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Lifts of extremal quasiconformal mappings of arbitrary Riemann surfaces

By

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§ 0. Introduction

Let R and S be Riemann surfaces and f a quasiconformal mapping of R onto S. Let $\tilde{R}$ be a normal covering surface of R. Then f can be lifted to a quasiconformal mapping $\tilde{f}$ of $\tilde{R}$ onto a normal covering surface $\tilde{S}$ of S, and $\tilde{f}$ is uniquely determined by f up to covering transformations of $\tilde{R}$. Suppose f is extremal. Is then $\tilde{f}$ also extremal? In case the universal covering surface of R is the Riemann sphere $\boldsymbol{C} \cup \{\infty\}$ or the complex plane $\boldsymbol{C}$, this problem is not of interest. In fact, suppose R is conformally equivalent to $\boldsymbol{C} \cup \{\infty\}$, $\boldsymbol{C}$, or the one-punctured complex plane $\boldsymbol{C}$-$\{0\}$, if f is extremal then f is conformal, hence $\tilde{f}$ is extremal. Suppose R is a torus and $\tilde{R}=\boldsymbol{C}$ or $\boldsymbol{C}$-$\{0\}$, then $\tilde{f}$ is extremal if and only if f is conformal, And suppose R and $\tilde{R}$ are tori, from the classical Teichmüller's theorem, $\tilde{f}$ is extremal if and only if so is f. E. Blum [8] showed that if R is a doubly connected bounded domain, $\tilde{R}$ is the unit disk U, and f is a horizontal streching, then $\tilde{f}$ is extremal. On the other hand, for the case R is a compact Riemann surface of genus $g \geqq 2$, and $\tilde{R}=U$, K. Strebel [22] gave, together with some examples, a conjecture: even if f is extremal, $\tilde{f}$ is never extremal except for the trivial case that f is conformal.

The aims of this paper are to give a condition for a lift mapping $\tilde{f}$ of extremal f to be extremal or not to be extremal (Theorem 3), and to prove the following theorem, which is a generalization of Blum's.

Theorem 1. *Suppose that R and S are two (arbitrary) Riemann surfaces whose universal covering surfaces are the unit disks and f is an extremal quasiconformal mapping of R onto S. And suppose that $\tilde{R}$ is a normal covering surface of R whose covering transformation group Γ is a finitely generated Abelian group, and a lift $\tilde{f}$ of f is a quasiconformal mapping of $\tilde{R}$ onto a normal covering surface of S. Then $\tilde{f}$ is extremal.*

The above theorem will be proven in § 2.2. In § 1 we shall sum up related facts for our aims. The proofs of theorem and lemmas in § 1 will be omitted, since they are seen in many articles.

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§1. Preliminaries

1. Let R be a Riemann surface whose universal covering surface is the unit disk U, and let R be represented by a Fuchsian group G acting on U as $R=U/G$. Set $R^*=((\partial U-\Lambda(G))\cup U)/G$, where $\Lambda(G)$ is the limit set of G, then R^* is a Riemann surface whose interior is R. A quasiconformal mapping f of R onto another Riemann surface S has the unique extension f^* which is a topological mapping of R^* onto S^* (cf. Ahlfors [2], Lehto & Virtanen [16]). Two quasiconformal mappings f and g of R onto S are said to be *homotopic modulo the boundary*, when $f^*=g^*$ on R^*-R and there exists a homotopy between f^* and g^* which is constant on R^*-R. If R^*-R is empty: that is, if G is of the first kind, then this means that f is just homotopic to g. For a quasiconformal mapping f of R onto S, by $\mathscr{Q}(f;R,S)$ (or simply by $\mathscr{Q}(f)$) we denote the set of all quasiconformal mppings of R onto S which are homotopic to f modulo the boundary. A quasiconformal mapping f of R onto S is said to be *extremal in* $\mathscr{Q}(f;R,S)$ if f has the smallest maximal dilatation among the mappings in $\mathscr{Q}(f;R,S)$. There always exists at least one extremal mapping in $\mathscr{Q}(f;R,S)$. (See Bers [5], theorem 4.)

