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Singular perturbation approach to traveling waves in competing and diffusing species models

By

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1. Introduction.

In the field of population dynamics, since Fisher's model had been presented, there have been extensive studies of reaction-diffusion equations of the form

$$\frac{\partial \bar{u}}{\partial t}=D\Delta\bar{u}+\bar{f}(\bar{u}), \tag{1.1}$$

where $\bar{u}$ and $\bar{f}$ are n dimensional vectors and D is an $n\times n$ constant matrix. It is widely known that (1.1) exhibits a variety of interesting phenomena, in spite of its simplicity. One of them is the appearance of traveling wave fronts. This type of solution is represented by the form

$$\bar{U}(z)=\bar{u}(x-ct),$$

where c is a velocity vector. This function $\bar{U}$ necessarily satisfies the following system of ordinary differential equations

$$D\bar{U}''+c\bar{U}'+f(\bar{U})=0, \tag{1.2}$$

subject to appropriate boundary conditions imposed at $z=\pm\infty$, where $'=d/dz$. When $n=1$, the existence of $\bar{U}(z, c)$ and its stability were almost completely discussed by many authors. For $n=2\sim4$, there are some results on biological models such as Nagumo's equation, Hodgikin-Huxley's equation, and Field-Noyes's equation (see, for instance, [1, 5, 12]). However, there has not been as yet any powerful general theory for any n, except topological methods developed by Conley [3].

In the framework of (1.1), we discuss here a model of two competing and diffusing species described by

$$\begin{aligned}&\frac{\partial u}{\partial t}-d_1\frac{\partial^2 u}{\partial x^2}=f_0(u, v)u\\&\frac{\partial v}{\partial t}-d_2\frac{\partial^2 v}{\partial x^2}=g_0(u, v)v\end{aligned}, \tag{1.3}$$

where u and v are the population densities of the two species. It is assumed from the competitive interaction that f_0 and g_0 satisfy

$$f_0(0,0)>0\,,\quad g_0(0,0)>0\,,\quad \frac{\partial f_0}{\partial v}<0 \quad \text{and} \quad \frac{\partial g_0}{\partial u}<0\,.$$

Under further additional conditions on f_0 and g_0, Tang and Fife [16] proved the existence of solutions $(U(z), V(z))$ of (1.3) joining the stable rest state (u^*, v^*) (>0) satisfying $f_0(u^*, v^*)=g_0(u^*, v^*)=0$ at $z=+\infty$ with the unstable one $(0, 0)$ at $z=-\infty$, and Gardner [10], Conley and Gardner [4] have recently found a traveling wave solutions joining two stable rest states $(u_0, 0)$ and $(0, v_0)$ where u_0 and v_0 satisfy $f_0(u_0, 0)=g_0(0, v_0)=0$. The latter solution is of interest from an ecological point of view. Suppose that $(U(z), V(z))$ satisfy

$$\begin{aligned} &U(+\infty)=u_0\,, && V(+\infty)=0\,,\\ &U(-\infty)=0\,, && V(-\infty)=v_0\,. \end{aligned} \tag{1.4}$$

This specifies the habitats of two species at infinity $z\to\pm\infty$. If $c>0$ (resp. <0), both diffusing and competing species move in the right (resp. left) direction and then one of the species, [v] (resp. [u]) is dominant asymptotically and if $c=0$, they coexist. Thus, it is of ecological importance to know the sign of c.

In this paper we restrict the nonlinearities (f_0, g_0) to

$$\begin{cases} f_0(u, v)=a_1-b_1u-\dfrac{k_1v}{1+e_1u}\,,\\ g_0(u, v)=a_2-b_2v-\dfrac{k_2u}{1+e_2v} \end{cases} \tag{1.5}$$

where a_i, b_i, k_i and e_i $(i=1, 2)$ are all positive constants, and seek the sign of the velocity c of traveling wave solutions. In the absence of e_i $(i=1, 2)$, f_0 and g_0 are the classical competitive interaction term proposed by Volterra. The presence of e_i states that the intracompetition rate of each species decreases as the population number increases. If $e_i=+\infty$, (1.3) with (1.5) is formally reduced to Fisher's equation of the form

$$w_t=dw_{xx}+(a-bw)w \tag{1.6}$$

with positive constants a and b. Then in this case, it is well known that u (resp. v) moves in the right (resp. left) direction with any fixed velocity $c>2\sqrt{d_1a_1}$ (resp. $<-2\sqrt{d_2a_2}$) under the conditions (1.4). This situation also occurs in the case where $v\equiv 0$ (resp. $u\equiv 0$), i.e., only one species exists in the entire line. Murray [15], Gibbs [11] and Troy [17] discussed the system similar to (1.5) with $a_2=b_2=e_1=e_2=0$ derived from the Belousov-Zhabotinskii reaction and showed traveling wave solutions with some velocity $c>0$.

To make the discussion simple only, let us consider here a simplified model of (1.5)

$$\begin{aligned} &\frac{\partial u}{\partial t}-\varepsilon^2\frac{\partial^2 u}{\partial x^2}=\Big(a-bu-\frac{kv}{1+eu}\Big)u\equiv f(u, v)\\ &\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=(a-bv-ku)v\equiv g(u, v)\,. \end{aligned} \tag{1.7}$$

Unfortunately we must make the following assumption

(A.1) $$0 \leqq \varepsilon^2 \ll 1\,,$$

though this restriction was not needed in [4], to reduce the difficulty of the problem so that the singular perturbation technique developed by Fife [8] can be applied to (1.7). Following his asymptotic analysis, we can succeed in proving the existence of an ε-family of solutions $(U(z, c(\varepsilon)), V(z, c(\varepsilon)))$ and finding the sign of $c(\varepsilon)$ under some conditions on the coefficients a, b, k and e.

2. Formulation.

We are concerned with traveling wave solutions of (1.7), that is, $(U(z), V(z))$ where $z=x-c(\varepsilon)t$ of

$$\begin{aligned} \varepsilon^2 U''+c(\varepsilon)U'+f(U, V)=0 \\ V''+c(\varepsilon)V'+g(U, V)=0 \end{aligned}, \quad z\in \boldsymbol{R}\,, \tag{2.1}$$

subject to the boundary conditions

$$\begin{aligned} &U(-\infty)=\frac{a}{b}\,, \qquad U(+\infty)=0\,, \\ &V(-\infty)=0\,, \qquad V(+\infty)=\frac{a}{b}\,. \end{aligned} \tag{2.2}$$

We make essential assumptions as follows:

(A.2) $$b<k\,,$$

which indicates that two rest states $P_-=(a/b, 0)$ and $P_+=(0, a/b)$ of the corresponding kinetic equations to (1.7) are asymptotically stable.

(A.3) $$c(\varepsilon)=O(\varepsilon)\,.$$

This restriction is required from the situation that, when e is large enough, the velocity of $[u]$ is expected to be of order ε. Then we regard $c(\varepsilon)$ as $\varepsilon c(\varepsilon)$ where $c(\varepsilon)=O(1)$. The resulting system from (2.1) is

$$\begin{aligned} \varepsilon^2 U''+\varepsilon c(\varepsilon)U'+f(U, V)=0 \\ V''+\varepsilon c(\varepsilon)V'+g(U, V)=0 \end{aligned}, \quad z\in \boldsymbol{R}\,. \tag{2.3}$$

Since solutions have translation invariance, we normalize U by

$$U(0)=\alpha\in\left(0, \frac{a}{b}\right)$$

for fixed α and furthermore we put

$$V(0)=\beta\in\left(0, \frac{a}{b}\right)$$

for some β which will be determined later as a function of ε. Our aim is to show the existence of slowly traveling wave solutions $(U(z), V(z))$ joining P_- to P_+.

Throughout this paper, we use the following function spaces:

(1) $X_\rho(I)=\{u(z)\,|\,\|u\|_{X_\rho(I)}\equiv\sup_{z\in I} e^{\rho|z|}|u(z)|<+\infty,\quad u\in C(I)\}$

(2) $X_\rho^m(I)=\left\{u(z)\,|\,\|u\|_{X_\rho^m(I)}\equiv\sum_{i=0}^m\left\|\left(\frac{d}{dz}\right)^i u\right\|_{X_\rho(I)}<+\infty,\qquad u\in C^m(I)\right\}$

(3) $X_{\rho,\varepsilon}^m(I)=\left\{u(z)\,|\,\|u\|_{X_{\rho,\varepsilon}^m(I)}\equiv\sum_{i=0}^m\left\|\left(\varepsilon\frac{d}{dz}\right)^i u\right\|_{X_\rho(I)}<+\infty,\qquad u\in C^m(I)\right\}$

(4) $\mathring{X}_\rho^m(I)=\{u(z)\,|\,u\in X_\rho^m(I),\quad u(0)=0\}$

(5) $\mathring{X}_{\rho,\varepsilon}^m(I)=\{u(z)\,|\,u\in X_{\rho,\varepsilon}^m(I),\quad u(0)=0\}$

(6) $Y_{\rho,\varepsilon}^m(I)=\left\{u(\zeta)\,|\,\|u\|_{Y_{\rho,\varepsilon}^m}=\sum_{i=0}^m\sup_{\zeta\in I} e^{\rho\varepsilon|\zeta|}\left|\left(\frac{d}{d\zeta}\right)^i u(\zeta)\right|<+\infty,\qquad u\in C^m(I)\right\}$

(7) $\mathring{Y}_{\rho,\varepsilon}^m(I)=\{u(\zeta)\,|\,u\in Y_{\rho,\varepsilon}^m,\quad u(0)=0\}$,

where I denotes $\boldsymbol{R}_+$, $\boldsymbol{R}_-$ or $\boldsymbol{R}$.

3. Reduced problem.

First we consider the reduced problem by putting $\varepsilon=0$ in (2.3). The resulting system is

$$\begin{matrix} f(U,V)=0 \\ V''+g(U,V)=0 \end{matrix},\qquad z\in\boldsymbol{R}, \tag{3.1}$$

subject to (2.2). From the first of (3.1), we define $U=h_\beta(V)$ by

$$U=h_\beta(V)=\begin{cases} h_+(V)\equiv 0 & \text{for}\quad V>\beta \\ h_-(V)=\{ae-b+[(ae+b)^2-4bkeV]^{1/2}\}/(2be) & \text{for}\quad 0<V<\beta. \end{cases} \tag{3.2}$$

Here $\beta\in I_0=I_+\cap I_-$ is arbitrarily fixed where $I_+=(0,a/b)$ and $I_-=(0,v_c)$ ($v_c=\max(a/k,(ae+b)^2/(4bke))$) (see Fig. 1).

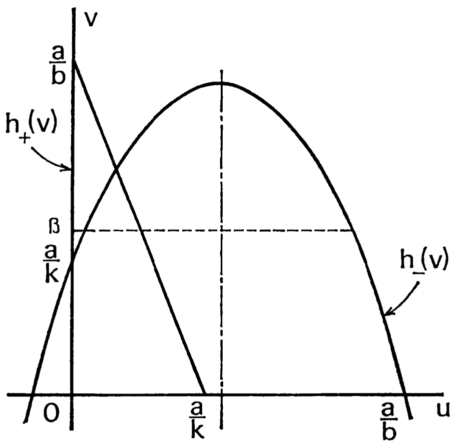

Fig. 1

Then, (3.1) is reduced to

$$V''+g_{\beta}(V)=0\,, \qquad z\in \boldsymbol{R}\,, \tag{3.3}$$

where $g_{\beta}(V)=g(h_{\beta}(V),\ V)$. The boundary conditions are

$$V(-\infty)=0\,, \qquad V(+\infty)=\frac{a}{b}. \tag{3.4}$$

We normalize $V(z)$ by putting

$$V(0)=\beta\,. \tag{3.5}$$

Now we consider the problems

$$\begin{cases} V''+g_{\pm}(V)=0\,, & z\in \boldsymbol{R}_{\pm} \\ V(0)=\beta\,, & V(\pm\infty)=v_{\pm}\,, \end{cases} \tag{3.6$_\pm$}$$

where $g_{\pm}(V)=g(h_{\pm}(V)V)$, $v_{+}=a/b$ and $v_{-}=0$.

Lemma 3.1. *Consider the problems* (3.6)$_{\pm}$ *under* (A.2). *There exist uniquely monotone increasing solutions* $V^{0}_{\pm}(z,\ \beta)$ $(z\in \boldsymbol{R}_{\pm})$ *satisfying*

$$V^{0}_{-}(z,\ \beta)\in X^{2}_{\mu_{-}}(\boldsymbol{R}_{-}) \quad and \quad \left(\frac{a}{b}-V^{0}_{+}(z,\ \beta)\right)\in X^{2}_{\mu_{+}}(\boldsymbol{R}_{+})\,,$$

where $\mu_{\pm}=\sqrt{-g'_{\pm}(v_{\pm})}$.

