leqq Cr^{-\varepsilon/2}\|f\|_{(1+\varepsilon)/2}\|\phi\|_{L_2(\Omega)}.
\end{aligned}$$

Then, if we put $\alpha(r) = C\|\mathscr{D}_+u\|_{E_r} + Cr^{-\varepsilon/2}(\|u\|_{-(1+\varepsilon)/2} + \|f\|_{(1+\varepsilon)/2})$, the required inequality holds for $\phi \in C^\infty(\Omega)$, and $\alpha(r) \downarrow 0$ as $r \to \infty$ since $\mathscr{D}_+u = \mathscr{D}_+R(\lambda + i0)f \in L_2(E_1)$ by assumption. Finally, the inequality is obtained for $\phi \in H_1(\Omega)$ by approximating it by smooth functions.

Proposition 3.6. *Let the assumptions* (V) *and* (b) *be satisfied and* $f \in L_{2,(1+\varepsilon)/2}$. *Then,* $\{r_m^{-1/2}\,\mathrm{grad}_\Omega\,\mathscr{F}(\lambda,\ r_m)f\}_m$ *is bounded in* $L_2(\Omega)$ *for any sequence* $\{r_m\}$ *which satisfies* (3.10), (3.11) *with* $u = R(\lambda + i0)f$ *and* $\eta = 1$.

Proof. By straightforward computation, we have

$$\frac{1}{C(\lambda)}\operatorname{grad}_\Omega \mathscr{F}(\lambda, r)f$$

$$= r^{(n+1)/2}e^{-i\sqrt{\lambda}r}\{\operatorname{grad}(e^{iA}u) - \tilde{x}\partial_r(e^{iA}u)\}$$

$$= r^{(n+1)/2}e^{-i\sqrt{\lambda}r+iA}\{Du - i(b - \operatorname{grad} A)u - \tilde{x}D_r u\}$$

$$= r^{(n+1)/2}e^{-i\sqrt{\lambda}r+iA}\{D_T u - i(b - \operatorname{grad} A)u\}.$$

Hence, by using Proposition 2.1 (1), we have

$$\|\operatorname{grad}_\Omega \mathscr{F}(\lambda, r_m)f\|_{L_2(\Omega)}$$

$$\leqq C\{r_m^{(n+1)/2}\|D_T u(r_m\cdot)\|_{L_2(\Omega)} + r_m^{n/2-\delta}\|u(r_m\cdot)\|_{L_2(\Omega)}\}$$

$$\leqq C r_m^{1/2}\left\{\sqrt{r_m\int_{S_{r_m}}|\mathscr{D}_+u|^2dS} + \sqrt{r_m^{-2\delta}\int_{S_{r_m}}|u|^2dS}\right\}.$$

Consequently, since ε denotes a positive constant smaller than $\delta/2$, we have the assertion of the proposition.

Proof of Lemma 3.2. Seeing that $\mathscr{F}(\lambda, r)f = (2i\sqrt{\lambda})^{-1}(w_-(r) - w_+(r))$, we have by Proposition 3.5

$$\text{(3.20)}\qquad |(\mathscr{F}(\lambda, r)f - \mathscr{F}(\lambda, s)f, \phi)_{L_2(\Omega)}|$$
$$\leqq \lambda^{-1/2}\alpha(r)(\|\phi\|_{L_2(\Omega)} + r^{-1/2}\|\operatorname{grad}_\Omega \phi\|_{L_2(\Omega)}).$$

for $\phi \in H_1(\Omega)$ and r, s $(1 \leqq r \leqq s)$. Consequently, $(\mathscr{F}(\lambda, r)f, \phi)_{L_2(\Omega)}$ is convergent when $r\to\infty$. Since $\{\mathscr{F}(\lambda, r)f\}$ is bounded with respect to r (Proposition 3.4) and $H_1(\Omega)$ is dense in $L_2(\Omega)$, we have the weak convergence in $L_2(\Omega)$ of $\{\mathscr{F}(\lambda, r)f\}$. Let $\mathscr{F}(\lambda; f)$ denote this weak limit. Then, letting $s\to\infty$ in (3.20), we have

$$\text{(3.21)}\qquad |(\mathscr{F}(\lambda, r)f - \mathscr{F}(\lambda; f), \phi)_{L_2(\Omega)}|$$
$$\leqq \lambda^{-1/2}\alpha(r)(\|\phi\|_{L_2(\Omega)} + r^{-1/2}\|\operatorname{grad}_\Omega \phi\|_{L_2(\Omega)}).$$