Let $\mathscr{L}^1(G)$ denote the Banach space of integrable measurable quadratic differentials on $R=U/G$; that, is, the set of measurable functions v on U such that

$$v(Az)A'(z)^2=v(z) \quad \text{for all } A\in G,\ a.e.\,z\in U, \quad \text{and}$$

$$\|v\|_R\equiv\iint_R |v(z)||dz\wedge d\bar{z}|<\infty.$$

Let $\mathscr{A}(G)$ denote the Banach space of integrable holomorhic quadratic differentials on R: that is,

$$\mathscr{A}(G)=\{\phi\in\mathscr{L}^1(G)\,|\,\phi \text{ is holomorphic on } U.\}.$$

And let $\mathscr{A}_1(G)$ denote the closed unit ball of $\mathscr{A}(G)$.

The following theorem due to R. S. Hamilton, E. Reich, K. Strebel and others is characteristic of extremal quasiconformal mappings.

Theorem 2. *A quasiconformal mapping f of R is extremal in $\mathscr{Q}(f)$ if and only if*

$$\sup_{\phi\in\mathscr{A}_1(G)}\left|\iint_R \mu(z)\phi(z)dz\wedge d\bar{z}\right|=\|\mu\|_\infty,$$

where μ is the Beltrami differential of f.

The proof is found in many articles, for example, see Strebel [23], theorem 5 and Hamilton [10], theorem 1.

For the Beltrami differential μ of an extremal quasiconformal mapping of R, a sequence $\{\phi_n\}_{n=1}^{\infty}$ of holomorphic quadratic differentials on R with norm $\|\phi_n\|_R=1$ is called *a Hamilton sequence for* μ *if*

$$\lim_{n\to\infty}\iint_R \mu\phi_n|dz\wedge d\bar{z}| = \|\mu\|_\infty.$$

For a holomorphic quadratic differential ϕ (not necessarily of finite norm) on R and for a real $k(0<k<1)$, a quasiconformal mapping of R whose Beltrami differential is $k\bar{\phi}/|\phi|$ is said to be *of Teichmüller type associated with* ϕ, moreover if ϕ has the finite norm, it is called *a Teichmüller mapping*. It follows from theorem 2 that a Teichmüller mapping f is extremal in $\mathscr{Q}(f)$.

2. A regular (i.e. complete and smooth) covering surface $\tilde{R}$ of R is said to be *normal* when the fundamental group $\pi_1(\tilde{R})$ of $\tilde{R}$ is isomorphic to a normal subgroup of $\pi_1(R)$. We here especially study the extremality of lifts to normal covering surfaces, so througbout the rest of this paper, unless otherwise mentioned, we assume the following.

I) *R is an arbitrary Riemann surface whose universal covering surface is the unit disk U, and R is represented by a Fuchsian group G, without ellitpic elements, acting on U as $R=U/G$.*

II) *$\tilde{R}$ is a normal covering surface of R, and $\tilde{R}$ is represented by a normal subgroup $\tilde{G}$ of G as $\tilde{R}=U/\tilde{G}$.*

III) *Γ is the covering transformation group of $\tilde{R}$.*

Under the above assumption we have $R=\tilde{R}/\Gamma$, and $\Gamma\cong G/\tilde{G}$. (See Ahlfors and Sario [3], Chapter I, § 3.)

