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Segal-Becker theorem for K_G-theory

By

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§1. Introduction.

Let G be a finite group and $\tilde{K}_G$ be the reduced equivariant K-theory of Atiyah-Segal [7]. By the Bott periodicity theorem [2], $\tilde{K}_G$ is the 0-th term of the reduced equivariant cohomology theory $\tilde{K}_G^*$ (cf. [5]). We denote by BU_G a representation space of $\tilde{K}_G^*$, that is

$$K_G(X)=\tilde{K}_G^0(X_+)=[X_+, BU_G]_G$$

for any compact G-space X, where $[\ ,\ ]_G$ denotes the set of G-homotopy classes of based G-maps. By ω we denote the complex regular representation of G. Let $\boldsymbol{CP}_G^\infty$ be the equivariant infinite dimensional complex projective space, which consists of lines in ω^∞ with a G-action induced from that of ω^∞. Then $\boldsymbol{CP}_G^\infty$ is a classifying space of G-line bundles.

The infinite loop space structure of BU_G defines an infinite loop map

$$\xi: Q_G(BU_G) \longrightarrow BU_G$$

where $Q_G(X)=\operatorname{Colim}_n \Omega^{n\omega}\Sigma^{n\omega}X$ for any pointed G-space X. The canonical G-line bundle over $\boldsymbol{CP}_G^\infty$ defines a based map

$$j: \boldsymbol{CP}_{G+}^\infty \longrightarrow BU_G .$$

We put

$$\lambda=\xi\circ Q_G(j): Q_G(\boldsymbol{CP}_{G+}^\infty) \longrightarrow Q_G(BU_G) .$$

The infinite loop map λ defines a transformation of cohomology theories

$$\lambda_*: P_G^* \longrightarrow K_G^* ,$$

where P_G^* is an equivariant cohomology theory defined by $Q_G(\boldsymbol{CP}_{G+}^\infty)$. Then we have

Theorem 1. *For any compact G-space X,*

$$\lambda_*: P_G(X) \longrightarrow K_G(X)$$

is a split epimorphism, where $P_G(X)=\tilde{P}_G^0(X_+)$.

§ 2. Induced representation and transfer.

Recall the result in [4]. Let $K=U(n)$ and $L=U(1)\times U(n-1)$. Let ι_n be the identity representation of $U(n)$ and β_n the one dimensional representation of L defined by the first projection. Let

$$\mathrm{Ind}_L^K : R(L) \longrightarrow R(K)$$

be the induction homomorphism defined by Segal [6]. The following was proved in [4].

Lemma 2. $\mathrm{Ind}_L^K(\beta_n)=\iota_n$.

Next we suppose G and K are any compact Lie groups, and that L is a closed subgroup of K. Let E be a compact $G\times K$-space which is free as a K-space.

For an L-module M we define $\alpha(M)$ to be a G-vector bundle

$$E\times_L M \longrightarrow E/L .$$

The correspondence $M\mapsto\alpha(M)$ induces a homomorphism

$$\alpha : R(L) \longrightarrow K_G(E/L) .$$

Then we have the following:

Lemma 3. *The following diagram is commutative:*

$$\begin{array}{ccc} R(L) & \xrightarrow{\ \alpha\ } & K_G(E/L) \\ \downarrow{\scriptstyle \mathrm{Ind}_L^K} & & \downarrow{\scriptstyle p_!} \\ R(K) & \xrightarrow{\ \alpha\ } & K_G(E/K) , \end{array}$$

where $p_!$ is the Nishida transfer for $p: E/L\to E/K$ (cf. [5]).

Proof. Let M be a $G\times K$-module. The diagonal G-action on $E\times M$ induces a G-action on $E\times_L M$ and the projection is a G-map. So the vector bundle $\alpha'(M)=(E\times_L M\to E/L)$ is a G-vector bundle. Moreover the correspondence $M\mapsto \alpha'(M)$ induces a homomorphism

$$\alpha' : R(G\times L) \longrightarrow K_G(E/L) .$$

Let $q: G\times L\to L$ be the second projection. Then

$$\alpha=\alpha'\circ q^* : R(L) \longrightarrow K_G(E/L) ,$$

where $q^*: R(L)\to R(G\times L)$. We consider the following diagram:

$$\begin{array}{ccccc} R(L) & \xrightarrow{\ q^*\ } & R(G\times L) & \xrightarrow{\ \alpha'\ } & K_G(E/L) \\ \downarrow{\scriptstyle \mathrm{Ind}_L^K} & & \downarrow{\scriptstyle \mathrm{Ind}_{G\times L}^{G\times K}} & & \downarrow{\scriptstyle p_!} \\ R(K) & \xrightarrow{\ q^*\ } & R)G\times K) & \xrightarrow{\ \alpha'\ } & K_G(E/K) . \end{array}$$

First we prove the commutativity of the left hand square. Let $T(K/L)$ be the (co) tangent bundle of K/L. By definition we have

$$\mathrm{Ind}_L^K=\mathrm{ind}\circ\Phi: R(L)\longrightarrow R(K),$$

where $\Phi: R(L)=K_K(K/L)\to K_K(T(K/L))$ is the Thom homomorphism and ind is the index homomorphism (cf. [3], [6]). Then the commutativity follows from the naturality of the Thom homomorphisms and the axiom of the index homomorphisms (cf. [3]).

On the other hand using the isomorphism $K_{G\times K}(E)=K_G(E/K)$ instead of $K_K(E)=K(E/K)$ in [6], we can prove the commutativity of the right-hand square similarly.

§ 3. Proof of Theorem 1.

Proof of Theorem 1. It is sufficient to prove the theorem in the case X/G is connected. So we may assume that $x\in K_G(X)$ is an n-dimensional G-vector bundle over X. Let E be the total space of the associated principal $K(=U(n))$-bundle. E is a compact $G\times K$-space which is free as a K-space.

Clearly $x=\alpha(\iota_n)$. By Lemma 2 and Lemma 3, we have

$$x=\alpha(\mathrm{Ind}_L^K(\beta_n))=p_!\circ\alpha(\beta_n).$$

Since $\alpha(\beta_n)$ is a line bundle over E/L, there exists $a\in P_G(E/L)$ such that $\lambda_*(a)=\alpha(\beta_n)$. Then

$$x=p_!\circ\lambda_*(a)=\lambda_*\circ p_!(a),$$

since λ_* is a transformation of cohomology theories, so that $\lambda_*: P_G(X)\to K_G(X)$ is epimorphic. Since BU_G is a colimit of compact G-spaces, this shows Theorem 1.

The Real analogue to Theorem 1 can be proved by a parallel argument. Let G be a finite group with an involution τ. We denote by $\tilde{G}=\boldsymbol{Z}/2\times_\tau G$, the semidirect product of G with $\boldsymbol{Z}/2$. Then we have

Theorem 1′. *Let X be a compact $\tilde{G}$-space. Then there exists a split epimorphism*

$$\lambda_*: PR_G(X)\longrightarrow KR_G(X).$$

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