The proof is seen in Fife [Lemma 2.1, 7].

(A.3) $\quad J(\beta)=\int_{v_{-}}^{v_{+}} g_{\beta}(s)ds$ has a unique isolated zero at $\beta=\beta^{*}\in I_{0}$.

Remark. If $(ae+b)^{2}/(4bke)>a/b$, (A.3) is satisfied.

Lemma 3.2. *Consider the problem* (3.3)$\sim$(3.5). *When* $\beta=\beta^{*}$, *there exists a unique monotone increasing solution* $V^{0}(z,\ \beta^{*})\in C^{1}(\boldsymbol{R})$ *which is constructed by*

$$V^{0}(z,\ \beta^{*})=\begin{cases} V^{0}_{+}(z,\ \beta^{*})\,, & z\in \boldsymbol{R}_{+}\,, \\ V^{0}_{-}(z,\ \beta^{*})\,, & z\in \boldsymbol{R}_{-}\,. \end{cases}$$

Moreover, $V^{0}(z,\ \beta^{*})$ *satisfies*

$$V^{0}(z,\ \beta^{*})\in X^{2}_{\mu}(\boldsymbol{R}_{-}) \quad and \quad \left(\frac{a}{b}-V^{0}(z,\ \beta^{*})\right)\in X^{2}_{\mu}(\boldsymbol{R}_{+})\,,$$

where $\mu=\min(\mu_{+},\ \mu_{-})$.

The proof is the direct consequence of Lemma 3.1.

From the function $V^{0}(z,\ \beta^{*})$, we define $U^{0}(z,\ \beta^{*})$ by

$$U^{0}(z,\ \beta^{*})=\begin{cases} h_{+}(V^{0}(z,\ \beta^{*}))\,, & z\in \boldsymbol{R}_{+}\,, \\ h_{-}(V^{0}(z,\ \beta^{*}))\,, & z\in \boldsymbol{R}_{-}\,. \end{cases}$$

Since $U^{0}(z,\ \beta^{*})$ is discontinuous at $z=0$ only, one may expect that $(U^{0}(z,\ \beta^{*}),$

$V^0(z, \beta^*)$) play a nice approximation to a solution of (2.3) and (2.2) outside the neighborhood of $z=0$ (Fig. 2).

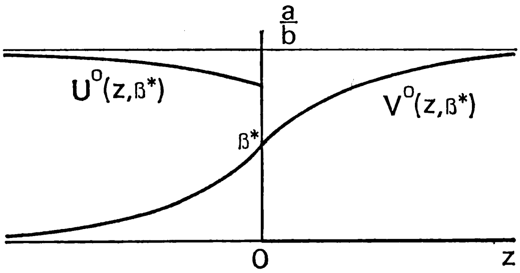

Fig. 2

4. Boundary layer solutions.

Since $U^0(z, \beta^*)$ has a discontinuity of the first kind at $z=0$, we must modify $U^0(z, \beta^*)$ to become an approximation to a solution in the neighborhood of $z=0$. For this purpose, we introduce the stretched variable $\zeta=z/\varepsilon$ in this neighborhood and define boundary layer corrections $W_\pm(\zeta, c, \beta)$ by solutions of the problems

$$\begin{cases} \ddot{W}_\pm+c\dot{W}_\pm+f(h_\pm(\beta)+W_\pm, \beta)=0\,, & \zeta\in \boldsymbol{R}_\pm\,, \\ W_\pm(0)=\alpha-h_\pm(\beta)\,, \\ W_\pm(\pm\infty)=0\,, \end{cases} \tag{4.1$_\pm$}$$

where $\cdot=d/d\zeta$ and α is a fixed constant satisfying $\alpha\in(h_+(\beta), h_-(\beta))$. Here we assume that $a/k<\xi$ $(=(ae+b)^2/(4bke))$. For any $\beta\in(a/k, \xi)$, there exists some $h_0(\beta)\in(h_+(\beta), h_-(\beta))$ such that

$$\begin{cases} f(h_0(\beta), \beta)=0\,, \\ f(u, \beta)<0 & \text{for} \quad h_+(\beta)<u<h_0(\beta)\,, \\ f(u, \beta)>0 & \text{for} \quad h_0(\beta)<u<h_-(\beta)\,, \\ f_u(h_\pm(\beta), \beta)<0\,. \end{cases} \tag{4.2}$$

Lemma 4.1. *Consider the problem*

$$\begin{cases} \ddot{W}+c\dot{W}+f(W, \beta)=0\,, & \zeta\in\boldsymbol{R}\,, \\ W(\pm\infty)=h_\pm(\beta) \quad \textit{and} \quad W(0)=\alpha\,, \end{cases} \tag{4.3}$$

for any fixed $\beta\in(a/k, \xi)$. Then there exists $c_0(\beta)$ such that (4.3) *has a unique strictly monotone decreasing solution $W(\zeta, c_0(\beta), \beta)$ satisfying*

$$|W(\zeta, c_0(\beta), \beta)-h_\pm(\beta)|\in X^2_{\tau_{0\pm}(\beta)} \qquad \textit{for} \quad \zeta\in\boldsymbol{R}_\pm\,,$$

where

$$\tau_{0\pm}(\beta)=\frac{1}{2}[c_0(\beta)\pm\{c_0(\beta)^2-4f_u(h_\pm(\beta), \beta)\}^{1/2}]$$

and

$$\operatorname{sign}(c_0(\beta))=\operatorname{sign}\Big(\int_{h_+}^{h_-}f(s, \beta)ds\Big)\,.$$

The proof is seen in, for example, Fife and McLeod [9].

(A.4) $$\beta^*\in\left(\frac{a}{k},\ \xi\right).$$

Remark. (A.4) is satisfied if $k/b>3$ and $e\gg 1$.

Lemma 4.2. *Let* c^* *and* $\tau_\pm(c,\ \beta)$ *be*

$$c^*=c_0(\beta^*) \quad \text{and} \quad \tau_\pm(c,\ \beta)=\frac{1}{2}[c\pm\{c^2-4f_u(h_\pm(\beta),\ \beta)\}^{1/2}].$$

Under (A.1)$\sim$(A.4), *there exists* $\delta>0$ *such that for any fixed* $(c,\ \beta)\in\Lambda_\delta \equiv\{(c,\ \beta)\,|\,|c-c^*|+|\beta-\beta^*|\leqq\delta\}$, $(4.1)_\pm$ *have unique strictly monotone decreasing solutions* $W_\pm(\zeta,\ c,\ \beta)$ *satisfying*

$$|W_\pm(\zeta,\ c,\ \beta)-h_\pm(\beta)|\in X^2_{\bar\tau_\pm}(\boldsymbol{R}),$$

where $\bar\tau_+=\inf_{(c,\beta)\in\Lambda_\delta}\tau_+(c,\ \beta)$ *and* $\bar\tau_-=\sup_{(c,\beta)\in\Lambda_\delta}\tau_-(c,\ \beta)$. *Furthemore,* $W_\pm(\zeta,\ c,\ \beta)$ are *continuous with respect to* $(c,\ \beta)\in\Lambda_\delta$ *in the* $X^2_{\bar\tau_\pm}$*-topology and*

$$\left[\frac{\partial}{\partial c}\left(\frac{dW_+}{d\zeta}(0,\ c,\ \beta)\right)-\frac{\partial}{\partial c}\left(\frac{dW_-}{d\zeta}(0,\ c,\ \beta)\right)\right]_{\substack{c=c^*\\ \beta=\beta^*}}\neq 0. \tag{4.5}$$

The proof is delegated to Appendices.

5. The existence of solutions in half lines $R_\pm$.

In this section, we consider the following problems

$$\begin{aligned}\varepsilon^2U''_\pm+\varepsilon cU'_\pm+f(U_\pm,\ V_\pm)=0\\ V''_\pm+\varepsilon cV'_\pm+g(U_\pm,\ V_\pm)=0\end{aligned},\qquad z\in\boldsymbol{R}_\pm, \tag{5.1$_\pm$}$$

$$\begin{aligned}&U_\pm(0)=\alpha, && V_\pm(0)=\beta,\\ &U_\pm(\pm\infty)=h_\pm(v_\pm), && V_\pm(\pm\infty)=v_\pm.\end{aligned} \tag{5.2$_\pm$}$$

Here we assume that $(c,\ \beta)$ is close to $(c^*,\ \beta^*)$. We seek solutions $(U_\pm(z),\ V_\pm(z))$ of $(5.1)_\pm$ and $(5.2)_\pm$ in the form

$$\begin{aligned}U_\pm(z,\ \varepsilon,\ c,\ \beta)&=U^0_\pm(z,\ \beta)+W_\pm(\zeta,\ c,\ \beta)+r_\pm(z,\ \varepsilon,\ c,\ \beta)\\ V_\pm(z,\ \varepsilon,\ c,\ \beta)&=V^0_\pm(z,\ \beta)+\varepsilon^2Y_\pm(\zeta,\ \varepsilon,\ c,\ \beta)+s_\pm(z,\ \varepsilon,\ c,\ \beta)\end{aligned},\qquad z\in\boldsymbol{R}_\pm. \tag{5.3}$$

Here $Y_\pm$ are defined by

$$Y_\pm(\zeta,\ \varepsilon,\ c,\ \beta)=Y_{1\pm}(\zeta,\ c,\ \beta)-Y_{1\pm}(0,\ c,\ \beta)e^{\mp\tilde\mu\varepsilon\zeta}, \tag{5.4}$$

where

$$Y_{1\pm}(\zeta,\ c,\ \beta)=-\int_\zeta^{\pm\infty}\int_\eta^{\pm\infty}[g(h_\pm(\beta)+W_\pm(\eta_1,\ c,\ \beta),\ \beta)-g(h_\pm(\beta),\ \beta)]d\eta_1 d\eta$$

for arbitrarily fixed $\tilde\mu$ ($\geqq\mu_\pm$). It is noted that

$$Y_{\pm}(0, \varepsilon, c, \beta)=0 \quad \text{and} \quad Y_{1\pm}\in X^2_{\tau_\pm}(\boldsymbol{R}_\pm).$$

In the following, we discuss the case of (U_+, V_+) only, because (U_-, V_-) can be treated in the almost same way. Therefore we omit the subindex + without confusion.

Put $t=(r, s)$ and rewrite $(5.1)_+$ and $(5.2)_+$ as

$$T(t, \varepsilon, \lambda)=\begin{pmatrix}\varepsilon^2 r''+c\varepsilon r'+f_u r+f_v s+N_1(r, s)+F_1\\ s''+c\varepsilon s'+g_u r+g_v s+N_2(r, s)+F_2\end{pmatrix}=0, \qquad z\in\boldsymbol{R}_+, \tag{5.5}$$

and

$$t(0, \varepsilon, \lambda)=t(+\infty, \varepsilon, \lambda)=0, \tag{5.6}$$

where $\lambda=(\beta, c)$, $f_u=\partial f/\partial u(U^0+W, V_0+\varepsilon^2 Y)$, f_v, g_u and g_v are defined similarly, N_1 and N_2 are higher order terms with respect to t and F_1 and F_2 are represented by

$$\begin{cases} F_1=\varepsilon^2 U^{0\prime\prime}+c\varepsilon U^{0\prime}+\ddot{W}+c\dot{W}+f(U^0+W, V^0+\varepsilon^2 Y)\\ F_2=V^{0\prime\prime}+c\varepsilon V^{0\prime}+\ddot{Y}+c\varepsilon^2\dot{Y}+g(U^0+W, V+\varepsilon^2 Y)\end{cases}, \qquad z\in\boldsymbol{R}_+. \tag{5.7}$$

Lemma 5.1. *There exist some $\varepsilon_0>0$ and $\delta_0>0$ such that for any $\varepsilon\in(0, \varepsilon_0)$ and $\lambda\in\Lambda_{\delta_0}$ it holds that*

$$\|F_i\|_{X\mu_+}\leqq K_i\varepsilon|\log\varepsilon| \qquad (i=1, 2), \tag{5.8}$$

where K_i is a constant independent of ε and λ $(i=1, 2)$.

For the study of (5.5) and (5.6), we introduce two Banach spaces

$$\mathring{X}_\varepsilon(\boldsymbol{R}_+)=\mathring{X}^2_{\rho,\varepsilon}(\boldsymbol{R}_+)\times\mathring{X}^2_\rho(\boldsymbol{R}_+) \quad \text{and} \quad Y(\boldsymbol{R}_+)=X_\rho(\boldsymbol{R}_+)\times X_\rho(\boldsymbol{R}_+).$$

Here ρ is an arbitrarily fixed constant satisfying $0<\rho<\mu$ $(=\min(\mu_+, \mu_-))$.

We define $T(t, \varepsilon, \lambda)$ by a mapping from $\mathring{X}_\varepsilon(\boldsymbol{R}_+)$ into $Y(\boldsymbol{R}_+)$.