3. For $z, \zeta\in U$, we set

$$K(z, \zeta) = 3i/\{2\pi(1 - z\bar{\zeta})^4\},$$

then it follows from a simple calculation that $K(z, \zeta)dz^2\wedge d\bar{\zeta}^2$ is Möb(U)-invariant: that is, for any $T\in$ Möb(U)

$$K(Tz, T\zeta)T'(z)^2\overline{T'(\zeta)}^2 = K(z, \zeta),$$

where Möb(U) is the set of all conformal self-mapping of U, and it follows that

$$\iint_U |K(z, \zeta)|\,|dz\wedge d\bar{z}| = 3\lambda(\zeta)^2,$$

where λ is the Poincaré metric on U. We set

$$F(z, \zeta) = \sum_{A\in\tilde{G}} K(Az, \zeta)A'(z)^2,$$

then the right hand side coverges absolutely and uniformly on compact subsets of U, and $F(z, \zeta)dz^2\wedge d\bar{\zeta}^2$ is $N(\tilde{G})$-invariant, where $N(\tilde{G})$ is the normalizer of $\tilde{G}$ in Möb(U), in particular, it is G-invariant, and

$$\iint_{\tilde{R}} F(z, \zeta)|\,|dz\wedge d\bar{z}| \leqq 3\lambda(\zeta)^2.$$

Lemma 1. *Let H be a Fuchsian group which contains $\tilde{G}$ and is contained in $N(\tilde{G})$. For each $\nu\in\mathscr{L}^1(H)$*, define

$$\beta[v](z) \equiv \iint_{\tilde{R}} \lambda(\zeta)^{-2} F(z, \zeta) v(\zeta) d\zeta \wedge d\bar{\zeta}.$$

Then β is a bounded linear projection of $\mathscr{L}^1(H)$ onto $\mathscr{A}(H)$, in particular, for any $\phi \in \mathscr{A}(H)$

$$\beta[\phi](z) = \phi(z).$$

(For this section see Kra [12], proposition 5.1 of Chap. III.)

§2. Main results

1. For any $\gamma \in \Gamma$, let $T_\gamma \in G$ be one of the lifts of γ: that is, T_γ is a representative of the coset, which is corresponding to γ, of the quotient group $G/\tilde{G} \cong \Gamma$. Let v be an element of $\mathscr{L}^1(\tilde{G})$, then $v(T_\gamma z) T'_\gamma(z)^2$ is independent of any choice of the representatives of the coset corresponding to γ. Set

$$(\Theta_\Gamma v)(z) = \sum_{\gamma \in \Gamma} v(T_\gamma z) T'_\gamma(z)^2, \qquad \text{for} \quad v \in \mathscr{L}^1(\tilde{G}).$$

Then the right hand side converges in $\mathscr{L}^1(G)$. Hence we can define a mapping Θ_Γ of $\mathscr{L}^1(\tilde{G})$ to $\mathscr{L}^1(G)$. $\Theta_\Gamma v$ is called, in this paper, *the Poincaré series of* v. If $\tilde{G}$ is the trivial group, then $\Theta_\Gamma v$ coincides with the usual Poincaré series $\Theta_G v$.

Proposition 1. *The mapping Θ_Γ defined above has the following properties:*
i) *Θ_Γ is a continuous linear mapping of norm 1, furthermore, $\Theta_\Gamma(\mathscr{A}(\tilde{G})) = \mathscr{A}(G)$, and for every $\phi \in \mathscr{A}(G)$, there is a $\Phi \in \mathscr{A}(\tilde{G})$ such that $\phi = \Theta_\Gamma \Phi$, and $\|\Phi\|_{\tilde{R}} \leqq 3\|\phi\|_R$.*
ii) $\Theta_G = \Theta_\Gamma \circ \Theta_{\tilde{G}}$.
iii) $\beta \circ \Theta_\Gamma = \Theta_\Gamma \circ \beta$.

Proof. Properties ii) and iii) follow form simple calculations, and i) follows from the same argument as in the case of the usual Poincaré series, and from ii).

