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Vanishing of $\mathrm{Ext}_A^i(M, A)$

By

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1. Let A denote a Noetherian ring. The purpose of this note is to establish a kind of vanishing theorem on $\mathrm{Ext}_A^i(M, A)$ over Gorenstein rings. Our result is

Theorem 1. *The following conditions are equivalent.*

(1) *A is a Gorenstein ring.*

(2) *For every finitely generated A-module M there exists an integer n depending on M such that*

$$\mathrm{Ext}_A^i(M, A)=(0)$$

for all $i \geqq n$.

In case A has finite Krull-dimension, say d, it is well-known by Bass [2] that A is a Gorenstein ring if and only if $\mathrm{Ext}_A^i(M, A)=(0)$ for every finitely generated A-module M and for every integer $i>d$. This doesn't make sense if A has infinite Krull-dimension and of course our theorem remains valid even in this case.

In their lecture [1] Auslander and Bridger introduced the concept of Gorenstein-dimension of finitely generated modules and gave a characterization of Gorenstein local rings (and hence of Gorenstein rings with finite Krull-dimension) in terms of Gorenstein-dimension. By virtue of Theorem 1 we can easily extend their result to an assertion about arbitrary Noetherian rings:

Corollary 2. *A is a Gorenstein ring if and only if every finitely generated A-module has finite Gorenstein-dimension.*

As a direct consequence of this fact we have the following

Corollary 3. *A is a regular ring if and only if every finitely generated A-module has finite projective dimension.*

2. First we note

Lemma 4. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be an exact sequence of finitely generated A-modules. Assume that the assertion (2) *of Theorem* 1 *holds for two of the A-modules M_1, M_2 and M_3. Then this holds also for the rest of them.*

Proof. This assertion comes from the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_A^i(M_1, A) \longrightarrow \mathrm{Ext}_A^i(M_2, A) \longrightarrow \mathrm{Ext}_A^i(M_3, A) \longrightarrow \cdots$$

of extensions.

Lemma 5. *Suppose that A is a Gorenstein local ring. Let M be a Cohen-Macaulay A-module. Then*

$$\mathrm{Ext}_A^i(M, A)=(0) \qquad \textit{for all} \quad i>0$$

if $\dim_A M=\dim A$.

Proof. See, e. g., [3], Korollar 6.8.

Proof of Theorem 1.

(2)⇒(1) Let p be a prime ideal of A. Then, as $\mathrm{Ext}_A^i(A/p, A)=(0)$ for every sufficiently large integer i, we see that A_p is a local Gorenstein ring (c. f. [2]). Hence by definition A is a Gorenstein ring.

(1)⇒(2) By virtue of induction on the number of generators of M together with Lemma 4 we may reduce our problem to the case where M is cyclic. Assume that our assertion fails to hold for $M=A/I$ and choose the ideal I to be maximal among such counterexamples. Notice that I is a primary ideal. For it suffices to show that I is an irreducible ideal. Let J and K be ideals of A with $I=J\cap K$ and assume that $J\neq I$ and $K\neq I$. Consider the following exact sequence

$$0 \longrightarrow A/I \longrightarrow A/J\oplus A/K \longrightarrow A/J+K \longrightarrow 0$$

and we find by Lemma 4 and by the maximality of I that the assertion (2) of Theorem 1 holds for $M=A/I$. Of course this is impossible and so I must be an irreducible ideal of A.

Claim. I is a prime ideal of A.

For we put $p=\sqrt{I}$. Assume that $p\neq I$ and choose an element f of A so that $p=I:f$. Consider the exact sequence

$$0 \longrightarrow A/p \longrightarrow A/I \longrightarrow A/I+fA \longrightarrow 0$$

and we get by Lemma 4 and by the maximality of I that $M=A/I$ satisfies the condition (2) of Theorem 1. This contradicts the choice of I and hence I is a prime ideal.