Lemma 5.2. *Define a linear operator M_ε by*

$$M_\varepsilon\equiv\frac{d^2}{dz^2}+c\varepsilon\frac{d}{dz}+g_v(U^0+W, V^0+\varepsilon^2 Y).$$

Suppose that M_ε is a mapping from $\mathring{X}^2_\rho(\boldsymbol{R}_+)$ into $X_\rho(\boldsymbol{R}_+)$. Then there exist $\varepsilon_M>0$ and $\delta_M>0$ such that M_ε has an inverse bounded uniformly in $\varepsilon\in(0, \varepsilon_M)$ and $\lambda\in\Lambda_{\delta_M}$.

Lemma 5.3. *Define a linear operator L_ε by*

$$L_\varepsilon\equiv\varepsilon^2\frac{d^2}{dz^2}+c\varepsilon\frac{d}{dz}+f_u(U^0+W, V^0+\varepsilon^2 Y).$$

Suppose that L_ε is a mapping from $\mathring{X}^2_{\rho,\varepsilon}(\boldsymbol{R}_+)$ into $X_\rho(\boldsymbol{R}_+)$. Then under (A.1)~(A.4), *there exist $\varepsilon_L>0$ and $\delta_L>0$ such that L_ε has an inverse bounded uniformly in $\varepsilon\in(0, \varepsilon_L)$ and $\lambda\in\Lambda_{\delta_L}$.*

The proofs of Lemmas 5.1~5.3 are delegated to Appendices. From Lemmas 5.2 and 5.3, it follows that

Lemma 5.4. *There exists* $\varepsilon_T>0$ *such that for any* $\varepsilon\in(0, \varepsilon_T)$ $(\varepsilon_T=\min(\varepsilon_M, \varepsilon_L))$ *and* $\lambda\in\Lambda_{\delta_T}$ $(\delta_T=\min(\delta_M, \delta_L))$, $T(t, \varepsilon, \lambda)$ *has the following properties:*

(i) *There exists* $K_1>0$ *independent of* ε *and* λ *such that*

$$\|T_t(t_1, \varepsilon, \lambda)-T_t(t_2, \varepsilon, \lambda)\|_{\mathring{X}_\varepsilon\to Y}\leqq K_1\|t_1-t_2\|_{\mathring{X}_\varepsilon}$$

for any $t_1, t_2\in\mathring{X}_\varepsilon$, *where* T_t *is the Frechét derivative of* T *with respect to* t.

(ii) *For sufficiently small* $\sigma_+=\sup_{z\in R_+} g_u(U^0(z, \beta^*), V^0(z, \beta^*))$, $T_t(0, \varepsilon, \lambda)$ *has an inverse bounded uniformly in* ε *and* λ.

(iii) *There exists* $K_2>0$ *independent of* ε *and* λ *such that*

$$\|T(0, \varepsilon, \lambda)\|_Y\leqq K_2\varepsilon|\log\varepsilon|,$$

where $\mathring{X}_\varepsilon=\mathring{X}_\varepsilon(\boldsymbol{R}_+)$ *and* $Y=Y(\boldsymbol{R}_+)$.

Proof. (i) is obvious and (iii) is a direct consequence of Lemma 5.1. We show (ii) in the similar way to the proof in [Lemma 15, 14]. Let us consider the linear problem

$$T_t(0, \varepsilon, \lambda)t=\begin{pmatrix} L_\varepsilon & f_v(U^0+W, V^0+\varepsilon^2Y) \\ g_u(U^0+W, V^0+\varepsilon^2Y) & M_\varepsilon \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}=F \tag{5.9}$$

for $F={}^t(F_r, F_s)\in Y(\boldsymbol{R}_+)$. By the invertibilities of M_ε and L_ε (Lemmas 5.2 and 5.3), (5.9) is reduced to

$$\begin{cases} r=-L_\varepsilon^{-1}(f_vs-F_r) & (5.10) \\ s=-M_\varepsilon^{-1}(g_ur-F_s)\,. & (5.11) \end{cases}$$

Substituting (5.10) into (5.11), we have the integral equation for s:

$$s=M_\varepsilon^{-1}g_uL_\varepsilon^{-1}f_vs+M_\varepsilon^{-1}(F_s-g_uL_\varepsilon^{-1}F_r)\,. \tag{5.12}$$

Now we examine the operator $\Omega_\varepsilon\equiv M_\varepsilon^{-1}g_uL_\varepsilon^{-1}f_v$ which is written as

$$\begin{aligned} \Omega_\varepsilon s&=M_\varepsilon^{-1}g_u(U^0, V^0)L_\varepsilon^{-1}f_vs+M_\varepsilon^{-1}\Delta g_uL_\varepsilon^{-1}f_vs\,, \\ &\equiv\Omega_{1\varepsilon}s+\Omega_{2\varepsilon}s\,, \end{aligned}$$

where $\Delta g_u\equiv g_u(U^0+W, V^0+\varepsilon^2Y)-g_u(U^0, V^0)$. It is easily found that $\Omega_{1\varepsilon}s$ satisfies

$$\|\Omega_{1\varepsilon}s\|_{X_\rho}\leqq K_M\cdot\sigma_+K_LK_f\|s\|_{X_\rho}\,, \tag{5.13}$$

where K_M and K_L are bounds of M_ε^{-1} and L_ε^{-1} respectively and

$$K_f=\sup_{z\in R_+}|f_u(U^0+W, V^0+\varepsilon^2Y)|.$$

We next estimate $\Omega_{2\varepsilon}s$ with the aid of the representation of M_ε^{-1} as

$$M_\varepsilon^{-1}w=\int_0^{+\infty}G_\varepsilon(z, \xi)w(\xi)d\xi\,, \tag{5.14}$$

since Lemma 5.2 implies the existence of such Green's kernel $G_\varepsilon(z, \xi)$ satisfying

$$|G_\varepsilon(z, \xi)| \leqq \begin{cases} c_1 e^{-\mu_\varepsilon^+(z-\xi)} & (0 \leqq \xi \leqq z) \\ c_2 e^{-\mu_\varepsilon^-(\xi-z)} & (z \leqq \xi < +\infty), \end{cases}$$

where c_1 and c_2 are some positive constants and

$$\mu_\varepsilon^\pm = \frac{1}{2}\left| -c\varepsilon \pm \sqrt{(c\varepsilon)^2 - g_v(h_+(v_+), v_+)} \right|,$$

(see Appendix 8.3). Since (5.14) is applied to $\Omega_{2\varepsilon}s$, it holds that

$$\|\Omega_{2\varepsilon}s\|_{X_\rho} \leqq \int_0^{+\infty} |G_\varepsilon(z, \varepsilon)\Delta g_u| e^{\rho(z-\xi)}(e^{\rho\xi}|L_\varepsilon^{-1}f_v s|)d\xi$$

$$\leqq \int_0^{+\infty} |G_\varepsilon(z, \varepsilon)||\Delta g_u| e^{\rho(z-\xi)} d\xi \|L_\varepsilon^{-1}f_v s\|_{X_\rho}.$$

Noting that

$$\begin{aligned} |\Delta g_u| &\leqq |g_{uu}(U^0+\theta W, V^0+\varepsilon^2\theta Y)||W| \\ &\quad + |g_{uv}(U^0+\theta W, V^0+\varepsilon^2\theta Y)||\varepsilon^2 Y| \\ &\leqq K_3(e^{-(\tau+/\varepsilon)z} + \varepsilon^2 e^{-\mu z}) \end{aligned}$$

for some positive K_3 and $0<\theta<1$, we have

$$\begin{aligned} \|\Omega_{2\varepsilon}s\|_{X_\rho} &\leqq K_3\Big[c_1\int_0^z e^{-(\mu_\varepsilon^+ +\rho)(\xi-z)}(e^{-(\tau+/\varepsilon)\xi} + \varepsilon^2 e^{-\mu\xi})d\xi \\ &\quad + c_2\int_z^{+\infty} e^{-(\mu_\varepsilon^- +\rho)(\xi-z)}(e^{-(\tau+/\varepsilon)\xi} + \varepsilon^2 e^{-\mu\xi})d\xi\Big] \|L_\varepsilon^{-1}f_v s\|_{X_\rho} \\ &\leqq \varepsilon K_4 K_L \cdot K_f \|s\|_{X_\rho} \end{aligned} \tag{5.15}$$

for some positive K_4 and any fixed $\rho (0<\rho\leqq\mu_\varepsilon^+)$. Thus, from (5.14) and (5.15), we know that

$$\|\Omega_\varepsilon s\|_{X_\rho} \leqq K_L \cdot K_f(K_M\sigma_+ + K_4\varepsilon)\|s\|_{X_\rho},$$

which shows that Ω_ε is a contracting mapping in X_ρ for any $\varepsilon \in (0, \varepsilon_T)$ if σ_+ and ε_T satisfy the condition

$$K_L \cdot K_f(K_M\sigma_+ + K_4\varepsilon_T) < 1. \tag{5.16}$$

Hence, under the assumption (5.16), (5.12) has a solution $s \in X_\rho$ and there exists some positive constant K_5 such that

$$\|s\|_{X_\rho} \leqq K_5 \|F\|_{Y_\rho}. \tag{5.17}$$

On the other hand, from (5.10) and (5.11), it holds that

$$\begin{cases} \|r\|_{\mathring{X}^2_{\rho,\varepsilon}} \leqq K_L(K_f\|s\|_{X_\rho} + \|F_r\|_{X_\rho}), \\ \|s\|_{\mathring{X}^2_\rho} \leqq K_M(K_g\|r\|_{X_\rho} + \|F_s\|_{X_\rho}), \end{cases}$$

where $K_g = \sup_{z \in R_+} |g_u(U^0+W, V^0+\varepsilon^2 Y)|$. These estimates combined with (5.17) lead to

$$\|t\|_{\mathring{X}_\varepsilon} \leqq K_T \|F\|_Y$$

for some positive constant K_T independent of $\varepsilon \in (0, \varepsilon_T)$ and $\lambda \in \Lambda_{\delta_1}$. Thus, the proof is completed.

Now, by the use of Lemma 5.4, we can apply the implicit function theorem (Fife [6]) to the problem (5.4), (5.5).

Theorem 5.5. *Suppose that* (A.1)$\sim$(A.4) *hold and that* σ_+ *is small enough. Then there exist* $\varepsilon_0>0$ *and* $\delta_0>0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *and* $\lambda \in \Lambda_{\delta_0}$, *there exists* $t(\varepsilon, \lambda) \in X_\varepsilon$ *satisfying*

(i) $T(t(\varepsilon, \lambda), \varepsilon, \lambda)=0$,

(ii) $\lim_{\varepsilon \downarrow 0} \|t(\varepsilon, \lambda)\|_{\mathring{X}_\varepsilon}=0$ *uniformly in* $\lambda \in \Lambda_{\delta_0}$

and

(iii) $t(\varepsilon, \lambda)$ *is uniformly continuous with respect to* ε *and* λ *in the* $\mathring{X}_\varepsilon$*-topology.*

Consequently, we found that $(5.1)_+$ and $(5.2)_+$ has a solution $(U_+(z, \varepsilon, c, \beta)$, $(V_+(z, \varepsilon, c, \beta))$ in $\boldsymbol{R}_+$ for any $\varepsilon \in (0, \varepsilon_0)$ and $(c, \beta) \in \Lambda_{\delta_0}$.

In the almost same way to the discussion on $(5.1)_+$ and $(5.2)_+$, we also know the existence of a solution $(U_-(z, \varepsilon, c, \beta), V_-(z, \varepsilon, c, \beta))$ of $(5.1)_-$ and $(5.2)_-$.

