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Hopf τ-spaces and τ-homotopy groups

By

Kouyemon IRIYE

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1. Introduction

Let G be a finite group. A pointed G-space X with a base point preserving equivariant map $m: X\times X\to X$ is said to be a Hopf G-space if the restriction $X\vee X\to X$ of m is equivariantly homotopic to $1\vee 1$ ([4], II-4).

In this paper we restrict ourselves to the case $G=\boldsymbol{Z}/2$. So by a Hopf τ-space we mean a Hopf $\boldsymbol{Z}/2$-space according to [3] and we use notation and elementary results of [1, 3] freely.

The purpose of this paper is to prove the followings.

Theorem 1.1. *The only non-trivial real representation spaces of* $\boldsymbol{Z}/2$ *whose unit spheres are Hopf* τ*-spaces are* $\boldsymbol{R}^{1,1}$, $\boldsymbol{R}^{2,2}$ *and* $\boldsymbol{R}^{4,4}$.

Theorem 1.2. *Up to* τ*-homotopy,* $S^{1,1}$ *admits only one multiplication. Up to* τ*-homotopy,* $S^{2,2}$ *and* $S^{4,4}$ *admit infinitely many multiplications.*

$\pi_{4,2}(\Sigma^{2,1})$ (resp. $\pi_{8,6}(\Sigma^{4,3})$) is in one-to-one correspondence with the set of the equivariant homotopy classes of the multiplications of $S^{2,2}$ (resp. $S^{4,4}$). Then we have

Theorem 1.3.

$$\psi: \pi_{4,2}(\Sigma^{2,1}) \longrightarrow \pi_6(S^3)$$

and

$$\psi\oplus\phi: \pi_{8,6}(\Sigma^{4,3}) \longrightarrow \pi_{14}(S^7)\oplus\pi_6(S^3)$$

are epimorphic.

In §2 we discuss the canonical multiplications of $S^{1,1}$, $S^{2,2}$ and $S^{4,4}$. Next we prove Theorem 1.1. In §3 we compute some τ-homotopy groups of $\Sigma^{r,s}$. Then, in §4, we prove Theorem 1.2. In §5 we prepare the equivariant version of the James' exact sequence to prove Theorem 1.3, which is proved in §6.

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2. Proof of Theorem 1.1.

We regard the complex numbers $\boldsymbol{C}$ as a τ-space by the conjugation and identify $\boldsymbol{C}=\boldsymbol{R}^{1,1}$ as usual. The product of the complex numbers gives rise to a τ-map $m_1: \boldsymbol{R}^{1,1}\times\boldsymbol{R}^{1,1}\to\boldsymbol{R}^{1,1}$. Restricting this τ-map to $S^{1,1}\times S^{1,1}$, we get a τ-map

$$\mu_1: S^{1,1}\times S^{1,1}\longrightarrow S^{1,1}. \tag{2.1}$$

Clearly $(0, 1)\in S^{1,1}$ is a two sided unit. Thus $S^{1,1}$ is a Hopf τ-space.

By the Hopf construction to μ_1 we obtain a τ-map

$$\hat{\eta}_{1,1}: \Sigma^{2,1}\approx_\tau S^{1,1}*S^{1,1}\longrightarrow \Sigma^{0,1}S^{1,1}\approx_\tau\Sigma^{1,1}.$$

Put

$$\hat{\eta}_{p,q}=\Sigma_*^{p-1,q-1}\hat{\eta}_{1,1}\in\pi_{p+1,q}(\Sigma^{p,q}) \tag{2.2}$$

for $p\geqslant 1$ and $q\geqslant 1$. Then we have that $\psi(\hat{\eta}_{p,q})=\eta_{p+q}\in\pi_{p+q+1}(S^{p+q})$ and $\phi(\hat{\eta}_{p,q})=2\iota_q\in\pi_q(S^q)$.

Identify the quaternions $\boldsymbol{H}$ with $\boldsymbol{R}^{2,2}$ by an involution $\tau(q)=-iqi$ for $q\in\boldsymbol{H}$. The product of the quaternions gives rise to a τ-map $m_2: \boldsymbol{R}^{2,2}\times\boldsymbol{R}^{2,2}\to\boldsymbol{R}^{2,2}$. Restricting this τ-map to $S^{2,2}\times S^{2,2}$, we get a τ-map

$$\mu_2: S^{2,2}\times S^{2,2}\longrightarrow S^{2,2}. \tag{2.3}$$

Thus $S^{2,2}$ is a Hopf τ-space with the canonical multiplication μ_2.