Remark. We have defined here the Poincaré series as a mapping not only of $\mathscr{A}(\tilde{G})$ but also of $\mathscr{L}^1(\tilde{G})$, so the norm of the Poincaré series is one. We, however, have much interest in the Poincaré series as a mapping of $\mathscr{A}(\tilde{G})$ onto $\mathscr{A}(G)$, so by $\|\Theta_\Gamma\|$ we mean the norm of the Poincaré series as a mapping of $\mathscr{A}(\tilde{G})$ onto $\mathscr{A}(G)$: that is,

$$\|\Theta_\Gamma\| = \sup_{\Phi \in \mathscr{A}(\tilde{G})} \frac{\|\Theta_\Gamma \Phi\|_R}{\|\Phi\|_{\tilde{R}}},$$

Set $b(G) = \{\phi \in \mathscr{A}(G) \mid \|\phi\|_R = 1\}$, and for every $\phi \in b(G)$, define

$$I(\phi) = \inf \{\|\Phi\|_{\tilde{R}} | \Theta_\Gamma \Phi = \phi, \ \Phi \in \mathscr{A}(\tilde{G})\},$$

$$\underline{I}(\phi) = \underline{\lim}_{\psi \to \phi} I(\psi).$$

Then I (resp. $\underline{I}$) is upper (resp. lower) semi-continuous function on $b(G)$. Obviously, $I(\phi) \geqq \underline{I}(\phi)$ for any $\phi \in (G)$,

$$\inf_{\phi \in b(G)} I(\phi) = \inf_{\phi \in b(G)} \underline{I}(\phi) = \|\Theta_\Gamma\|^{-1} \geqq 1.$$

Now the following is one of the main results of this paper.

Theorem 3. I) *Suppose that* $\underline{I}(\phi_0)=1$ *for some* $\phi_0 \in b(G)$. *Let* f *be a Teichmüller mapping of* R *associated with* ϕ_0 *(*f *is, of course, extremal in* $\mathscr{Q}(f)$*.), and* $\tilde{f}$ *be a lift to* $\tilde{R}$ *of* f. *Then* $\tilde{f}$ *is extremal in* $\mathscr{Q}(\tilde{f})$.

II) *Suppose that* $\{\phi \in b(G) \mid I(\phi)=1\}$ *is dense in* $b(G)$, *so* $\{\phi \in b(G) \mid \underline{I}(\phi)=1\} = b(G)$. *Let* f *be an arbitrary extremal quasiconformal mapping of* R *(not necessarily, a Teichmüller mapping), and* $\tilde{f}$ *be a lift to* $\tilde{R}$ *of* f. *Then* $\tilde{f}$ *is extremal in* $\mathscr{Q}(\tilde{f})$.

III) *If* $\dim \mathscr{A}(G) < \infty$, *then the converses of* I) *and* II) *are valid. That is, the following* III.i) *and* III.ii) *are concluded.*

III.i) *Suppose that* $\dim \mathscr{A}(G) < \infty$. *Let* f *be a Teichmüller mapping of* R *associated with some* $\phi_0 \in b(G)$, *and* $\tilde{f}$ *a lift to* $\tilde{R}$ *of* f. *If* $\tilde{f}$ *is extremal in* $\mathscr{Q}(\tilde{f})$, *then* $\underline{I}(\phi_0)=1$.

III.ii) *Suppose that* $\dim \mathscr{A}(G) < \infty$. *If a lift* $\tilde{f}$ *to* $\tilde{R}$ *of each extremal quasiconformal mapping* f *of* R *is extremal in* $\mathscr{Q}(\tilde{f})$, *then*

$$\{\phi \in b(G) \mid \underline{I}(\phi)=1\} = b(G).$$

Proof. I) Suppose that $\underline{I}(\phi_0)=1$ for some $\phi_0 \in b(G)$. Let μ be the Beltrami differential of a Teichmüller mapping f associated with ϕ_0. Then there is a sequence $\{\phi_n\}_{n=1}^{\infty} \subset b(G)$ which satisfies the next three conditions;

a) ϕ_n converges to ϕ_0 in $\mathscr{L}^1(G)$.

b) For each ϕ_n, there is a $\Phi_n \in \mathscr{A}(\tilde{G})$ such that $\phi_n = \Theta_\Gamma \Phi_n$.

c) $\lim_{n\to\infty} \|\Phi_n\|_{\tilde{R}} = 1$.