6. The existence of solutions in the entire line $\boldsymbol{R}$.

In this section, we intend to match (U_+, V_+) with (U_-, V_-) at $z=0$ in the C^1-sense, by choosing β and c appropriately. In order to do this, we define two functions Φ and Ψ by

$$\begin{cases} \Phi(\varepsilon, c, \beta)=\dfrac{d}{d\zeta}U_+(0, \varepsilon, c, \beta)-\dfrac{d}{d\zeta}U_-(0, \varepsilon, c, \beta) \\ \Psi(\varepsilon, c, \beta)=\left(\dfrac{d}{dz}V_+(0, \varepsilon, c, \beta)\right)^2-\left(\dfrac{d}{dz}V_-(0, \varepsilon, c, \beta)\right)^2. \end{cases} \tag{6.1}$$

Setting D as $D=\{(\varepsilon, c, \beta) \mid \varepsilon \in (0, \varepsilon_0), (\beta, c) \in \Lambda_{\delta_0}\}$ for sufficiently small ε_0 and δ_0, we know from Theorem 5.5 that $\Phi(\varepsilon, c, \beta)$ and $\Psi(\varepsilon, c, \beta)$ are uniformly continuious in D. Therefore, Φ and Ψ can be continuously extended in a way that they are defined in $\bar{D}$. From this extension, (ii) of Theorem 5.5 rewrites (6.1) for $\varepsilon=0$ as

$$\begin{cases} \Phi(0, c, \beta)=\dfrac{d}{d\zeta}W_+(0, c, \beta)-\dfrac{d}{d\zeta}W_-(0, c, \beta) \\ \Psi(0, c, \beta)=\left(\dfrac{d}{dz}V_+^0(0, \beta)\right)^2-\left(\dfrac{d}{dz}V_-^0(0, \beta)\right)^2. \end{cases} \tag{6.2}$$

Noting that

(i) $\Phi(0, c^*, \beta^*)=\Psi(0, c^*, \beta^*)=0$,

(ii) $\Phi(0, c, \beta^*)$ has an isolated zero $c=c^*$,

and

(iii) $\Psi(0, c, \beta)=2J(\beta)$ has an isolated zero $\beta=\beta^*$,

we can apply the implicit function theorem [Theorem 4.3, 6] to (6.1) and then we have

Lemma 6.1. *For sufficiently small $\varepsilon>0$, there exist $\beta(\varepsilon)$ and $c(\varepsilon)$ such that*

$$\Phi(\varepsilon, c(\varepsilon), \beta(\varepsilon))=\Psi(\varepsilon, c(\varepsilon), \beta(\varepsilon))=0$$

and

$$\lim_{\varepsilon\downarrow 0}\beta(\varepsilon)=\beta^*, \qquad \lim_{\varepsilon\downarrow 0}c(\varepsilon)=c^*.$$

Thus, this lemma directly leads to the main theorem.

Theorem 6.2. *Suppose that* (A.1)~(A.4) *hold and that* $\sigma=\min(\sigma_+, \sigma_-)$ *is fixed small enough. Then, for small enough ε, there exists a solution* $(U(z, c(\varepsilon)), V(z, c(\varepsilon)))$ *of the problem* (2.3) *and* (2.2), *satisfying*

$$\|U-(U^0+W)\|_{X^1_{\rho,\varepsilon}(R)}+\|V-V^0\|_{X^1_\rho(R)}\to 0 \quad as \quad \varepsilon\downarrow 0.$$

Moreover, the velocity $c(\varepsilon)$ satisfies

$$c(\varepsilon)\to c^* \quad as \quad \varepsilon\downarrow 0.$$

7. Numerical Simulations.

We have found the existence of an ε-family of traveling wave solutions $(U(z, \varepsilon), V(z, \varepsilon))$ of (1.7) (i.e., (2.1)) subject to boundary conditions (2.2). In this section, let us show some pictures of traveling wave solutions. The curves of $f=g=0$ for $a=4.0$, $b=1.0$, $k=4.0$ and $e=4.0$ are drawn in Fig. 3 where the dashed line is $v=\beta^*=1.18668$ and $\int_{h_+(\beta)}^{h_-(\beta)}f(u, \beta^*)du>0$. For these values of the parameters numerical simulations were carried out by the use of the usual explicit difference scheme for the initial value problems of (1.7). Fig. 4 shows that the piecewise linear initial distribution

$$u_0(x)=\begin{cases}4 & x<-1.5,\\ -\dfrac{4}{3}x+2 & -1.5<x<1.5,\\ 0 & x>1.5,\end{cases} \qquad v_0(x)=\begin{cases}0 & x<-1.5,\\ \dfrac{4}{3}x+2 & -1.5<x<1.5,\\ 4 & x>1.5,\end{cases}$$

generates a traveling wave for $\varepsilon^2=0.01$. In this case, the velocity of the front is computed as $c=0.2$ which is approximately of order ε. Another example is drawn in Fig. 5 where $\varepsilon^2=0.04$ and the piecewise linear initial data is

$$u_0(x)=\begin{cases}4 & x<-4,\\ -2x-4 & -4<x<-2,\\ 0 & x>-2,\end{cases} \qquad v_0(x)=\begin{cases}0 & x<3,\\ 2x-6 & 3<x<5,\\ 4 & x>5.\end{cases}$$

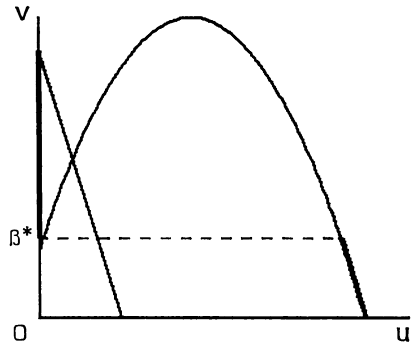

Fig. 3

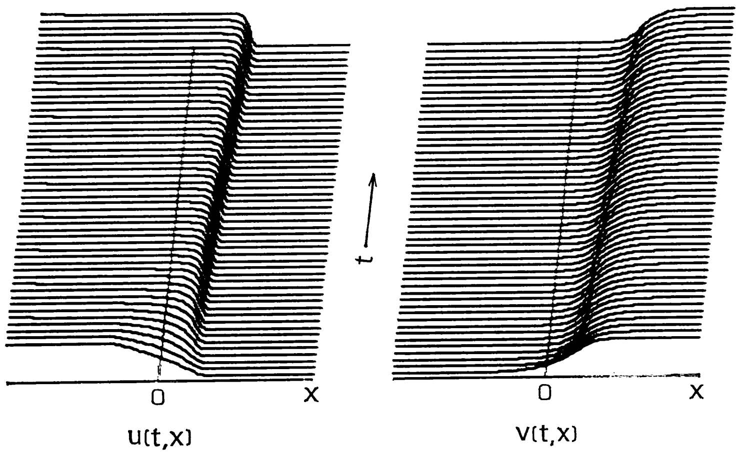

Fig. 4

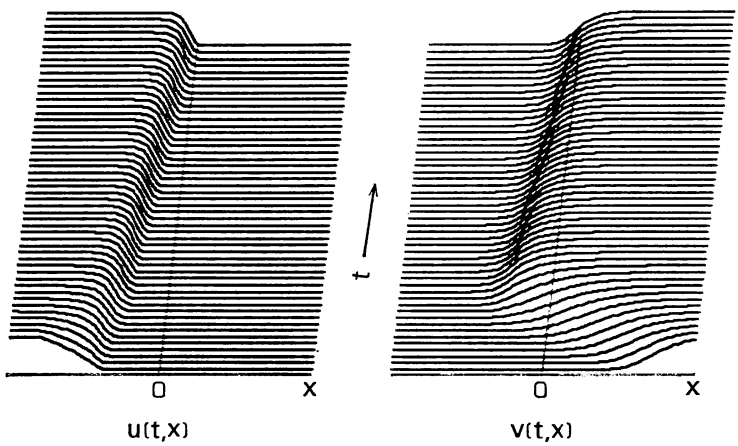

Fig. 5

This figure illustrates clearly that at the first stage, where the competitive interaction does not work, the fronts of U and V propagate independently with the same speed as that of Fisher's model and then, at the next stage where two species are encountered and compete, the fronts of U and V move together from the left to the right with the same speed, as predicted by our result.

8. Appendices.

8.1. The proof of Lemma 4.2.

We consider the case $(4.1)_+$ only. Define a nonlinear operator $R(W_+, c, \beta)$ by

$$R(W_+, c, \beta)=\frac{d^2}{d\zeta^2}W_+ + c\frac{d}{d\zeta}W_+ + f(h_+(\beta)+W_+, \beta) \tag{8.1}$$

and regard it as a mapping from $X^2_{\tau+}(\boldsymbol{R}_+)\times\Lambda_\delta$ into $X_{\tau+}(\boldsymbol{R}_+)$. We first note $R(W_+(\zeta, c^*, \beta^*), c^*, \beta^*)=0$, and that the Frechét derivative of R with respect to W_+, $R_W(W_+, c, \beta)$ is continuous in the neighborhood of $(W_+(\zeta, c^*, \beta^*), c^*, \beta^*)$. Let us show that the linear operator $R_W(W_+(\zeta, c^*, \beta^*), c^*\beta^*)$ mapping $\mathring{X}^2_{\tau+}$ into $X_{\tau+}$ is invertible. To do so, it is sufficient to prove the existence of a unique solution $w(\zeta)\in\mathring{X}^2_{\tau+}(\boldsymbol{R}_+)$ of

$$R_W(W_+(\zeta, c^*, \beta^*), c^*, \beta^*)w=k \tag{8.2}$$

for any $k\in X_{\tau+}$. Since $\phi_+(\zeta)=\frac{d}{d\zeta}W_+(\zeta, c^*, \beta^*)$ (<0) satisfies $R_W\cdot\phi_+=0$, we easily obtain a unique solution $w(\zeta)$ of (8.2) in the form

$$w(\zeta)=-\phi_+(\zeta)\int_0^\zeta\frac{e^{-c^*\eta}}{\phi_+(\eta)^2}\int_\eta^{+\infty}e^{c^*\xi}\phi_+(\xi)k(\xi)d\xi d\eta\,. \tag{8.3}$$

Here we note that $w(\zeta)\in\mathring{X}^2_{\tau+}(\boldsymbol{R}_+)$ for any $k(\zeta)\in X_{\tau+}$. Thus, by the use of the implicit function theorem, we know that there exists some δ such that $(4.1)_+$ has a solution $W_+(\zeta, c, \beta)$ for any fixed $(c, \beta)\in\Lambda_\delta$. We can also discuss the regularity of $W_+(\zeta, c, \beta)$ with respect to (c, β), since $R(W_+, c, \beta)$ is at least of the C^1-class. The monotonicity of $W_+(\zeta, c, \beta)$ can be easily shown by a phase plane analysis.

Remark. Using the general theory of ordinary differential equations, we can conclude that

$$W_+(\zeta, c, \beta)\in X^2_{\tau+(c,\beta)}(\boldsymbol{R}_+)\,.$$

(See, for example, Coddington and Levinson [2]).

We next show (4.5). Differentiating $R(W, c, \beta)=0$ with respect to c, we find that $W_c=\frac{\partial}{\partial c}W_+(\zeta, c, \beta)$ satisfies

$$R_W(W_\pm, c, \beta)W_c=-\frac{d}{d\zeta}W_+(\zeta, c, \beta) \tag{8.4}$$

so that W_c is explicitly represented by (8.3) when k is replaced by $-\frac{d}{d\zeta}W_+(\zeta, c, \beta)$, because $W_c(0)=0$. Differentiating it with respect to ζ and then putting $\zeta=0$ and $(c, \beta)=(c^*, \beta^*)$, we obtain

$$\frac{d}{d\zeta}W_c(0, c^*, \beta^*)=\frac{1}{\phi_+(0)}\int_0^{+\infty} e^{c^*\xi}\phi_+^2(\xi)d\xi . \tag{8.5}$$

On the other hand, it is easily proved that

$$\frac{d}{d\zeta}W_c(0, c^*, \beta^*)=\frac{\partial}{\partial c}\frac{d}{d\zeta}W_+(0, c^*, \beta^*) .$$

In the same way as the above, we also obtain

$$\frac{\partial}{\partial c}\frac{d}{d\zeta}W_-(0, c^*, \beta^*)=\frac{1}{\phi_-(0)}\int_0^{-\infty} e^{c^*\xi}\phi_-^2(\xi)d\xi . \tag{8.6}$$

Therefore it follows from (8.5) and (8.6) that

$$\frac{\partial}{\partial c}\frac{d}{d\zeta}W_+(0, c^*, \beta^*)-\frac{\partial}{\partial c}\frac{d}{d\zeta}W_-(0, c^*, \beta^*)=\frac{1}{\phi(0)}\int_{-\infty}^{\infty} e^{c^*\xi}\phi^2(\xi)d\xi\neq 0 ,$$

where $\phi(\zeta)=\frac{d}{d\zeta}W(\zeta, c^*, \beta^*)$. Thus, the proof is completed.