By the Hopf construction to μ_2 we obtain a τ-map

$$\hat{\nu}_{2,2}: \Sigma^{4,3}\longrightarrow\Sigma^{2,2}.$$

Put

$$\hat{\nu}_{p,q}=\Sigma_*^{p-2,q-2}\,\hat{\nu}_{2,2}\in\pi_{p+2,q+1}(\Sigma^{p,q}) \tag{2.4}$$

for $p\geqslant 2$ and $q\geqslant 2$. We see that $\psi(\hat{\nu}_{p,q})$ is a generator of $\pi_{p+q+3}(S^{p+q})$ and that $\phi(\hat{\nu}_{p,q})=\eta_q\in\pi_{q+1}(S^q)$.

Let $\mathfrak{C}$ be the Cayley numbers and $e_0=1, e_1,\dots, e_7$ the canonical basis of $\mathfrak{C}$ over $\boldsymbol{R}$. Identify the Cayley numbers with $\boldsymbol{R}^{4,4}$ by an involution τ defined by

$$\tau(\sum a_ie_i)=\sum_{0<i<3}a_ie_i-\sum_{4<i<7}a_ie_i.$$

The product of the Cayley numbers gives rise to a τ-map $m_4: \boldsymbol{R}^{4,4}\times\boldsymbol{R}^{4,4}\to\boldsymbol{R}^{4,4}$. Restricting this τ-map to $S^{4,4}\times S^{4,4}$, we get a τ-map

$$\mu_4: S^{4,4}\times S^{4,4}\longrightarrow S^{4,4}. \tag{2.5}$$

Thus $S^{4,4}$ is a Hopf τ-space with the canonical multiplication μ_4.

By the Hopf construction to μ_4 we get a τ-map

$$\hat{\sigma}_{4,4}: \Sigma^{8,7}\longrightarrow\Sigma^{4,4}.$$

Put

(2.6) $$\hat{\sigma}_{p,q}=\Sigma_*^{p-4,q-4}\hat{\sigma}_{4,4}\in\pi_{p+4,q+3}(\Sigma^{p,q})$$

for $p\geqslant 4$ and $q\geqslant 4$. We have that $\psi(\hat{\sigma}_{p,q})$ is a generator of $\pi_{p+q+7}(S^{p+q})$ and that $\phi(\hat{\sigma}_{p,q})$ is a generator of $\pi_{q+3}(S^q)$.

Now we prove Theorem 1.1.

Proof of Theorem 1.1. By the above discussion $S^{1,1}$, $S^{2,2}$ and $S^{4,4}$ are Hopf τ-spaces. If $S^{p,q}$ is a Hopf τ-space, then S^{p+q-1} and S^{q-1} are Hopf spaces. So it is sufficient to prove that $S^{3,1}$, $S^{7,1}$ and $S^{6,2}$ are not Hopf τ-spaces.

We assume that $S^{3,1}$ is a Hopf τ-space and μ: $S^{3,1}\times S^{3,1}\to S^{3,1}$ is its multiplication. The Hopf construction to μ gives a τ-map $H(\mu)$: $\Sigma^{6,1}\to\Sigma^{3,1}$. The element of $\pi_{3,0}^S$ represented by $H(\mu)$ is also denoted by the same letter. Then we see that $\psi(H(\mu))$ is a generator of π_3^S. But this contradicts the fact that $\psi(\pi_{3,0}^S)\neq\pi_3^S$ ([5], 11.3 or [1], 10.11). Thus $S^{3,1}$ is not a Hopf τ-space.

By the same method making use of the fact that $\psi(\pi_{7,0}^S)\neq\pi_7^S$ and $\psi(\pi_{6,1}^S)\neq\pi_7^S$ ([5], 11.4 or [1], 14.18), we conclude that $S^{7,1}$ and $S^{6,2}$ are not Hopf τ-spaces.

We finish this section with presenting an application of Theorem 1.1.

Let G be a finite group. A smooth G-manifold M is said to be equivariantly parallelizable if its tangent bundle TM is isomorphic to a product bundle $M\times V$ for some real G-module V.

Proposition 2.7. *The only equivariantly parallelizable spheres with linear involutions are* $S^{0,2}$, $S^{0,4}$, $S^{0,8}$, $S^{2,0}$, $S^{4,0}$, $S^{8,0}$, $S^{1,1}$, $S^{2,2}$, *and* $S^{4,4}$.

Proof. It is obvious that $S^{n,0}$ is equivariantly parallelizable if and only if $n=2$, 4 or 8.