Putting $r = r_m$ and $\phi = \mathscr{F}(\lambda, r_m)f$ in (3.21) and using the boundedness of $\{\mathscr{F}(\lambda, r)f\}$ (Proposition 3.4) and Proposition 3.6 for a sequence $\{r_m\}$ which satifises (3.10) and (3.11) with $u = R(\lambda + i0)f$ and $\eta = 1$ (the existence of such a sequence is guaranteed by Proposition 3.3 since $u \in L_{2,-(1+\varepsilon)/2}$ by Theorem 2.3 and $\mathscr{D}_+u \in L_2(E_1)$ by assumption), one can see that there exists a constant M independent of m such that the following inequality holds:

$$|(\mathscr{F}(\lambda, r_m)f - \mathscr{F}(\lambda; f), \mathscr{F}(\lambda, r_m)f)| \leqq \alpha(r_m)M.$$

According to this and the weak convergence of $\{\mathscr{F}(\lambda, r)f\}$ to $\mathscr{F}(\lambda; f)$, we have

$$\lim_{m\to\infty}\|\mathscr{F}(\lambda, r_m)f\|^2_{L_2(\Omega)} = \|\mathscr{F}(\lambda; f)\|^2_{L_2(\Omega)},$$

for some sequence $\{r_m\}$. Since $\|\mathscr{F}(\lambda, r)f\|_{L_2(\Omega)}$ converges when $r\to\infty$ by Proposition 3.4, we have further

$$\lim_{r\to\infty}\|\mathscr{F}(\lambda, r)f\|^2_{L_2(\Omega)} = \|\mathscr{F}(\lambda; f)\|^2_{L_2(\Omega)}.$$

From this and the weak convergence, the strong convergence of $\mathscr{F}(\lambda, r)f$ to $\mathscr{F}(\lambda; f)$ follows.

Lemma 3.7. *Suppose that the assumptions* (V') *and* (b) *are satisfied. Let* $f \in L_{2,1} \subset L_{2,(1+\varepsilon)/2}$. *Then, in addition to all the statements of Theorem* 2.3, *we* have

$$\mathscr{D}(\lambda \pm i0) R(\lambda \pm i0) f \in L_2(E_1),$$

$$\|\mathscr{D}(\lambda \pm i0) R(\lambda \pm i0) f\|_{E_1} \leqq C\|f\|_1,$$

where C *is a constant independent of* f *and remains bounded when* λ *varies in a compact set of* $\boldsymbol{R}^+$.

For the proof, we need the following proposition which is a version of Lemma 2.1 in Ikebe-Saitō [6].

Proposition 3.8. *Assume that* V *is a bounded measurable function and* b_j *are continuously differentiable. Then, for any* $u \in H_{2,loc}$ *and* $R>0$, *the following inequality holds with a positive constant* C *independent of* u *and* R:

$$\|Du\|_{B_R}^2 \leqq C(\|u\|_{B_{2R}}^2 + \|Lu\|_{B_{2R}}^2). \tag{3.22}$$

Proof. For ψ, a real-valued smooth function with compact support, we have by partial integration

$$\begin{aligned}\|D(\psi u)\|^2 &= -(\psi u, \sum_j D_j D_j(\psi u)) \\ &= -(\psi u, \psi(\sum_j D_j D_j u)) - 2(\psi u, \langle \operatorname{grad} \psi, Du\rangle) - (\psi u, (\Delta\psi)u) \\ &= (\psi u, \psi Lu) - (\psi u, \psi V u) - 2(u \operatorname{grad} \psi, D(\psi u)) + \\ &\qquad + 2\|u \operatorname{grad} \psi\|^2 - (\psi u, (\Delta \psi) u).\end{aligned}$$

Hence, noting that $2|(u \operatorname{grad} \psi, D(\psi u))| \leqq 2\|u \operatorname{grad} \psi\|^2 + \frac{1}{2}\|D(\psi u)\|^2$ by the use of Schwarz' inequality, we have

$$\frac{1}{2}\|D(\psi u)\|^2 \leqq |(\psi u, \psi L u)| + |(\psi u, \psi V u)| + 4\|u \operatorname{grad} \psi\|^2 + |(\psi u, (\Delta\psi) u)|.$$

(3.22) is obtained from this inequality by taking $\psi(x) = \rho(x/R)$ where ρ is a smooth function on $\boldsymbol{R}^n$ such that $\rho(x) = 1(|x| \leqq 1)$ and $\rho(x) = 0$ $(|x| \geqq 2)$.