8.2. The proof of Lemma 5.1.

From $(3.6)_+$, $(4.1)_+$ and (5.4), F_1 and F_2 in (5.7) can be rewitten as

$$\begin{cases} F_1=\varepsilon^2 U^{0\prime\prime}+c\varepsilon U^{0\prime}+f(h(V^0)+W, V^0+\varepsilon^3 Y)-f(h(\beta)+W, \beta) , \\ F_2=c\varepsilon V^{0\prime}+c\varepsilon^2 Y-\varepsilon^2\mu^2 Y_1(0, c, \beta)e^{-\mu\varepsilon\zeta}-g(h(V^0), V^0) \\ \qquad -[g(h(\beta)+W, \beta)-g(h(\beta), \beta)]+g(h(V^0)+W, V^0+\varepsilon^2 Y) . \end{cases} \tag{8.7}$$

Now we divide $\boldsymbol{R}_+=\{z|z\geqq 0\}$ into $I_1^\varepsilon=[0, -A\varepsilon\log\varepsilon)$ and $I_2^\varepsilon=[-A\varepsilon\log\varepsilon, +\infty)$ for any fixed $A>0$ and estimate F_1 and F_2 on each interval. We know that

$$|F_1|\leqq\varepsilon^2|U^{0\prime\prime}|+c\varepsilon|U^{0\prime}|+\left|\frac{\partial\bar f}{\partial u}\frac{d\bar h}{dv}\frac{d\bar V^0}{dz}\cdot z+\frac{\partial\bar f}{\partial v}\left[\left(\frac{d\bar V^0}{dz}\right)z+\varepsilon^2 Y(\zeta)\right]\right|, \tag{8.8}$$

where

$$\frac{\partial\bar f}{\partial u}=\frac{\partial f}{\partial u}(h(V^0)+W+\theta_1(h(\beta)-h(V^0)), V^0+\varepsilon^2 Y) ,$$

$$\frac{\partial\bar f}{\partial v}=\frac{\partial f}{\partial v}(h(\beta)+W, V^0+\varepsilon^2 Y+\theta_2(\beta-V^0-\varepsilon^2 Y)) ,$$

$$\frac{d\bar h}{dV}=\frac{dh}{dV}(V^0(z)+\theta_3(\beta-V^0(z)))$$

and

$$\frac{d\bar V^0}{dz}=\frac{dV^0}{dz}(\theta_4 z)$$

for some $0<\theta_i<1$ $(i=1, \cdots, 4)$. Thus, (8.8) is estimated as

$$|F_1|\leqq\varepsilon^2|U^{0\prime\prime}|+c\varepsilon|U^{0\prime}|+K_3(z+\varepsilon^2K_4)$$
$$\leqq\varepsilon^2|U^{0\prime\prime}|+c\varepsilon|U^{0\prime}|+K_3\varepsilon(-A\log\varepsilon+\varepsilon K_4)\qquad\text{on}\quad I_1^{\varepsilon}$$

for some constants K_3 and K_4. Thus, it follows from $U^0\in X_{\mu+}^2$ that

$$|F_1|=\mathrm{O}(-A\varepsilon\log\varepsilon)\,.$$

On the other hand, it is obvious from the first of (8.7) that

$$|F_1|\leqq\varepsilon^2|U^{0\prime\prime}|+c\varepsilon|U^{0\prime}|+|f(h(V^0)+W,\ V^0+\varepsilon^2Y)$$
$$-f(h(V^0),\ V^0)+f(h(\beta),\ \beta)-f(h(\beta)+W,\ \beta)|$$
$$\leqq\varepsilon^2|U^{0\prime\prime}|+c\varepsilon|U^{0\prime}|+\left|\frac{\partial\bar{f}}{\partial u}\cdot W+\varepsilon^2\frac{\partial\bar{f}}{\partial v}Y\right|+\left|\frac{\partial\bar{\bar{f}}}{\partial u}W\right|,$$

where

$$\frac{\partial\bar{f}}{\partial u}=\frac{\partial f}{\partial u}(h(V^0)+\theta_5W,\ V^0+\varepsilon Y^2)\,,$$
$$\frac{\partial\bar{f}}{\partial v}=\frac{\partial f}{\partial v}(h(V^0),\ V_0+\theta_6\varepsilon^2Y^2)\,,$$
$$\frac{\partial\bar{\bar{f}}}{\partial u}=\frac{\partial f}{\partial u}(h(\beta)+\theta_7W,\ \beta)\,,$$

for some θ_i ($i=5\sim7$). Noting that

$$|W(\zeta)|\leqq c_1e^{-\tau+\zeta}\leqq c_1e^{A\tau+\log\varepsilon}\leqq c_1\varepsilon^{A\tau+}\qquad(\zeta/\varepsilon\in I_2^{\varepsilon})$$

for some c_1, we find that, choosing A sufficiently large as $A\geqq 1/\tau_+$,

$$|W(\zeta)|\leqq c_2\varepsilon e^{-\mu+z}\qquad(z\in I_2^{\varepsilon})$$

for some c_2. Then, by using $U',\,U''\in X_{\mu+}$, we obtain $|F_1|=\mathrm{O}(\varepsilon)$ on I_2^{ε}. Thus, we find

$$\|F_1\|_{X_{\mu+}(\boldsymbol{R}_+)}\leqq K_1\varepsilon|\log\varepsilon|$$

for some K_1. In the similar way to the above, we can prove (5.7) for F_2. The details were seen in Hosono and Mimura [14].

8.3. The proof Lemma 5.2.

For brevity we omit the index $+$ and write $\mathring{X}_\rho^2(\boldsymbol{R}_+)$ and $X_\rho(\boldsymbol{R}_+)$ as $\mathring{X}_\rho^2$ and X_ρ simply. For the proof, it is sufficient to show that a mapping from $\mathring{X}_\rho^2$ into X_ρ

$$M_\varepsilon^0=\frac{d^2}{dz^2}+c\varepsilon\frac{d}{dz}+g_v(U^0+W,\ V^0)$$

is invertible. Because, M_ε is rewritten as

$$M_\varepsilon=M_\varepsilon^0+(M_\varepsilon-M_\varepsilon^0)\,,$$

$(M_\varepsilon-M_\varepsilon^0)$ is regarded as a perturbation since $\|M_\varepsilon-M_\varepsilon^0\|_{\mathring{X}_\rho^2\to X_\rho}\leqq K\varepsilon^2$ for some K.

We first define M_0 by

$$M_0 \equiv \frac{d^2}{dz^2} + [g_v(U^0, V^0) + g_u(U^0, V^0)h'(V_0)]$$

which is a mapping from $\mathring{X}^2_{\rho'}$ into $X_{\rho'}$ for any fixed ρ' $(0 \leqq \rho' \leqq \mu)$.

Lemma 8.1. *Let β $(\in I_0)$ be fixed arbitrarily. Consider the problem*

$$M_0\psi = k_0 \qquad (z \in \boldsymbol{R}_+) \tag{8.9}$$

for any $k_0 \in X_{\rho'}$. Then M_0 is invertible.

Proof. It is easy to see that $\psi_1 = \dfrac{dV^0}{dz} \in X^2_{\mu_+}(\boldsymbol{R}_+)$ satisfies

$$M_0\psi_1 = 0 \quad \text{and} \quad \psi_1 > 0\,.$$

Then, by using $\psi_1(z)$ and

$$\psi_2(z) \equiv \psi_1(z)\int_0^z \frac{dy}{\psi_1(y)^2} \qquad (\in X^2_{\mu_+})\,,$$

the Green function $G(z, \xi)$ of M_0 can be explicitly written as

$$G(z, \xi) = \begin{cases} \psi_1(z)\psi_2(\xi) & (0 \leqq \xi < z)\,, \\ \psi_1(\xi)\psi_2(z) & (z \leqq \xi < +\infty)\,, \end{cases} \tag{8.10}$$

where

$$\begin{cases} |G(z, \xi)| \leqq c_1 e^{-\mu_+(z-\xi)} & (0 \leqq \xi \leqq z)\,, \\ |G(z, \xi)| \leqq c_2 e^{-\mu_+(\xi - z)} & (z \leqq \xi < +\infty)\,, \end{cases}$$

for some c_1 and c_2. Thus, a solution of (8.9) can be represented by

$$\psi(z) = M_0^{-1}k_0 \equiv \int_0^{+\infty} G(z, \xi)k_0(\xi)d\xi \qquad (\in \mathring{X}^2_{\rho'})\,,$$

which implies the invertibility of M_0. Thus, the proof is completed.

We next consider the problem

$$M^0_\varepsilon\phi = k \qquad (z \in \boldsymbol{R}_+)\,. \tag{8.11}$$

By the transformation of

$$\phi = e^{-(c\varepsilon/2)z}\tilde{\phi}\,, \tag{8.12}$$

(8.11) is reduced to

$$\tilde{M}^0_\varepsilon\tilde{\phi} \equiv \left[\frac{d^2}{dz^2} + \left\{g_v(U^0 + W, V^0) - \frac{(c\varepsilon)^2}{4}\right\}\right]\tilde{\phi} = \tilde{k}\,, \tag{8.13}$$

where $\tilde{k} = e^{(c\varepsilon/2)z}k$. Write $\tilde{M}^0_\varepsilon$ as

$$\tilde{M}^0_\varepsilon = M_0 + (\tilde{M}^0_\varepsilon - M_0)\,.$$

Then, it holds from Lemma 8.1 that for $\tilde{\phi} \in \mathring{X}^2_{\rho'}$ and $\tilde{k}_0 \in X_{\rho'}$ that

$$\tilde{\phi} = -M_0^{-1}(M^0_\varepsilon - M_0)\tilde{\phi} + M_0^{-1}k\,, \tag{8.14}$$

where

$$M_0^{-1}(\tilde{M}_\varepsilon^0-M_0)\tilde{\phi}=\int_0^{+\infty}G(z,\xi)\Big[g_v\Big(U^0(\xi)+W\Big(\frac{\xi}{\varepsilon}\Big),V^0(\xi)\Big)-g_v(U^0(\xi),V^0(\xi))$$
$$-g_u(U^0(\xi),V^0(\xi))\frac{dh}{dV}(V^0(\xi))-\frac{(c\varepsilon)^2}{4}\Big]\tilde{\phi}(\xi)d\xi. \tag{8.15}$$

By noting that $\dfrac{dh}{dV}\equiv 0$ in $\boldsymbol{R}_+$ and

$$\Big|g_v\Big(U^0(\xi)+W\Big(\frac{\xi}{\varepsilon}\Big),V^0(\xi)\Big)-g_v(U^0(\xi),V^0(\xi))\Big|\leqq c_3e^{-\tau+\xi/\varepsilon}$$

for some c_3, it follows from (8.10) that

$$\|M_0^{-1}(M_\varepsilon^0-M_0)\tilde{\phi}\|_{X_{\rho'}}$$
$$\leqq\int_0^{+\infty}|G(z,\xi)|c_3e^{-\tau+\xi/\varepsilon}e^{\rho'(z-\xi)}e^{\rho'\xi}|\tilde{\phi}(\xi)|d\xi+c_4\frac{(c\varepsilon)^2}{4}\|\tilde{\phi}\|_{X_{\rho'}}$$
$$\leqq\Big(c_5\varepsilon+c_4\frac{(c\varepsilon)^2}{4}\Big)\|\tilde{\phi}\|_{X_{\rho'}}$$

for some c_4 and c_5. Then (8.14) or (8.13) has a solution $\tilde{\phi}\in\mathring{X}^2_{\rho'}$ for any $k\in X_{\rho'}$ when ε is appropriately small, that is, there exists some c_6 such that

$$\|\tilde{\phi}\|_{\mathring{X}^2_{\rho'}}\leqq c_6\|\tilde{k}\|_{X_{\rho'}}.$$

Thus, by putting ρ' as

$$\rho'=\rho-\frac{c\varepsilon}{2},$$

(8.12) and (8.13) lead to

$$\|\phi\|_{\mathring{X}^2_\rho}\leqq c_6\|k\|_{X_\rho}. \tag{8.16}$$

Here (8.16) is valid for $0<\varepsilon<\varepsilon_M$ if ε_M is chosen as

$$\frac{\varepsilon_M}{2}(|c^*|+\delta_1)<\rho<\rho+\frac{\varepsilon_M}{2}(|c^*|+\delta_1)<\mu.$$

Thus, the proof is completed.

Remark. In the proof of (8.16), we used a special property, i.e. $\dfrac{dh_+}{dV}\equiv 0$. Since $\dfrac{dh_-}{dV}\not\equiv 0$ on $z\in\boldsymbol{R}_-$, the proof must be carried out under the assumption that $\sigma_-=\sup_{z\in\boldsymbol{R}_-}|g_u(U^0(z,\beta^*),V^0(z,\beta^*))|$ is sufficiently small in (8.15).

8.4. The proof of Lemma 5.3.

We define L^0 by

$$L_\varepsilon^0=\frac{d^2}{d\zeta^2}+c\frac{d}{d\zeta}+f_u(U^0(\varepsilon\zeta)+W(\zeta),V^0(\varepsilon\zeta)).$$

Here we write

$$f_u(U^0(\varepsilon\zeta)+W(\zeta),V^0(\varepsilon\zeta))=-(q_0+q_1+\gamma_0),$$

where

$$-q_0(\zeta)=f_u(U^0(0)+W(\zeta),\ V^0(0))-f_u(U^0(0),\ V^0(0)),$$

$$-q_1(\zeta,\ \varepsilon)=f_u(U^0(\varepsilon\zeta)+W(\zeta),\ V^0(\varepsilon\zeta))-f_u(U^0(0)+W(\zeta),\ V^0(0))$$

and

$$-\gamma_0=f_u(U_0(0),\ V_0(0))<0\,.$$

Lemma 8.2. *There exists* $\varepsilon_0>0$ *such that for any* $\varepsilon\in[0,\ \varepsilon_0)$,

(i) $-(q_1+\gamma_0)\equiv-\gamma_\varepsilon(\zeta)\leqq-\theta^2<0$,

(ii) $|q_1|\leqq K_1\varepsilon\zeta$ *and* $\left|\dfrac{d}{d\zeta}q_1\right|\leqq K_2\varepsilon$,

(iii) $|q_0|\leqq K_3e^{-\bar{\tau}_+\zeta}$,

where θ *and* K_i $(i=1,\ 2,\ 3)$ *are some positive constants independent of* ε *and* λ.