Then

$$\begin{aligned}\sup_{\Phi \in \mathscr{A}_1(\tilde{G})} \left|\iint_{\tilde{R}} \mu \Phi \, dz \wedge d\bar{z}\right| &\geqq \overline{\lim_{n\to\infty}} \left|\iint_{\tilde{R}} \mu \Phi_n dz \wedge d\bar{z}\right| \Big/ \|\Phi_n\|_{\tilde{R}} \\ &= \overline{\lim_{n\to\infty}} \left|\iint_R \mu \phi_n dz \wedge d\bar{z}\right| \\ &= \left|\iint_R \mu \phi_0 dz \wedge d\bar{z}\right| \\ &= \|\mu\|_\infty.\end{aligned}$$

Hence $\tilde{f}$ is extremal in $\mathscr{Q}(\tilde{f})$.

II) Let μ be the Beltrami differential of an arbitrary extremal quasiconformal mapping f of R, and let $\{\phi_n\}_{n=1}^{\infty} \subset b(G)$ be a Hamilton sequence for μ. Then there is a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset \mathscr{A}(\tilde{G})$ such that $\lim_{n\to\infty} \|\Phi_n\|_{\tilde{R}} = 1$, and $\lim_{n\to\infty} \|\phi_n - \Theta_\Gamma \Phi_n\|_R = 0$. This means that

$$\begin{aligned}\sup_{\Phi \in \mathscr{A}_1(\tilde{G})} \left|\iint_{\tilde{R}} \mu \Phi \, dz \wedge d\bar{z}\right| &\geqq \overline{\lim_{n\to\infty}} \left|\iint_{\tilde{R}} \mu \Phi_n dz \wedge d\bar{z}\right| \Big/ \|\Phi_n\|_{\tilde{R}} \\ &= \overline{\lim_{n\to\infty}} \left|\iint_R \mu \Theta_\Gamma \Phi_n dz \wedge d\bar{z}\right|\end{aligned}$$

$$= \lim_{n\to\infty} \left| \iint_R \mu\phi_n dz \wedge d\bar{z} \right|$$

$$= \|\mu\|_\infty .$$

Hence $\tilde{f}$, a lift to $\tilde{R}$ of f, is extremal in $\mathscr{Q}(\tilde{f})$.

III.i) In case $\dim \mathscr{A}(G) < \infty$, it turns out that every Hamilton sequence $\{\Phi_n\}_{n=1}^\infty \subset b(G)$ for $\mu = k\overline{\phi_0}/|\phi_0|\,(0<k<1)$ coverges to ϕ_0 in $\mathscr{L}^1(G)$. Since $\tilde{f}$ is extremal in $\mathscr{Q}(\tilde{f})$, there is a Hamilton sequence $\{\Phi_n\}_{n=1}^\infty \subset b(\tilde{G})$ for μ. Then

$$\|\mu\|_\infty = \lim_{n\to\infty} \iint_{\tilde{R}} \mu\Phi_n |dz \wedge d\bar{z}|$$

$$= \lim_{n\to\infty} \iint_R \mu\Theta_\Gamma\Phi_n |dz \wedge d\bar{z}|.$$

From this, $\lim_{n\to\infty} \|\Theta_\Gamma\Phi_n\| =_R 1$, and $\{\Theta_\Gamma\Phi_n/\|\Theta_\Gamma\Phi_n\|_R\}_{n=1}^\infty \subset b(G)$ is a Hamilton sequence for μ, so it converges to ϕ_0 in $\mathscr{L}^1(G)$. Hence

$$1 \leqq \underline{I}(\phi_0) \leqq \varliminf_{n\to\infty} I(\Theta_\Gamma\Phi_n/\|\Theta_\Gamma\Phi_n\|_R)$$

$$\leqq \lim_{n\to\infty} \|\Phi_n\|_{\tilde{R}}/\|\Theta_\Gamma\Phi_n\|_R = 1.$$

III.ii) This follows from III.i). Q. E. D.