Proof of Lemma 3.7. Let $\{z_m\}$ be a sequence of complex numbers such that $\mathscr{I}_m z_m > 0$ and $z_m \to \lambda$ as $m \to \infty$. Then for $u_m \equiv R(z_m) f \in L_2$ we have by Theorem 2.3

$$\|u_m\|_{-(1+\varepsilon)/2} \leqq C\|f\|_{(1+\varepsilon)/2} \leqq C\|f\|_1, \tag{3.23}$$

$$u_m \longrightarrow u = R(\lambda + i0) f \quad \text{strongly in} \quad L_{2,-(1+\varepsilon)/2}. \tag{3.24}$$

Since $f = (L - z_m) u_m$, applying Proposition 3.8 with $u = u_m$, we have

$$\int_{1<|x|<R} |\mathscr{D}(z_m)u_m|^2 dx \leqq C\{\|f\|^2_{B_{2R}} + \|u_m\|^2_{B_{2R}}\}, \tag{3.25}$$

with C independent of R, f and m. Since $u_m \in L_2$ and $f \in L_{2,1}$, $\mathscr{D}(z_m)u_m \in L_2(E_1)$ by (3.25). Let ρ be a smooth function of $r=|x|$ on $\boldsymbol{R}^n$ such that $\rho(1)=0$ and $\rho(r)=r$ $(r \geqq 2)$. Then, by multiplying $(L-z_m)u_m=f$ by $\rho\mathscr{D}_r(z_m)u_m$ and integrating by parts, the following equality can be obtained:

$$\int_{E_1}\Big[\Big\{(\mathscr{I}m\sqrt{z_m})\rho+\frac{1}{2}\frac{\partial\rho}{\partial r}\Big\}|\mathscr{D}(z_m)u_m|^2+$$

$$+\Big(\frac{\rho}{r}-\frac{\partial\rho}{\partial r}\Big)(|\mathscr{D}(z_m)u_m|^2-|\mathscr{D}_r(z_m)u_m|^2)\Big]dx$$

$$=-\mathscr{R}e\Big[\int_{E_1}\rho\tilde{V}u_m\overline{\mathscr{D}_r(z_m)u_m}dx\Big]+\mathscr{R}e\Big[\int_{E_1}\rho f\overline{\mathscr{D}_r(z_m)u_m}dx\Big]-$$

$$-\mathscr{I}m\Big[\int_{E_1}\sum_{j,k=1}^{n}\rho B_{jk}\mathscr{D}_j(z_m)u_m\tilde{x}_k\overline{u_m}dx\Big].$$

(For the details of the computation see Ikebe-Saitō [6].) From this equality, noting that $\mathscr{I}m\sqrt{z_m}>0$, $\frac{\partial\rho}{\partial r}=1$ and $\frac{\rho}{r}-\frac{\partial\rho}{\partial r}=0$ if $r\geqq 2$, $|\mathscr{D}_r(z_m)u_m|^2 \leqq |\mathscr{D}(z_m)u_m|^2$, $\rho\tilde{V}=O(|x|^{-1/2-\delta})$ and $\rho B_{jk}=O(|x|^{-1/2-\delta})$, we have by the use of Schwarz' inequality

$$\|\mathscr{D}(z_m)u_m\|^2_{E_1}$$

$$=\int_{1<|x|<2}|\mathscr{D}(z_m)u_m|^2dx+\int_{E_2}|\mathscr{D}(z_m)u_m|^2dx$$

$$\leqq\int_{1<|x|<2}|\mathscr{D}(z_m)u_m|^2dx+2\int_{E_2}\Big((\mathscr{I}m\sqrt{z_m})\rho+\frac{1}{2}\frac{\partial\rho}{\partial r}\Big)|\mathscr{D}(z_m)u_m|^2dx$$

$$\leqq C\int_{1<|x|<2}|\mathscr{D}(z_m)u_m|^2dx+C\|(1+|x|)^{-1/2-\delta}u_m\|\,\|\mathscr{D}(z_m)u_m\|_{E_1}+$$