Proof. We first show (i). We divide $\boldsymbol{R}_+=\{\zeta\,|\,\zeta\geqq0\}$ into $I_1^\varepsilon=[0,\ -A\log\varepsilon)$ and $I_2^\varepsilon=[-A\log\varepsilon,\ +\infty)$, for any fixed $A>0$. Since

$$-q_1(\zeta,\ \varepsilon)=\Big(\bar{f}_{uu}\frac{d}{dz}\bar{U}^0+\bar{f}_{uv}\frac{d}{dz}\bar{V}^0\Big)\varepsilon\zeta\,, \tag{8.17}$$

where

$$\bar{f}_{uu}=f_{uu}(U^0(\varepsilon\zeta)+\theta_1(U^0(0)-U^0(\varepsilon\zeta))+W(\zeta),\ V^0(\varepsilon\zeta)),$$

$$\bar{f}_{uv}=f_{uv}(U^0(0)+W(\zeta),\ V^0(\varepsilon\zeta)+\theta_2(V^0(0)-V^0(\varepsilon\zeta))),$$

$$\frac{d}{dz}\bar{U}^0=\frac{d}{dz}U^0(\theta_3\varepsilon\zeta)\quad\text{and}\quad\frac{d}{dz}\bar{V}^0=\frac{d}{dz}V^0(\theta_4\varepsilon\zeta)$$

for some θ_i $(0<\theta_i<1,\ i=1\sim4)$, it turns out that

$$|q_1(\zeta,\ \varepsilon)|\leqq K_4\varepsilon|\log\varepsilon|\qquad\text{in}\quad I_1^\varepsilon \tag{8.18}$$

for some $K_4>0$. On the other hand, it follows from $W\in X_{\bar{\tau}_+}(\boldsymbol{R}_+)$ that

$$-q_1(\zeta,\ \varepsilon)\leqq f_u(U^0(\varepsilon\zeta),\ V^0(\varepsilon\zeta))-f_u(U^0(0),\ V^0(0))+K_5\varepsilon\qquad\text{in}\quad I_2^\varepsilon$$

for some $K_5>0$. Here we note that

$$f_u(U^0(\varepsilon\zeta),\ V^0(\varepsilon\zeta))-f_u(U^0(0),\ V^0(0))=\Big(\bar{f}_{uu}\frac{d\bar{h}_+}{dV}+\bar{f}_{uv}\Big)\frac{d\bar{V}^0}{dz}\cdot\varepsilon\zeta\,,$$

where

$$\bar{f}_{uu}=f_{uu}(U^0(\varepsilon\zeta)+\theta_5(U^0(0)-U^0(\varepsilon\zeta)),\ V^0(\varepsilon\zeta)),$$

$$\bar{f}_{uv}=f_{uv}(U^0(0),\ V^0(\varepsilon\zeta)+\theta_6(V^0(0)-V^0(\varepsilon\zeta))),$$

$$\frac{d\bar{h}_+}{dV}=\frac{dh_+}{dV}(V^0(0)+\theta_7(V^0(\varepsilon\zeta)-V^0(0)))\quad\text{and}\quad\frac{d\bar{V}}{dz}=\frac{dV}{dz}(\theta_8\varepsilon\zeta)$$

for some θ_i $(i=5\sim8)$. Therefore, by using

$$\frac{dh_+}{dV}\equiv0\,,\quad f_{uv}(u,\ v)=-\frac{b}{(1+eu)^2}<0\quad\text{and}\quad\frac{dV}{dz}>0\quad\text{in}\quad\boldsymbol{R}_+\,,$$

it is easy to see

$$-q_1(\zeta, \varepsilon) \leqq K_6 \varepsilon \qquad \text{in} \quad I_2^{\varepsilon} \tag{8.19}$$

for some $K_6>0$. Thus, (8.18) and (8.19) lead to (i) when ε is chosen sufficiently small. Differentiating $-q_1$ with respect to ζ, we have

$$\begin{aligned} -\frac{\partial q_1}{\partial \zeta} &= f_{uu}(U^0(\varepsilon\zeta)+W(\zeta),\ V^0(\varepsilon\zeta))\Big(\frac{dU^0}{dz}\cdot\varepsilon+\frac{dW}{d\zeta}\Big) \\ &\quad + f_{uv}(U^0(\varepsilon\zeta)+W(\zeta),\ V^0(\varepsilon\zeta))\frac{dV^0}{dz}\cdot\varepsilon \\ &\quad - f_{uu}(U^0(0)+W(\zeta),\ V^0(0))\frac{dW}{d\zeta}, \end{aligned}$$

and then

$$\begin{aligned} \Big|\frac{\partial q_1}{\partial \zeta}\Big| &\leqq K_7\varepsilon e^{-\mu_+\varepsilon\zeta} + \Big|\{f_{uu}(U^0(\varepsilon\zeta)+W(\zeta),\ V^0(\varepsilon\zeta)) \\ &\qquad - f_{uu}(U^0(0)+W(\zeta),\ V^0(0))\}\frac{dW}{d\zeta}\Big| \\ &\leqq K_7\varepsilon e^{-\mu_+\varepsilon\zeta} + K_8\varepsilon\zeta e^{-\tau_+\zeta} \\ &\leqq K_9\varepsilon \end{aligned}$$

for some $K_i>0$ ($i=7, 8, 9$), which implies the second of (ii). (iii) is obvious. Thus, Lemma 8.2 is proved.

Remark. For the proof of Lemma 8.2 in the case of $\boldsymbol{R}_-$, it is sufficient to show

$$\Big(f_{uu}\frac{dh_-}{dV}+f_{uv}\Big) \geqq 0\,. \tag{8.20}$$

If follows from an elementary calculation that

$$\begin{aligned} f_{uu}+f_{uv}\frac{dV}{dU} &= \frac{ea-b-4beU-2be^2U^2}{(1+eU)^2} \\ &\leqq -\frac{ea-b}{(1+eU)} < 0\,. \end{aligned}$$

Here we used $U>\dfrac{(ea-b)}{(2be)}>0$. Thus, by noting $\dfrac{dh_-}{dV}<0$, (8.20) can be proved.

Let us rewrite the problem

$$\begin{cases} L_\varepsilon^0 r = k & (\zeta \in R_+)\,, \\ r(0)=0\,, & r(+\infty)=0\,, \end{cases} \tag{8.21}$$

as

$$\begin{cases} L_\varepsilon \bar{r} = \Big\{\dfrac{d}{d\zeta}-(A_\varepsilon+B_0)\Big\}\bar{r}=\bar{k} & (\zeta\in R_+)\,, \\ r(0)=0\,, \quad r(+\infty)=0\,, \end{cases} \tag{8.22}$$

where $\bar{r}={}^t\Big(r,\ \dfrac{dr}{d\zeta}\Big)$,

$$A_\varepsilon(\zeta)=\begin{bmatrix} 0 & 1 \\ \gamma_\varepsilon(\zeta) & -c \end{bmatrix}, \qquad B_0(\zeta)=\begin{bmatrix} 0 & 0 \\ q_0(\zeta) & 0 \end{bmatrix}$$

and $\bar{k}={}^t(0, k)$. Since $A_\varepsilon(\zeta)$ has two real distinct eigenvalues

$$\lambda_\varepsilon^\pm(\zeta)=\frac{-c\pm\sqrt{c^2+\gamma_\varepsilon}}{2},$$

A_ε can be transformed into the diagonal form D_ε

$$P_\varepsilon^{-1}A_\varepsilon P_\varepsilon=D_\varepsilon=\begin{bmatrix} \lambda_\varepsilon^+ & 0 \\ 0 & \lambda_\varepsilon^- \end{bmatrix},$$

by using the regular matrix uniformly in ε and ζ

$$P_\varepsilon(\zeta)=\begin{bmatrix} 1 & 1 \\ \lambda_\varepsilon^+(\zeta) & \lambda_\varepsilon^-(\zeta) \end{bmatrix}.$$

Thus, by the change of the variable $\bar{r}=P_\varepsilon\bar{w}$ with $\bar{w}={}^t(w_1, w_2)$, (8.22) is reduced to the convenient first order system

$$\begin{cases} \tilde{L}_\varepsilon\bar{w}=\left\{\dfrac{d}{d\zeta}-D_\varepsilon-\tilde{B}_\varepsilon+C_\varepsilon\right\}\bar{w}=P_\varepsilon^{-1}\bar{k} \qquad (\zeta\in\boldsymbol{R}_+), \\ w_1(0)+w_2(0)=0, \quad w_1(+\infty)+w_2(+\infty)=0, \end{cases} \tag{8.23}$$

where $\tilde{B}_\varepsilon=P_\varepsilon^{-1}B_0P_\varepsilon$ and $C_\varepsilon=P_\varepsilon^{-1}\dfrac{dP_\varepsilon}{d\zeta}$. By setting $\varepsilon=0$ in (8.22) and (8.23), we define the operators $\bar{L}_0$ and $\tilde{L}_0$ by

$$\bar{L}_0=\frac{d}{d\zeta}-A_0-B_0 \quad\text{and}\quad \tilde{L}_0=\frac{d}{d\zeta}-D_0-\tilde{B}_0,$$

respectively. Here, let us introduce Banach spaces

$$\mathring{\bar{Y}}{}^1_{\rho,\varepsilon}\equiv\{\bar{w}\,|\,\bar{w}\in Y^1_{\rho,\varepsilon}(\boldsymbol{R}_+)\times Y^1_{\rho,\varepsilon}(\boldsymbol{R}_+),\ w_1(0)+w_2(0)=0\}$$

and

$$\bar{Y}_{\rho,\varepsilon}\equiv\{\bar{w}\,|\,\bar{w}\in Y_{\rho,\varepsilon}(\boldsymbol{R}_+)\times Y_{\rho,\varepsilon}(\boldsymbol{R}_+)\}.$$

Lemma 8.3. *Let $\tilde{L}_0$ be a linear mapping from $\mathring{\bar{Y}}{}^1_{\rho,\varepsilon}$ into $\bar{Y}_{\rho,\varepsilon}$ for any ε and any fixed ρ satisfying $0\leqq\varepsilon\leqq\varepsilon_0$ and $0\leqq\rho\leqq\mu$ respectively. There exists $\delta_0>0$ such that $\tilde{L}_0$ has an inverse bounded uniformly in $\lambda\in\Lambda_{\delta_0}$.*

Proof. Using the solution $\phi_+(\zeta)$ of $R_W\cdot\phi_+=0$ (in (8.1)), we define ϕ_i, Φ_i $(i=1, 2)$ and Φ by

$$\phi_1(\zeta)\equiv\phi_+(\zeta)\in X_{\tau_+(\boldsymbol{R}_+)},$$

$$\phi_2(\zeta)=\phi_1(\zeta)\int_0^\zeta e^{-c\eta}(\phi_1(\eta))^{-2}d\eta\in X_{-\tau_-(\boldsymbol{R}_+)},$$

$$\Phi_i(\zeta)={}^t\left(\phi_i(\zeta),\ \frac{d}{d\zeta}\phi_i(\zeta)\right) \qquad (i=1, 2)$$

and

$$\Phi(\zeta)=(\Phi_1(\zeta),\ \Phi_2(\zeta))\ .$$

Since $\Phi(\zeta)$ is a fundamental matrix of $\bar{L}_0$, a general solution $\bar{r}_0={}^t(r_{01},\ r_{02})$ of $\bar{L}_0\bar{r}_0=\bar{k}_0$ is represented by

$$\bar{r}_0(\zeta)=\Phi(\zeta)\Phi(0)^{-1}\bar{r}_0(0)+\int_0^{\zeta}\Phi(\zeta)\Phi^{-1}(\eta)\bar{k}_0(\eta)d\eta\ .$$

Let us define $\Psi(\zeta,\ \eta)$ by

$$\begin{aligned}\Psi(\zeta,\ \eta)&=\Phi(\zeta)\Phi^{-1}(\eta)\\ &=e^{c\eta}\begin{pmatrix}\phi_1(\zeta)\phi_2(\eta)-\phi_2(\zeta)\phi_1(\eta) & -\phi_1(\zeta)\phi_2(\eta)+\phi_2(\zeta)\phi_1(\eta)\\ \phi_1(\zeta)\phi_2(\eta)-\phi_2(\zeta)\phi_1(\eta) & -\phi_1(\zeta)\phi_2(\eta)+\phi_2(\zeta)\phi_1(\eta)\end{pmatrix}\end{aligned}$$

and decompose it into

$$\Psi(\zeta,\ \eta)=\Psi_1(\zeta,\ \eta)+\Psi_2(\zeta,\ \eta)\ ,$$

where

$$\Psi_1(\zeta,\ \eta)=e^{c\eta}\begin{pmatrix}\phi_1(\zeta)\phi_2(\eta) & -\phi_1(\zeta)\phi_2(\eta)\\ \phi_1(\zeta)\phi_2(\eta) & -\phi_1(\zeta)\phi_2(\eta)\end{pmatrix}$$

and

$$\Psi_2(\zeta,\ \eta)=e^{c\eta}\begin{pmatrix}-\phi_2(\zeta)\phi_1(\eta) & \phi_2(\zeta)\phi_1(\eta)\\ -\phi_2(\zeta)\phi_1(\eta) & \phi_2(\zeta)\phi_1(\eta)\end{pmatrix}.$$

Here, we note that

$$\begin{cases}|\Psi_1(\zeta,\ \eta)|\leqq c_1e^{-\tau_+(\zeta-\eta)} & (0\leqq\eta\leqq\zeta)\ ,\\ |\Psi_2(\zeta,\ \eta)|\leqq c_2e^{-\tau_-(\zeta-\eta)} & (\eta\geqq\zeta)\ ,\end{cases}$$

where $|\cdot|$ is an appropriate matrix norm.