Let f be an element of A not contained in I. Clearly f is regular on A/I and so there is an exact sequence

$$0 \longrightarrow A/I \xrightarrow{\ f\ } A/I \longrightarrow A/I+fA \longrightarrow 0$$

of A-modules. Using this sequence and the fact that the A-module $A/I+fA$ satisfies the condition (2) of Theorem 1 we find that there must be an integer n such that for every $i \geqq n$ the element f acts on the A-module $\mathrm{Ext}_A^i(A/I, A)$ bijectively, i. e., the canonical map

$$\mathrm{Ext}_A^i(A/I, A) \longrightarrow \mathrm{Ext}_{A_f}^i(A_f/IA_f, A_f) = A_f \underset{A}{\otimes} \mathrm{Ext}_A^i(A/I, A)$$

is an isomorphism. To get a contradiction this fact allows us to localize the ring A freely by a single element f not contained in I.

We put $r = \mathrm{ht}_A I$, the height of I. Recall that $r = \mathrm{grade}_I A$ as A is a Cohen-Macaulay ring and we see that I contains an A-regular sequence $a_1, a_2, \cdots, a_r$ of length r. We put $J = (a_1, a_2, \cdots, a_r)$ and $\bar{A} = A/J$. Then it is well-known that for every integer i and for every finitely generated $\bar{A}$-module N there is a natural isomorphism

$$\mathrm{Ext}_{\bar{A}}^i(N, \bar{A}) \cong \mathrm{Ext}_A^{i+r}(N, A).$$

Therefore, after passing through $\bar{A}$, we may assume $r=0$, i. e., I is a minimal prime ideal of A. Let $\mathrm{Min}\, A$ denote the set of all the minimal prime ideals of A. Suppose that $\sharp \mathrm{Min}\, A \geqq 2$ and choose an element f of $\bigcap_{p \in \mathrm{Min}\, A \setminus \{I\}} p$ not contained in I. Then as $\mathrm{Min}\, A_f = \{IA_f\}$ we may assume that $\mathrm{Min}\, A = \{I\}$ after passing through A_f. Now let us choose an integer $n>0$ so that $I^n \neq (0)$ and $I^{n+1} = (0)$. Then as A/I is an integral domain we can find a suitable element f of A not contained in I so that $I^i A_f / I^{i+1} A_f$ is a free A_f/IA_f-module for every $1 \leqq i \leqq n$. Therefore we may assume further that I^i/I^{i+1} is a free A/I-module. In this situation we obtain

Claim. A/I is a Cohen-Macaulay ring.

In fact let p be a prime ideal of A and put $t = \mathrm{depth}\, A_p/IA_p$. Notice that $\mathrm{depth}_{A_p} I^i A_p / I^{i+1} A_p = t$ because $I^i A_p / I^{i+1} A_p$ is a free A_p/IA_p-module. Consider the exact sequences

$$0 \longrightarrow I^i A_p / I^{i+1} A_p \longrightarrow A_p / I^{i+1} A_p \longrightarrow A_p / I^i A_p \longrightarrow 0$$

$(1 \leqq i \leqq n)$ of A_p-modules and we have by induction on i that $\mathrm{depth}\, A_p / I^i A_p = t$ for every $1 \leqq i \leqq n+1$. In particular $\mathrm{depth}\, A_p = t$ as $I^{n+1} A_p = (0)$ by our choice of n, which implies that A_p/IA_p is a Cohen-Macaulay local ring.

Now we are in position to finish the proof of Theorem 1. Let p be a prime ideal of A. Then by the above claim A_p/IA_p is a Cohen-Macaulay A_p-module with $\dim A_p/IA_p = \dim A_p$. From this it follows by Lemma 5 that $\mathrm{Ext}_{A_p}^i(A_p/IA_p, A_p) = (0)$ for all $i>0$. Therefore

$$\mathrm{Ext}_A^i(A/I, A) = (0)$$

for every integer $i>0$. This is the final contradiction and we have completed the proof of Theorem 1.

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References

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