Remark. It is easily seen from theorem 2 that if $\|\Theta_\Gamma\| < 1$, then $\tilde{f}$, a lift to $\tilde{R}$ of an aribitrary extremal quasiconformal mapping f of R, is not extremal in $\mathscr{Q}(\tilde{f})$ except for the case that f is conformal. On the other hand, if $\dim \mathscr{A}(G) < \infty$ and if $\|\Theta_\Gamma\| = 1$, then there is a ϕ_0 in $b(G)$ such that $1 = \|\Theta_\Gamma\|^{-1} = \underline{I}(\phi_0)$. From I) of theorem 3, there is an extremal quasiconformal mapping of R which is not conformal and whose lift to $\tilde{R}$ is extremal. Hence, in case $\dim \mathscr{A}(G) < \infty$, $\|\Theta_\Gamma\| < 1$ if and only if a lift to $\tilde{R}$ of an arbitrary extremal quasiconformal but not conformal mapping is not extremal. The above argument tells us that Strebel's conjecture stated in the introduction is equivalent to the following one.

Conjecture. *Let G be a Fuchsian group acting on the unit disk U such that U/G is a compact Riemann surface of genus $g \geqq 2$. Then the operator norm of the Poincaré series as a mapping of $\mathscr{A}(1)$ onto $\mathscr{A}(G)$ is strictly less than one, where 1 is the trivial group.*

But no Fuchsian group G such that $\|\Theta_G\| < 1$ is known.

2. Now we are in position to prove theorem 1. It is sufficient to do this in the case that the covering transformation group Γ is cyclic. In case Γ is a finite group, theorem 1 is trivial, since $\mathscr{A}(G) \subset \mathscr{A}(\tilde{G})$, and for any $\phi \in \mathscr{A}(G)$, $\sharp(\Gamma)\|\phi\|_R = \|\phi\|_{\tilde{R}}$. In order to show theorem 1 in the case that Γ is an infinite cyclic group, we need the next two lemmas.

For the covering transformation group Γ of $\tilde{R}$, which may not be a cyclic group, an open subset ω of $\tilde{R}$ which satisfies the following three conditions is called, in this paper, *a fundamental region for Γ in $\tilde{R}$.*

(i) *Whenever $\gamma(p)=q$ for some p, $q \in \omega$, $\gamma \in \Gamma$, then $\gamma = id$.*

(ii) *For every point $q \in \tilde{R}$, there is a $\gamma \in \Gamma$, and a $p \in \bar{\omega}$ such that $\gamma(p)=q$.*

(iii) *The (two dimensional) Lebesgue measure of $\bar{\omega}-\omega$ is zero.*

A fundamental region for Γ in $\tilde{R}$ always exists, and we can take such an open set as a connected set, i.e. a region. For example, the projection to $\tilde{R}$ of a usual Dirichlet fundamental region for G in U is such a region.

Lemma 2. *Suppose that the covering transformation group Γ is an infinite cyclic group generated by a conformal self-mapping γ of $\tilde{R}$. Let ω be a fundamental region for $\Gamma=\langle\gamma\rangle$ in $\tilde{R}$ and let χ_n be the characteristic function of $\pi^{-1}(\omega_n)$, where π is the projection of U onto $\tilde{R}=U/\tilde{G}$, and $\omega_n=\sum_{|k|\leqq n}\gamma^k(\omega)$. And for an arbitrary $\phi \in \mathscr{A}(G)$, set*

$$\Phi_n=(2n+1)^{-1}\chi_n\phi.$$

Then $\Phi_n \in \mathscr{L}^1(\tilde{G})$, $\|\Phi_n\|_{\tilde{R}}=\|\phi\|_R$, *and*

$$\lim_{n\to\infty}\|\Phi_n-\beta[\Phi_n]\|_{\tilde{R}}=0.$$

Proof. The first two assertions are obvious. And

$$\|\Phi_n-\beta[\Phi_n]\|_{\tilde{R}}=\iint_{\tilde{R}}|\Phi_n(z)-\beta[\Phi_n](z)|\,|dz\wedge d\bar{z}|$$

$$=\iint_{\omega_n}|\Phi_n(z)-\beta[\Phi_n](z)|\,|dz\wedge d\bar{z}|+\iint_{\tilde{R}-\overline{\omega_n}}|\beta[\Phi_n](z)|\,|dz\wedge d\bar{z}|$$