$$+C\|(1+|x|)f\|\,\|\mathscr{D}(z_m)u_m\|_{E_1}$$

$$\leqq C\int_{1<|x|<2}|\mathscr{D}(z_m)u_m|^2dx+C'(\|u_m\|^2_{-(1+\varepsilon)/2}+\|f\|^2_1)+$$

$$+\frac{1}{2}\|\mathscr{D}(z_m)u_m\|^2_{E_1},$$

where C, C' are constants depending only on n, ρ and C_0 in the assumptions (V') and (b). Hence, applying (3.25) with $R=2$ and using (3.23), we have

$$\|\mathscr{D}(z_m)u_m\|^2_{E_1}\leqq C''\|f\|^2_1.$$

Consequently, there exists a subsequence $\{m_j\}$ such that $\{\mathscr{D}(z_{m_j})u_{m_j}\}$ is weakly convergent in $L_2(E_1)$ to some $w\in L_2(E_1)$. On the other hand, according to (3.24), $\{\mathscr{D}(z_m)u_m\}$ converges to $\mathscr{D}_+u$ in the distributional sense. Therefore, $\mathscr{D}_+u$ coincides with w and we have

$$\|\mathscr{D}_+u\|_{E_1}\leqq\liminf_{j\to\infty}\|\mathscr{D}(z_{m_j})u_{m_j}\|_{E_1}\leqq C''\|f\|_1.$$

We have thus concluded the proof of Lemma 3.7.

We conclude this section with a theorem which can be obtained by combining Lemma 3.7 with Lemma 3.2.

Theorem 3.9. *Suppose that the assumptions* (V') *and* (b) *are satisfied. Then, there exists an operator* $\mathscr{F}(\lambda)$: $L_{2,1} \to L_2(\Omega)$ ($\lambda > 0$) *such that the following assertions hold*:

(1) $$\mathscr{F}(\lambda)f = \operatorname*{s-lim}_{r\to\infty} \mathscr{F}(\lambda, r)f \quad \text{in} \quad L_2(\Omega) \quad (f \in L_{2,1}).$$

(2) $$\|\mathscr{F}(\lambda)f\|^2_{L_2(\Omega)} = \frac{1}{2\pi i}(R(\lambda+i0)f - R(\lambda-i0)f, f) \quad (f \in L_{2,1}).$$

Proof. Under the assumptions (V') and (b), $f \in L_{2,1}$ satisfies the assumption of Lemma 3.2 for every positive number λ according to Lemma 3.7. Hence, we can define $\mathscr{F}(\lambda)f$ by $\mathscr{F}(\lambda; f) = \lim_{r\to\infty} \mathscr{F}(\lambda, r)f$ of Lemma 3.2 which obviously satisfies (1). (2) follows from (1) and (3.13) of Proposition 3.4.

4. Spectral representation for H

In this section, the spectral representation for H is obtained by means of the next lemma, a version of Theorem 3.9, where we impose a stronger condition on b but, instead, relax the condition on V.

Lemma 4.1. *Suppose that the assumptions* (V) *and* (b') *are satisfied. Then there exists a bounded operator* $\mathscr{F}(\lambda)$: $L_{2,(1+\varepsilon)/2} \to L_2(\Omega)$ ($\lambda > 0$) *such that the following assertions hold*:

(1) $$\mathscr{F}(\lambda)f = \operatorname*{s-lim}_{m\to\infty} \mathscr{F}(\lambda, r_m)f \qquad (f \in L_{2,(1+\varepsilon)/2}),$$

where $\{r_m\}$ *is any sequence satisfying*

$$(4.1)\qquad \begin{cases} r_m \longrightarrow \infty, \\ r_m^{-\varepsilon} \displaystyle\int_{S_{r_m}} |R(\lambda+i0)f|^2 dS \longrightarrow 0, \\ r_m^{\varepsilon} \displaystyle\int_{S_{r_m}} |\mathscr{D}(\lambda+i0)R(\lambda+i0)f|^2 dS \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \end{cases}$$

(2) $$\|\mathscr{F}(\lambda)f\|^2_{L_2(\Omega)} = \frac{1}{2\pi i}(R(\lambda+i0)f - R(\lambda-i0)f, f) \quad (f \in L_{2,(1+\varepsilon)/2}).$$

(3) $\mathscr{F}(\lambda)f$ is strongly continuous in λ for any $f \in L_{2,(1+\varepsilon)/2}$.