Thus, a bounded solution of $\bar{L}_0\bar{r}_0=\bar{k}_0$ is represented by

$$\bar{r}_0(\zeta)=\frac{r_{01}(0)}{\phi_1(0)}\Phi_1(\zeta)+\int_0^{\zeta}\Psi_1(\zeta,\ \eta)k_0(\eta)d\eta-\int_{\zeta}^{+\infty}\Psi_2(\zeta,\ \eta)\bar{k}_0(\eta)d\eta\ . \tag{8.24}$$

From the expression (8.24), any solution $\bar{w}_0={}^t(w_1,\ w_2)$ of $\tilde{L}_0\bar{w}_0=\bar{k}$ in $\mathring{\bar{Y}}^1_{\rho,\varepsilon}(\boldsymbol{R}_+)$ is given uniquely by

$$\bar{w}_0(\zeta)=\int_0^{\zeta}P_0^{-1}(\zeta)\Psi_1(\zeta,\ \eta)\bar{k}(\eta)d\eta-\int_{\zeta}^{+\infty}P_0^{-1}(\zeta)\Psi_2(\zeta,\ \eta)\bar{k}(\eta)d\eta\ , \tag{8.25}$$

which completes the proof.

Next, we consider the main part $\tilde{L}^0_{\varepsilon}\equiv\dfrac{d}{d\zeta}-D_{\varepsilon}$ of $\tilde{L}_{\varepsilon}$. Let $\xi^{\pm}_{\varepsilon}(\zeta,\ \eta)$ be solutions of

$$\begin{cases}\dfrac{d\xi^{\pm}_{\varepsilon}}{d\zeta}=\lambda^{\pm}_{\varepsilon}\xi^{\pm}_{\varepsilon}\ ,\\ \xi^{\pm}_{\varepsilon}(\eta,\ \eta)=1\ ,\end{cases} \tag{8.26}$$

then, they are represented by

$$\xi_\varepsilon^\pm(\zeta, \eta)=\exp\left(\int_\eta^\zeta \lambda_\varepsilon^\pm(\eta')d\eta'\right). \tag{8.27}$$

Lemma 8.4. *Let* $\theta_\varepsilon^\pm(\zeta, \eta)$ *be* $\xi_\varepsilon^\pm(\zeta, \eta)-\xi_0^\pm(\zeta, \eta)$. *Then, there exist* ε_0 *and* δ_0 *such that the following estimates hold for any* $0\leqq\varepsilon\leqq\varepsilon_0$ *and* $(c, \beta)\in\Lambda_{\delta_0}$:

$$\left|\left(\frac{d}{d\zeta}\right)^j\xi_\varepsilon^+(\zeta, \eta)\right|\leqq c_1e^{-\lambda_0(\eta-\zeta)} \qquad (\zeta\leqq\eta<+\infty),$$

$$\left|\left(\frac{d}{d\zeta}\right)^j\xi_\varepsilon^-(\zeta, \eta)\right|\leqq c_2e^{-\lambda_0(\zeta-\eta)} \qquad (0\leqq\eta\leqq\zeta),$$

$$\left|\left(\frac{d}{d\zeta}\right)^j\theta_\varepsilon^+(\zeta, \eta)\right|\leqq c_3\varepsilon e^{-\lambda_0(\eta-\zeta)}(\eta^2-\zeta^2+\zeta) \quad (\zeta\leqq\eta<+\infty),$$

$$\left|\left(\frac{d}{d\zeta}\right)^j\theta_\varepsilon^-(\zeta, \eta)\right|\leqq c_4\varepsilon e^{-\lambda_0(\zeta-\eta)}(\zeta^2+\zeta-\eta^2) \quad (0\leqq\eta\leqq\zeta),$$

for $j=0, 1$, *where* c_i $(i=1, \cdots, 4)$ *are some constants independent of* ε, β *and* c *and*

$$\lambda_0=\inf_{(\beta, c)\in\Lambda_{\delta_0}}\left|\frac{1}{2}(-c+\sqrt{c^2+4\theta^2})\right|.$$

Proof. See, for instance, Hoppensteadt [13].

By the use of this lemma, the uniform invertibility of $\tilde{L}_\varepsilon^0$: $\mathring{\bar{Y}}_{\rho,\varepsilon}^1\to\bar{Y}_{\rho,\varepsilon}$ is easily verified. In fact, a solution of $\tilde{L}_\varepsilon^0\bar{w}=\bar{k}$ is represented by

$$\bar{w}(\zeta)=\nu'\xi_\varepsilon^-(\zeta, 0)e_2+\int_0^\zeta H_\varepsilon^-(\zeta, \eta)\bar{k}(\eta)d\eta-\int_\zeta^{+\infty}H_\varepsilon^+(\zeta, \eta)\bar{k}(\eta)d\eta$$

where $e_2={}^t(0, 1)$, $\bar{k}={}^t(k_1, k_2)$,

$$H_\varepsilon^+(\zeta, \eta)=\begin{pmatrix}\xi_\varepsilon^+(\zeta, \eta) & 0\\ 0 & 0\end{pmatrix}, \qquad H_\varepsilon^-(\zeta, \eta)=\begin{pmatrix}0 & 0\\ 0 & \xi_\varepsilon^-(\zeta, \eta)\end{pmatrix}$$

and ν' is an arbitrary constant. Setting $\zeta=0$ in the above representation, we have

$$\begin{pmatrix}w_1(0)\\ w_2(0)\end{pmatrix}=\begin{pmatrix}0\\ \nu'\end{pmatrix}-\int_0^{+\infty}\begin{pmatrix}\xi_\varepsilon^+(0, \eta)k_1(\eta)\\ 0\end{pmatrix}d\eta,$$

so that, by the condition $w_1(0)+w_2(0)=0$, ν' is uniquely determined as

$$\nu'=\int_0^{+\infty}\xi_\varepsilon^+(0, \eta)k_1(\eta)d\eta.$$

Hence, a solution $\bar{w}$ of $\tilde{L}_\varepsilon^0\bar{w}=\bar{k}$ in $\mathring{\bar{Y}}_{\rho,\varepsilon}^1$ is uniquely given by

$$\bar{w}(\zeta)=(\tilde{L}_\varepsilon^0)^{-1}\bar{k}=\xi_\varepsilon^-(\zeta, 0)\left(\int_0^{+\infty}\xi_\varepsilon^+(0, \eta)k_1(\eta)d\eta\right)e_2$$
$$+\int_0^\zeta H_\varepsilon^-(\zeta, \eta)\bar{k}(\eta)d\eta-\int_\zeta^{+\infty}H_\varepsilon^+(\zeta, \eta)\bar{k}(\eta)d\eta. \tag{8.28}$$

Since the estimates in Lemma 8.4 hold uniformly in ε, (8.28) is valid for $\varepsilon=0$. By the use of (8.28), the problem (8.23) is reduced to solving the integral equation

$$\bar{w}_\varepsilon(\zeta)=(\tilde{L}_\varepsilon^0)^{-1}\{(\tilde{B}_\varepsilon-C_\varepsilon)\bar{w}_\varepsilon+P_\varepsilon^{-1}\bar{k}\}$$
$$=(\tilde{L}_0^0)^{-1}\tilde{B}_0\bar{w}_\varepsilon+\{(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon-C_\varepsilon)-(\tilde{L}_0^0)^{-1}\tilde{B}_0\}\,\bar{w}_\varepsilon+(\tilde{L}_\varepsilon^0)^{-1}P_\varepsilon^{-1}\bar{k}\,.$$

Operating $\tilde{L}_0^0$ in the above, we have

$$\tilde{L}_0\bar{w}_\varepsilon=\tilde{L}_0^0\{(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon-C_\varepsilon)-(\tilde{L}_0^0)^{-1}\tilde{B}_0\}\,\bar{w}_\varepsilon+\tilde{L}_0^0(\tilde{L}_\varepsilon^0)^{-1}P_\varepsilon^{-1}\bar{k}\,. \tag{8.29}$$

Thus, using Lemma 8.3, we arrive at the integral equation

$$\bar{w}_\varepsilon=Q_\varepsilon\bar{w}_\varepsilon+\tilde{k}\,, \tag{8.30}$$

where $Q_\varepsilon\equiv\tilde{L}_0^{-1}\tilde{L}_0^0\{(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon-C_\varepsilon)-(\tilde{L}_0^0)^{-1}\tilde{B}_0\}$ is a linear operator in $\mathring{\bar{Y}}_{\rho,\varepsilon}\equiv\{\bar{w}\,|\,\bar{w}\in Y_{\rho,\varepsilon}\times Y_{\rho,\varepsilon},\ w_1(0)+w_2(0)=0\}$ and $\tilde{k}=\tilde{L}_0^{-1}\tilde{L}_0^0(\tilde{L}_\varepsilon^0)^{-1}P_\varepsilon^{-1}\bar{k}$.

Lemma 8.5. *Let ρ be any fixed constant satisfying $0\leqq\rho\leqq\mu$. Then, there exist positive constants ε_0 and δ_0 such that*

$$\|Q_\varepsilon\|_{\mathring{\bar{Y}}_{\rho,\varepsilon}\to\mathring{\bar{Y}}_{\rho,\varepsilon}}\leqq K\cdot\varepsilon \tag{8.31}$$

for $0\leqq\varepsilon\leqq\varepsilon_0$ and $(c,\beta)\in\Lambda_{\delta_0}$ where K is some constant independent of ε, β and c.

Proof. $\|\tilde{L}_0^{-1}\|_{\bar{Y}_{\rho,\varepsilon}\to\bar{Y}^1_{\rho,\varepsilon}}$ and $\|\tilde{L}_0^0\|_{\bar{Y}^1_{\rho,\varepsilon}\to\bar{Y}_{\rho,\varepsilon}}$ are uniformly bounded in ε, β and c, hence it is sufficient to show

$$\|(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon-C_\varepsilon)-(L_0^0)^{-1}\tilde{B}_0\|_{\bar{Y}_{\rho,\varepsilon}\to\bar{Y}^1_{\rho,\varepsilon}}$$
$$\leqq\|(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon-C_\varepsilon-\tilde{B}_0)\|_{\bar{Y}^1_{\rho,\varepsilon}\to\bar{Y}^1_{\rho,\varepsilon}}+\|((\tilde{L}_\varepsilon^0)^{-1}-(L_0^0)^{-1})\tilde{B}_0\|_{\bar{Y}_{\rho,\varepsilon}\to\bar{Y}^1_{\rho,\varepsilon}}=O(\varepsilon)\,.$$

From the uniform invertibility of $\tilde{L}_\varepsilon^0$, we have

$$\|Q_1\bar{w}\|_{\mathring{\bar{Y}}^1_{\rho,\varepsilon}}\equiv\|(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon-C_\varepsilon-\tilde{B}_0)\bar{w}\|_{\bar{Y}^1_{\rho,\varepsilon}}\leqq c_1\|(\tilde{B}_\varepsilon-C_\varepsilon-\tilde{B}_0)\bar{w}\|_{\bar{Y}_{\rho,\varepsilon}} \tag{8.32}$$

Since $\tilde{B}_\varepsilon-\tilde{B}_0$ can be written as

$$\tilde{B}_\varepsilon-\tilde{B}_0=P_\varepsilon^{-1}B_0P_\varepsilon-P_0^{-1}B_0P_0$$
$$=-P_0^{-1}(P_\varepsilon-P_0)P_\varepsilon^{-1}B_0P_\varepsilon+P_0^{-1}B_0(P_\varepsilon-P_0)\,,$$

is holds that

$$|\tilde{B}_\varepsilon-\tilde{B}_0|\leqq c_2|P_\varepsilon-P_0|\,|B_0|.$$

Applying Lemma 8.2 to

$$P_\varepsilon-P_0=\begin{pmatrix}0 & 0\\ \lambda_\varepsilon^+-\lambda_0^+ & \lambda_\varepsilon^--\lambda_0^-\end{pmatrix},\quad \frac{dP_\varepsilon}{d\zeta}=\begin{pmatrix}0 & 0\\ \dfrac{d\lambda_\varepsilon^+}{d\zeta} & \dfrac{d\lambda_\varepsilon^-}{d\zeta}\end{pmatrix}$$

and B_0, we find that

$$|\tilde{B}_\varepsilon-\tilde{B}_0|\leqq c_3|q_1|\,|q_0|=O(\varepsilon)$$

and

$$|C_\varepsilon|\leqq c_4\left|\frac{dP_\varepsilon}{d\zeta}\right|\leqq c_5\left|\frac{dq_1}{d\zeta}\right|=O(\varepsilon)\,,$$

so that

$$\|(\tilde{L}_\varepsilon^0)^{-1}(\tilde{B}_\varepsilon - C_\varepsilon - \tilde{B}_0)\bar{w}\|_{\mathring{Y}^1_{\rho,\varepsilon}} \leqq c_6 \varepsilon \|\bar{w}\|_{Y_{\rho,\varepsilon}},$$

where c_i ($k=1\sim 6$) are some positive constants.