The first integral is equal to

$$(2n+1)^{-1}\iint_{\omega_n}\left|\phi(z)-\iint_{\omega_n}\lambda(\zeta)^{-2}F(z,\zeta)\phi(\zeta)d\zeta\wedge d\bar{\zeta}\right|\,|dz\wedge d\bar{z}|$$

$$=(2n+1)^{-1}\iint_{\omega_n}\left|\iint_{\tilde{R}-\overline{\omega_n}}\lambda(\zeta)^{-2}F(z,\zeta)\phi(\zeta)d\zeta\wedge d\bar{\zeta}\right|\,|dz\wedge d\bar{z}|$$

$$\leqq(2n+1)^{-1}\sum_{|j|\leqq n}\sum_{|k|>n}\iint_{\gamma^j(\omega)}\iint_{\gamma^k(\omega)}\lambda(\zeta)^{-2}|F(z,\zeta)|$$

$$\times|\phi(\zeta)|\,|d\zeta\wedge d\bar{\zeta}|\,|dz\wedge d\bar{z}|$$

$$=(2n+1)^{-1}\sum_{|j|\leqq n}\sum_{|k|>n}\iint_{\gamma^{j-k}(\omega)}\iint_{\omega}\lambda(\zeta)^{-2}|F(z,\zeta)|$$

$$\times|\phi(\zeta)|\,|d\zeta\wedge d\bar{\zeta}|\,|dz\wedge d\bar{z}|$$

$$=(2n+1)^{-1}\sum_{t=0}^{2n}\sum_{|s|>t}\iint_{\gamma^s(\omega)}\iint_{\omega}\lambda(\zeta)^{-2}|F(z,\zeta)|\,|\phi(\zeta)|\,|d\zeta\wedge d\bar{\zeta}|\,|dz\wedge d\bar{z}|$$

$$=(2n+1)^{-1}\sum_{t=0}^{2n}\iint_{\omega}\lambda(\zeta)^{-2}|\phi(\zeta)|\,|d\zeta\wedge d\bar{\zeta}|\iint_{\tilde{R}-\overline{\omega_t}}|F(z,\zeta)|\,|dz\wedge d\bar{z}|$$

The similar computation tells us that the second integral is not greater than the above last expression, too. Hence

$$\|\Phi_n-\beta[\Phi_n]\|_{\tilde{R}}\leqq 2(2n+1)^{-1}\sum_{t=0}^{2n}\iint_{\omega} a_t(\zeta)|\phi(\zeta)|\ |d\zeta\wedge d\bar{\zeta}|,$$

where

$$a_t(\zeta)=\lambda(\zeta)^{-2}\iint_{R-\overline{\omega_t}}|F(z,\zeta)|\ |dz\wedge d\bar{z}|.$$

It turns out from § 1.3 that for any non-negative integer t

$$\|a_t\|_\infty\leqq 3, \qquad \text{and}$$

for an arbitrary fixed $\zeta\in U$, $a_t(\zeta)\to 0$ as $t\to\infty$. From Lebesgue's theorem, $\lim_{t\to\infty}\iint_{\omega} a_t(\zeta)|\phi(\zeta)|\,|d\zeta\vee d\bar{\zeta}|=0$. Hence $\lim_{n\to\infty}\|\Phi_n-\beta[\Phi_n]\|_{\tilde{R}}=0$. Q.E.D.

Lemma 3. *Under the same assumption as in lemma 2, it is concluded that* $I\equiv 1$, *in particular,* $\underline{I}\equiv 1$, $\|\Theta_\Gamma\|=1$.