Proof. Let $f \in L_{2,(1+\varepsilon)/2}$. By Theorem 2.3, $u \equiv R(\lambda+i0)f \in L_{2,-(1+\varepsilon)/2}$ and $\mathscr{D}_+ u \in L_{2,(-1+\varepsilon)/2}$. Therefore, the existence of a sequence $\{r_m\}$ satisfying (4.1) is guaranteed by Proposition 3.3 with $\eta = \varepsilon$.

Let $\phi \in C^\infty(\Omega)$ and $v_\phi = v_\phi(x, \lambda)$ be defined as in Proposition 2.2. Then we have by Green's formula

$$(4.2)\qquad ((L-\lambda)u,\ v_\phi)_{B_r}-(u,\ (L-\lambda)v_\phi)_{B_r}$$
$$=\int_{S_r} u\overline{\mathscr{D}_{+,r}v_\phi}dS-\int_{S_r}\mathscr{D}_{+,r}u\overline{v_\phi}dS+2i\sqrt{\lambda}\int_{S_r}u\overline{v_\phi}dS.$$

Hence, since $(\mathscr{F}(\lambda,\ r)f,\ \phi)_{L_2(\Omega)}=C(\lambda)^2\int_{S_r}u\overline{v_\phi}dS$ $(C(\lambda)=\pi^{-1/2}\lambda^{1/4})$ in view of the definition of v_ϕ and $\mathscr{F}(\lambda,\ r)f$ ((2.1) and Definition 3.1), we obtain from (4.2) with $r=r_m$

$$(4.3)\qquad (\mathscr{F}(\lambda,\ r_m)f,\ \phi)_{L_2(\Omega)}=\frac{1}{2\pi i}\left\{\int_{S_{r_m}}u\overline{\mathscr{D}_{+,r}v_\phi}dS-\int_{S_{r_m}}\mathscr{D}_{+,r}u\overline{v_\phi}dS-\right.$$
$$\left.-((L-\lambda)u,\ v_\phi)_{B_{r_m}}+(u,\ (L-\lambda)v_\phi)_{B_{r_m}}\right\}.$$

From (4.1) and the estimates in Proposition 2.2, we have

$$(4.4)\qquad \int_{S_{r_m}}u\overline{\mathscr{D}_{+,r}v_\phi}dS\longrightarrow 0,\quad \int_{S_{r_m}}\mathscr{D}_{+,r}u\overline{v_\phi}dS\longrightarrow 0\quad\text{as}\quad m\longrightarrow\infty.$$

Because $u\in L_{2,-(1+\varepsilon)/2}$, $(L-\lambda)u=f\in L_{2,(1+\varepsilon)/2}$, $v_\phi\in L_{2,-(1+\varepsilon)/2}$ and $(L-\lambda)v_\phi\in L_{2,(1+\varepsilon)/2}$ (Proposition 2.2), we have in view of (4.4) the following equality by letting $m\to\infty$ in (4.3):

$$(4.5)\qquad \lim_{m\to\infty}(\mathscr{F}(\lambda,\ r_m)f,\ \phi)_{L_2(\Omega)}=\frac{1}{2\pi i}\{-(f,\ v_\phi)+(u,\ (L-\lambda)v_\phi)\}.$$

Similarly, by letting $r=r_m$ and $m\to\infty$ in (3.2), we have

$$(4.6)\qquad \lim_{m\to\infty}\|\mathscr{F}(\lambda,\ r_m)f\|^2_{L_2(\Omega)}=\frac{1}{2\pi i}(R(\lambda+i0)f-R(\lambda-i0)f,\ f).$$

Therefore, $\{\mathscr{F}(\lambda,\ r_m)f\}_m$ is bounded in $L_2(\Omega)$. Hence, since (4.5) holds for $\phi\in C^\infty(\Omega)$, which is a dense subspace of $L_2(\Omega)$, the weak convegence of $\{\mathscr{F}(\lambda,\ r_m)f\}$ in $L_2(\Omega)$ follows. Note that this weak limit is independent of the choice of the sequence $\{r_m\}$ because the right-hand side of (4.5) is so.