Next, we consider

$$\|Q_2\bar{w}\|_{\mathring{Y}^1_{\rho,\varepsilon}} \equiv \|((\tilde{L}_\varepsilon^0)^{-1} - (L_0^0)^{-1})\tilde{B}_0\bar{w}\|_{\mathring{Y}^1_{\rho,\varepsilon}}.$$

From (8.28), $e^{\rho\varepsilon\zeta}\left(\frac{d}{d\zeta}\right)^j Q_2\bar{w}$ ($j=0, 1$) is written as

$$\begin{aligned}
e^{\rho\varepsilon\zeta}\left(\frac{d}{d\zeta}\right)^j Q_2\bar{w} = e^{\rho\varepsilon\zeta}\Big[&\left(\frac{\partial}{\partial\zeta}\right)^j \xi_\varepsilon^-(\zeta, 0)\int_0^{+\infty} \xi_\varepsilon^+(0, \eta)e^{-\rho\varepsilon\eta}(\tilde{B}_0(\eta)e^{\rho\varepsilon\eta}\bar{w}(\eta))_1 d\eta \\
&-\left(\frac{\partial}{\partial\zeta}\right)^j \xi_0^-(\zeta, 0)\int_0^{+\infty} \xi_0^+(0, \eta)e^{-\rho\varepsilon\eta}(\tilde{B}_0(\eta)e^{\rho\varepsilon\eta}\bar{w}(\eta))_1 d\eta\Big]e_2 \\
&+\int_0^\zeta \Big[\left(\frac{\partial}{\partial\zeta}\right)^j (H_\varepsilon^-(\zeta, \eta) - H_0^-(\zeta, \eta))\Big]\tilde{B}_0(\eta)e^{\rho\varepsilon(\zeta-\eta)}\cdot e^{\rho\varepsilon\eta}\bar{w}(\eta)d\eta \\
&-\int_\zeta^{+\infty} \Big[\left(\frac{\partial}{\partial\zeta}\right)^j (H_\varepsilon^+(\zeta, \eta) - H_0^+(\zeta, \eta))\Big]\tilde{B}_0(\eta)e^{\rho\varepsilon(\zeta-\eta)}\cdot e^{\rho\varepsilon\eta}\bar{w}(\eta)d\eta \\
\equiv\ & Q_{21} + Q_{22} - Q_{23}
\end{aligned}$$

where $(\cdot)_1$ denotes the first component of the vectors. Here, we used the fact $\xi_\varepsilon^\pm(\zeta, \zeta) = \xi_0^\pm(\zeta, \zeta) = 1$. Now, we estimate Q_{2i} ($i=1, 2, 3$) with the aid of Lemmas 8.2 and 8.4 in the following. First it is shown that

$$\begin{aligned}
|Q_{21}| \leqq\ & \left|e^{\rho\varepsilon\zeta}\left(\frac{\partial}{\partial\zeta}\right)^j \theta_\varepsilon^-(\zeta, 0)\int_0^{+\infty} \xi_\varepsilon^+(0, \eta)e^{-\rho\varepsilon\eta}(\tilde{B}_0(\eta)e^{\rho\varepsilon\eta}\bar{w}(\eta))_1 d\eta\right| \\
&+\left|e^{\rho\varepsilon\zeta}\left(\frac{\partial}{\partial\zeta}\right)^j \xi_0^-(\zeta, 0)\int_0^{+\infty} \theta_\varepsilon^+(0, \eta)e^{-\rho\varepsilon\eta}(\tilde{B}_0(\eta)e^{\rho\varepsilon\eta}\bar{w}(\eta))_1 d\eta\right| \\
\leqq\ & C_7\varepsilon e^{-(\lambda_0-\rho\varepsilon)\zeta}(\zeta^2+\zeta)\int_0^{+\infty} e^{-\lambda_0\eta}e^{-\rho\varepsilon\eta}\cdot e^{-\tau_+\eta}d\eta\|\bar{w}\|_{Y_{\rho,\varepsilon}} \\
&+C_8\varepsilon e^{-(\lambda_0-\rho\varepsilon)\zeta}\int_0^{+\infty} e^{-\lambda_0\eta}\eta^2\cdot e^{-\rho\varepsilon\eta}\cdot e^{-\tau_+\eta}d\eta\|\bar{w}\|_{Y_{\rho,\varepsilon}} \\
\leqq\ & c_9\varepsilon\|\bar{w}\|_{Y_{\rho,\varepsilon}}.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
|Q_{22}| \leqq\ & C_{10}\varepsilon\int_0^\zeta e^{-\lambda_0(\zeta-\eta)}(\zeta^2+\zeta-\eta^2)\cdot e^{-\tau_+\eta}\cdot e^{\rho\varepsilon(\zeta-\eta)}d\eta\|\bar{w}\|_{Y_{\rho,\varepsilon}} \\
\leqq\ & C_{10}\varepsilon(\zeta^2+\zeta)\cdot e^{-(\lambda_0-\rho\varepsilon)\zeta}\int_0^\zeta e^{(\lambda_0-\tau_+-\rho\varepsilon)\eta}d\eta\|\bar{w}\|_{Y_{\rho,\varepsilon}} \\
\leqq\ & C_{11}\varepsilon(\zeta^2+\zeta)\cdot e^{-(\lambda_0-\rho\varepsilon)\zeta}\cdot e^{(\lambda_0-\tau_+-\rho\varepsilon)\zeta}\|\bar{w}\|_{Y_{\rho,\varepsilon}} \\
\leqq\ & C_{11}\varepsilon(\zeta^2+\zeta)\cdot e^{-\tau_+\zeta}\|\bar{w}\|_{Y_{\rho,\varepsilon}} \leqq C_{12}\varepsilon\|\bar{w}\|_{Y_{\rho,\varepsilon}}.
\end{aligned}$$

Analogously, we know that

$$|Q_{23}| \leqq C_{13}\varepsilon\int_{\zeta}^{+\infty} e^{-\lambda_0(\eta-\zeta)}(\eta^2-\zeta^2+\zeta)e^{-\tau+\eta}e^{\rho\varepsilon(\zeta-\eta)}d\eta\|\bar{w}\|_{\bar{Y}_{\rho,\varepsilon}}$$

$$\leqq C_{13}\varepsilon\int_{\zeta}^{+\infty} e^{-(\lambda_0+\rho\varepsilon)(\eta-\zeta)}(\eta^2+\eta)\cdot e^{-\tau+\eta}d\eta\|\bar{w}\|_{\bar{Y}_{\rho,\varepsilon}}$$

$$\leqq C_{14}\varepsilon\|\bar{w}\|_{\bar{Y}_{\rho,\varepsilon}}.$$

Thus, these estimates lead to

$$\|Q_2\bar{w}\|_{\mathring{\bar{Y}}^1_{\rho,\varepsilon}} \leqq C_{15}\varepsilon\|\bar{w}\|_{\bar{Y}_{\rho,\varepsilon}}. \tag{8.33}$$

Here C_i ($i=7, \cdots, 15$) are some positive constants. (8.32) and (8.33) show Lemma 8.5.

Lemma 8.5 implies that Q_ε is a contracting mapping in $\mathring{\bar{Y}}_{\rho,\varepsilon}$ for sufficiently small ε, so that we conclude that there exists a unique solution $\bar{w}\in\mathring{\bar{Y}}_{\rho,\varepsilon}$ of (8.30). Therefore, the problem (8.22) has a unique solution $\bar{r}=P_\varepsilon\bar{w}$ satisfying

$$\|\bar{r}\|_{\mathring{\bar{Y}}^1_{\rho,\varepsilon}} \leqq c\|\bar{k}\|_{\bar{Y}_{\rho,\varepsilon}},$$

where c denotes some positive constant independent of ε, λ and ρ. Namely, L^0_ε: $\mathring{X}^2_{\rho,\varepsilon}(\boldsymbol{R}_+)\to X_\rho(\boldsymbol{R}_+)$ is invertible uniformly in ε, λ and ρ.

Since L_ε can be written as $L_\varepsilon=L^0_\varepsilon+f_u(U^0+W, V^0+\varepsilon^2 Y)-f_u(U^0+W, V^0)$, it is also shown that L_ε: $\mathring{X}^2_{\rho,\varepsilon}(\boldsymbol{R}_+)\to X_\rho(\boldsymbol{R}_+)$ has an inverse bounded uniformly in ε, λ and ρ. This completes the proof of Lemma 5.3.

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References

[1] G. A. Carpenter, A Geometric Approach to Singular Perturbation Problems with Applications to Nerve Impulse Equations, J. Diff. Eq. **23** (1977), 335-367.

[2] E. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York (1955).

[3] C. Conley, Isolated Invariant Sets and the Morse Index, C. B. M. S. Regional Conference Series in Math., 38, A. M. S., Providence, Rhode Island (1978).

[4] C. Conley and R. A. Gardner, An Application of the Generalized Morse Index to Travelling Wave Solutions of a Competition Model, to appear.

[5] R. J. Field and W. C. Troy, The Existence of Solitary Traveling Wave Solutions of a Model of the Belousov-Zhabotinskii Reaction, SIAM J. Appl. Math., **37** (1979), 561-587.

[6] P. C. Fife, Boundary and Interior Transition Layer Phenomena for Pairs of Second-Order Differential Equations, J. Math. Anal. Appl., **54** (1976), 497-521.

[7] P. C. Fife, Semilinear Elliptic Boundary Value Problems with Small Parameters, Arch. Rat. and Mech. Anal., **52** (1973), 205-232.

[8] P. C. Fife, Asymptotic Analysis of Reaction-Diffusion Wave Fronts, Rocky Mountain J. Math., **7** (1977), 389-415.

[9] P. C. Fife and J. B. McLeod, The approach of Solutions of Nonlinear Diffusion Equation to Travelling Wave Solutions, Arch. Rat. Mech. and Anal., **65** (1977) 335-361.

[10] R. A. Gardner, Existence and Stability of Travelling Wave Solutions of Competition Models: A Degree Theoretic Approach, J. Diff. Eqs., 44 (1982) 343-364.

[11] R. G. Gibbs, Travelling Waves in the Belousov-Zhabotinskii Reaction, SIAM J. Appl. Math., **38** (1980), 422-444.

[12] S. P. Hastings, Some Mathematical Problems from Neurobiology, Amer. Math. Monthly, **82** (1975), 881-895.

[13] F. Hoppensteadt, Properties of Solutions of Ordinary Differential Equation with Small Parameters, Comm. Pure Appl. Math., **24** (1971), 807-840.

[14] Y. Hosono and M. Mimura, Singular Perturbations for Pairs of Two-Point Boundary Value Problems of Neumann Type, Lecture Notes in Numer. Appl. Anal., **2** (1980), 79-138.

[15] J. D. Murray, On Travelling Wave Solutions in a Model for the Belousov-Zhobitinskii Reaction, J. Theor. Biol., **156** (1976), 329-353.

[16] M. M. Tang and P. C. Fife, Propagating Fronts for Competing Species Equations with Diffusion, Arch. Rat. Mech. and Anal., **73** (1980), 69-77.

[17] W. C. Troy, The existence of Traveling Wave Front Solutions of a Model of the Belousov-Zhabotinskii Chemical Reaction, J. Diff. Eqs., **36** (1980), 89-98.