Proof. For any $\phi\in b(G)$, and for every non-negative integer n, $\beta[\Phi_n]\in\mathscr{A}(\tilde{G})$, and $\Theta_\Gamma\circ\beta[\Phi_n]=\beta[\Theta_\Gamma\Phi_n]=\beta[\phi]=\phi$. Hence $1\leqq I(\phi)\leqq\underline{\lim}_{n\to\infty}\|\beta[\Phi_n]\|_{\tilde{R}}=1$. Q.E.D.

From theorem 3 and the above two lemmas, we get theorem 1.

Remark. 1) We can prove theorem 1 from lemma 2 without the help of the Poincaré series.

2) Though it is a trivial case that a Fuchsian group itself is an Abelian group (even if it is an infinitely generated Kleinian group), it is not the case that the covering trnasformation group of a normal covering surface is a finitely generated Abelian group. In fact, there exists an Abelian covering transformation group generated by n elements for any positive integer n.

3) Even if a covering surface $\tilde{R}=U/\tilde{G}$ of $R=U/G$ is not normal, if $\tilde{R}$ is a complete covering surface of R and the number of sheets of $\tilde{R}$ is finite, then the conclusion of theorem 1 is valid. For, in this case, $\mathscr{A}(G)\subset\mathscr{A}(\tilde{G})$, and for any $\phi\in\mathscr{A}(G)$, $\|\phi\|_{\tilde{R}}=[G:\tilde{G}]\|\phi\|_R$.

By applying this fact and theorem 1 some times over, we get the following.

Theorem 1′. *Let R be an arbitrary Riemann surface whose universal covering surface is the unit disk and $\tilde{R}$ a complete covering surface of R. Suppose that there are a finite number of Riemann surfaces $R_n (n=0, 1,\dots, N)$ which satisfy* I)$\sim$III).

I) $R_0=R$, $R_N=\tilde{R}$.

II) *For $n=1,\dots, N$, each R_n is a complete covering surface of R_{n-1}.*

III) *For $n=1,\dots, N$, either* i) *or* ii) *is fulfilled;*

i) *The number of sheets of a covering surface R_n of R_{n-1} is finite.*

ii) *R_n is a normal covering surface of R_{n-1}, and the covering transformation group of R_n is a finitely generated Abelian group.*

Then the lifts to $\tilde{R}$ of extremal quasiconformal mappings of R are extremal.

3. Let $\boldsymbol{T}(R_0)$ be the Teichmüller space whose initial points is $(R_0, \text{id.})$. Let $\tilde{R}_0$ be a complete covering surface of R_0. Then we can naturally regard $\boldsymbol{T}(R_0)$ as a subset of $\boldsymbol{T}(\tilde{R}_0)$, in particular, we can regard $\boldsymbol{T}(R_0)$ as a subset of the universal Teichmüller space $\boldsymbol{T}(U)$. Concerning the Teichmüller metrics of $\boldsymbol{T}(R_0)$ and of $\boldsymbol{T}(R_0)\subset\boldsymbol{T}(\tilde{R}_0)$ (or of $\boldsymbol{T}(R_0)\subset\boldsymbol{T}(U)$), it is known that $\boldsymbol{T}(R_0)$ is closed in $\boldsymbol{T}(\tilde{R}_0)$ (or $\boldsymbol{T}(U)$) and the inclusion mapping is homeomorphism (see Bers [6].). We get from theorem 1′

Corollary. *Let R_0 be a Riemann surface whose universal covering surface is the unit disk, and let $\tilde{R}_0$ be a complete covering surface of R_0 as in the above theorem. Then the inclusion mapping of $\boldsymbol{T}(R_0)$ to $\boldsymbol{T}(\tilde{R}_0)$ is isometric. In particular, if R_0 is a doubly connected bounded domain. then the inclusion mapping of $\boldsymbol{T}(R_0)$ to $\boldsymbol{T}(U)$ is isometric.*

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