Let an operator $\mathscr{F}(\lambda)$: $L_{2,(1+\varepsilon)/2}\to L_2(\Omega)$ be defined by

$$(4.7)\qquad \mathscr{F}(\lambda)f=\text{w-}\lim_{m\to\infty}\mathscr{F}(\lambda,\ r_m)f$$

for $f\in L_{2,(1+\varepsilon)/2}$, where $\{r_m\}$ is any sequence satisfying (4.1). Then we have by (4.6) and Theorem 2.3 (2)

$$(4.8)\qquad \|\mathscr{F}(\lambda)f\|^2_{L_2(\Omega)}\leqq\frac{1}{2\pi i}(R(\lambda+i0)f-R(\lambda-i0)f,\ f)\leqq C\|f\|^2_{(1+\varepsilon)/2},$$

where C is a constant independent of f which can be taken uniformly bounded when λ varies in a compact set in $\boldsymbol{R}^+$. Hence $\mathscr{F}(\lambda)$ is a bounded operator: $L_{2,(1+\varepsilon)/2}\to L_2(\Omega)$.

Let us prove that (4.7) is a strong limit.

Let H_0 be the self-adjoint realization of $L_0=-\sum_{j=1}^n D_j^2$ in $\mathscr{H}=L_2(\boldsymbol{R}^n)$. Then the argument developed so far is applicable to H_0. That is, if $R_0(\lambda+i0)$ denotes the boundary value of the resolvent of H_0 and $\mathscr{F}_0(\lambda, r)$ is defined as $\mathscr{F}(\lambda, r)$ in Definition 3.1 with $R(\lambda+i0)$ replaced by $R_0(\lambda+i0)$, we have

$$(4.6') \qquad \lim_{m\to\infty} \|\mathscr{F}_0(\lambda, r_m)f\|^2_{L_2(\Omega)} = \frac{1}{2\pi i}(R_0(\lambda+i0)f - R_0(\lambda-i0)f, f),$$

$$(4.7') \qquad \mathscr{F}_0(\lambda)f = \text{w-}\lim_{m\to\infty} \mathscr{F}_0(\lambda, r_m)f,$$

for $f\in L_{2,(1+\varepsilon)/2}$ and for a sequence $\{r_m\}$ satisfying (4.1) with $R(\lambda+i0)$ replaced by $R_0(\lambda+i0)$, and $\mathscr{F}_0(\lambda)$ is a bounded operator: $L_{2,(1+\varepsilon)/2}\to L_2(\Omega)$. Since Theorem 3.9 applies to H_0, $\{\mathscr{F}_0(\lambda, r)f\}$ converges strongly in $L_2(\Omega)$ when $r\to\infty$ if $f\in L_{2,1}$. Thus, from (4.6′) and (4.7′), we have

$$(4.9) \qquad \|\mathscr{F}_0(\lambda)f\|^2_{L_2(\Omega)} = \frac{1}{2\pi i}(R_0(\lambda+i0)f - R_0(\lambda-i0)f, f),$$

for $f\in L_{2,1}$. But since both sides of (4.9) are continuous in $f\in L_{2,(1+\varepsilon)/2}$ and since $L_{2,1}$ is dense in $L_{2,(1+\varepsilon)/2}$, (4.9) holds for all $f\in L_{2,(1+\varepsilon)/2}$. (4.9) combined with (4.6′), (4.7′) leads to the strong convergence of $\{\mathscr{F}_0(\lambda, r_m)f\}$ to $\mathscr{F}_0(\lambda)f$ for $f\in L_{2,(1+\varepsilon)/2}$.

Next, noting that $H=H_0+V$ and V is a bounded linear operator: $L_{2,-(1+\varepsilon)/2}\to L_{2,(1+\varepsilon)/2}$ by the assumption (V), we have, passing to the limit in the usual resolvent equation,

$$(4.10) \qquad R(\lambda+i0) - R_0(\lambda+i0) = -R_0(\lambda+i0)VR(\lambda+i0).$$

Hence, applying (4.10) to $f\in L_{2,(1+\varepsilon)/2}$, we have the following equality for $f\in L_{2,(1+\varepsilon)/2}$ and $u=R(\lambda+i0)f\in L_{2,-(1+\varepsilon)/2}$:

$$u = R_0(\lambda+i0)(f-Vu).$$

From the definition of $\mathscr{F}_0(\lambda, r)$ and $\mathscr{F}(\lambda, r)$, we have

$$\mathscr{F}(\lambda, r)f = \mathscr{F}_0(\lambda, r)(f-Vu).$$

Noting that the condition (4.1) on $\{r_m\}$ is concerned only with $u=R(\lambda+i0)f=R_0(\lambda+i0)(f-Vu)$ and applying the result obtained for H_0 to $f-Vu\in L_{2,(1+\varepsilon)/2}$, we have the strong convergence in $L_2(\Omega)$ of the sequence $\{\mathscr{F}(\lambda, r_m)f\}=\{\mathscr{F}_0(\lambda, r_m)(f-Vu)\}$ for $f\in L_{2,(1+\varepsilon)/2}$ and for any sequence $\{r_m\}$ satisfying (4.1). Thus, we have proved that the weak limit (4.7) is also a strong limit. From (4.6) and the strong convergence of $\{\mathscr{F}(\lambda, r_m)f\}$, (2) of the lemma follows. Thus we have proved (1) and (2) of the lemma.

Finally, for obtaining the continuity in λ of $\mathscr{F}(\lambda)f$, it suffices to show the continuity of $(\mathscr{F}(\lambda)f, \phi)_{L_2(\Omega)}$ for all $\phi\in C^\infty(\Omega)$ since $\|\mathscr{F}(\lambda)f\|^2_{L_2(\Omega)}$ is continuous in λ as is seen from the right-hand side of (2) of the lemma and Theorem 2.3. We have by (4.5)

$$(4.11) \qquad (\mathscr{F}(\lambda)f, \phi)_{L_2(\Omega)}$$
$$= \frac{1}{2\pi i}\{-(f(\cdot), v_\phi(\cdot, \lambda)) + (R(\lambda+i0)f(\cdot), (L-\lambda)v_\phi(\cdot, \lambda))\}.$$

Let λ vary in an interval $[a_1, a_2]$ $(0<a_1<a_2<\infty)$. Then we have the following pointwise estimates by Proposition 2.2:

$$|v_\phi(x, \lambda)| \leqq C(1+|x|)^{-(n-1)/2},$$

$$|(L_0-\lambda)v_\phi(x, \lambda)| \leqq C(1+|x|)^{-(n-1)/2-1-\delta},$$

with C independent of $\lambda \in [a_1, a_2]$. Hence, since $|V(x)| \leqq C_0(1+|x|)^{-1-\delta}$ and $L=L_0+V$, we have

$$(1+|x|)^{-(1+\varepsilon)}|v_\phi(x, \lambda')-v_\phi(x, \lambda)|^2 \leqq 2C(1+|x|)^{-n-\varepsilon},$$

$$(1+|x|)^{1+\varepsilon}|(L-\lambda')v_\phi(x, \lambda')-(L-\lambda)v_\phi(x, \lambda)|^2 \leqq C'(1+|x|)^{-n-2\delta+\varepsilon},$$

for $\lambda, \lambda' \in [a_1, a_2]$. Therefore, noting that according to (2.1) $v_\phi(x, \lambda') \to v_\phi(x, \lambda)$ and $(L-\lambda')v_\phi(x, \lambda') \to (L-\lambda)v_\phi(x, \lambda)$ as $\lambda' \to \lambda$ for each $x \in \boldsymbol{R}^n$, we have by the use of the Lebesgue dominated convergence theorem

$$\|v_\phi(\cdot, \lambda')-v_\phi(\cdot, \lambda)\|_{-(1+\varepsilon)/2} \longrightarrow 0,$$

$$\|(L-\lambda')v_\phi(\cdot, \lambda')-(L-\lambda)v_\phi(\cdot, \lambda)\|_{(1+\varepsilon)/2} \longrightarrow 0,$$

as $\lambda' \to \lambda$ $(\lambda', \lambda \in [a_1, a_2])$. Thus, $v_\phi(\cdot, \lambda)$ and $(L-\lambda)v_\phi(\cdot, \lambda)$ are continuous for $\lambda>0$ in $L_{2,-(1+\varepsilon)/2}$ and $L_{2,(1+\varepsilon)/2}$, respectively. Hence, since $R(\lambda+i0)f$ is continuous in λ in $L_{2,-(1+\varepsilon)/2}$ (Theorem 2.3 (2)), we obtain the continuity of $(\mathscr{F}(\lambda)f, \phi)_{L_2(\Omega)}$ from (4.11). This completes the proof of (3) and thus Lemma 4.1.

We leave the proof of the next theorem, the spectral representation for H, to the reader, because it can be obtained in the same way as in the proof of Theorem 2.8 and Theorem 3.1 of Ikebe [4], by using Theorem 2.3, Lemma 4.1 and Proposition 2.2.

Theorem 4.2. *Suppose that the assumptions* (V) *and* (b') *are satisfied. Then the following assertions hold:*
(1) *Let* $P_{ac}=E(0, \infty)$ *be the projection onto the subspace* $\mathscr{H}_{ac}$ *of absolute continuity for* H. *Let* $\hat{\mathscr{H}}=L_2((0, \infty); L_2(\Omega))$ *be the Hilbert space of all* $L_2(\Omega)$*-valued square integrable functions over* $(0, \infty)=\boldsymbol{R}^+$. *For* $f \in L_{2,(1+\varepsilon)/2}$, *we define a mapping* $\mathscr{F}f: \boldsymbol{R}^+ \to L_2(\Omega)$ *by*

$$\mathscr{F}f(\lambda) \equiv \mathscr{F}(\lambda)f \quad (\lambda>0),$$

where $\mathscr{F}(\lambda)$ *is given in Lemma* 4.1. *Then for* $f, g \in L_{2,(1+\varepsilon)/2}$ *and for any Borel subset* B *of* $\boldsymbol{R}^+$, *we have*

$$(E(B)f, g)=\int_B (\mathscr{F}f(\lambda), \mathscr{F}g(\lambda))_{L_2(\Omega)}d\lambda, \tag{4.12}$$

where E *is the spectral measure for* H. *In particular, by letting* $B=\boldsymbol{R}^+$ *in* (4.12), *we have* $\mathscr{F}f \in \hat{\mathscr{H}}$ *and*

$$(P_{ac}f, g)=(\mathscr{F}f, \mathscr{F}g)_{\hat{\mathscr{H}}}=\int_0^\infty (\mathscr{F}f(\lambda), \mathscr{F}g(\lambda))_{L_2(\Omega)}d\lambda.$$

(2) *The operator* $\mathscr{F}$ *defined above on* $L_{2,(1+\varepsilon)/2}$ *can be uniquely extended to whole* $\mathscr{H}$ *(this will be denoted by* $\mathscr{F}$ *also).* $\mathscr{F}$ *is a partial isometry with the initial set* $\mathscr{H}_{ac}$ *and the final set* $\hat{\mathscr{H}}$ *(i.e.* $\mathscr{F}$ *is an unitary operator from* $\mathscr{H}_{ac}$ *to* $\hat{\mathscr{H}}$*). For a bounded Borel measureable function* $\alpha(\lambda)$ *on* $\mathbf{R}^+$, *we have for all* $f \in \mathscr{H}_{ac}$

$$(\mathscr{F}\alpha(H)f)(\lambda) = \alpha(\lambda)\mathscr{F}f(\lambda) \quad \text{a.e.} \quad \lambda > 0.$$

(3) *Let B be a relatively compact Borel subset of* $\mathbf{R}^+$. *Then* $\mathscr{F}_B^*$ *is defined by*

$$\mathscr{F}_B^* \hat{f} = \int_B \mathscr{F}(\lambda)^* \hat{f}(\lambda) d\lambda \qquad \text{for} \quad \hat{f} \in \hat{\mathscr{H}},$$

which is a partial isometry from $\hat{\mathscr{H}}$ *to* $\mathscr{H}_{ac}$ *and we have*

$$\mathscr{F}_B^* = E(B)\mathscr{F}^* = (\mathscr{F}E(B))^*.$$

The following inversion formula holds:

$$P_{ac}f = \operatorname*{s\text{-}lim}_{N\to\infty} \int_{1/N}^{N} \mathscr{F}(\lambda)^*(\mathscr{F}f)(\lambda) d\lambda.$$

(4) $\mathscr{F}(\lambda)^*: L_2(\Omega) \to L_{2,-(1+\varepsilon)/2}$ *is an eigenoperator of H with eigenvalue* λ *in the sense that for any* $\phi \in L_2(\Omega)$, $L\mathscr{F}(\lambda)^*\phi = \lambda\mathscr{F}(\lambda)^*\phi$ *in the distributional sense.*

Remark. One can obtain the spectral representation under the assumptions (V') and (b) on the basis of $\mathscr{F}(\lambda)$ defined in Theorem 3.9 except for the unitraity assertion.

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