If $S^{p,q}$, $q>0$, is equivariantly parallelizable, $TS^{p,q}$ is isomorphic to $S^{p,q}\times R^{p,q-1}$ since $\phi(TS^{p,q})=TS^{q-1}$ is isomorphic to $S^{q-1}\times R^{q-1}$. Then, by the parallel arguments to [6], I, 5.8, the assertion follows from Theorem 1.1.

3. τ-homotopy groups of spheres

Here we discuss some τ-homotopy groups of $\Sigma^{r,s}$. The basic tool of the computation is the forgetful exact sequence ([3], §12):

(3.1) $$\pi_{p+q}(S^{r+s})\xrightarrow{\delta^*}\pi_{p,q}(\Sigma^{r,s})\xrightarrow{\chi}\pi_{p-1,q}(\Sigma^{r,s})\xrightarrow{\psi}\pi_{p+q-1}(S^{r+s}).$$

The following is implicitly given in [3], §12.

Proposition 3.2. *Let* δ^*: $\pi_{p+q}(S^{r+s})\to\pi_{p,q}(\Sigma^{r,s})$ *be the homomorphism in the forgetful exact sequence* (3.1). *If* $p>0$ *we have*

$$\psi\circ\delta^*(\alpha)=\alpha+(-1)^p\{(-1)^r\iota_{r+s}\}\circ\alpha$$

for $\alpha\in\pi_{p+q}(S^{r+s})$, *where* ι_{r+s} *is the homotopy class of the identity mapping of* S^{r+s}.

By the routine arguments using the forgetful exact sequence as in [3], §12, we have the followings:

Proposition 3.3.
i) $\pi_{1,q}(\Sigma^{0,s})=0$ *for* $q\geqslant 0$.
ii) *For* $r>0$ *and* $q>0$, *we have an isomorphism*

$$\Phi\colon \pi_{1+q}(S^{r+s})\oplus\pi_q(S^s)\cong\pi_{1,q}(\Sigma^{r,s})$$

where $\Phi(\alpha,\beta)=\delta^*(\alpha)+\chi^{r-1}(\Sigma_*^{r,0}\beta)$ *for* $\alpha\in\pi_{1+q}(S^{r+s})$ *and* $\beta\in\pi_q(S^s)$.

Proposition 3.4. ([3], §12) *For* $p+q=r+s$, $p\equiv r \bmod 2$, $p>0$ *and* $q>0$, *we have an isomorphism*

$$\Phi\colon \boldsymbol{Z}\oplus\pi_q(S^s)\cong\pi_{p,q}(\Sigma^{r,s})$$

where $\Phi(n,\beta)=n\delta^*(\iota_{p+q})+\chi^{r-p}(\Sigma_*^{r,0}\beta)$ *for* $n\in\boldsymbol{Z}$ *and* $\beta\in\pi_q(S^s)$.

Proposition 3.5.
i) $\pi_{r+1,s-1}(\Sigma^{r,s})=0$ *for* $r\geqslant 0$ *and* $s>0$.
ii) *For* $p+q=r+s$, $p\not\equiv r \bmod 2$, $p\neq r+1$, $p>1$ *and* $q>0$, *we have an isomorphism*

$$\Phi\colon \boldsymbol{Z}/2\oplus\pi_q(S^s)\cong\pi_{p,q}(\Sigma^{r,s})$$

where $\Phi(n,\beta)=n\delta^*(\iota_{p+q})+\chi^{r-p}(\Sigma_*^{r,0}\beta)$ *for* $n\in\boldsymbol{Z}/2$ *and* $\beta\in\pi_q(S^s)$.

Next we compute $\pi_{p+1,q}(\Sigma^{p,q})$ *for* $p>0$ and $q>0$ for the latter use.

Proposition 3.6.
i) $\pi_{2,1}(\Sigma^{1,1})=\boldsymbol{Z}\cdot\mathring{\eta}_{1,1}\oplus\boldsymbol{Z}\cdot\rho\mathring{\eta}_{1,1}$, $\chi\mathring{\eta}_{1,1}=(1+\rho)\iota_{1,1}$ *and* $\delta^*(\eta_2)=(1-\rho)\mathring{\eta}_{1,1}$.
ii) $\pi_{p+1,q}(\Sigma^{p,q})=\boldsymbol{Z}\cdot\mathring{\eta}_{p,q}$ *and* $\rho\mathring{\eta}_{p,q}=\mathring{\eta}_{p,q}$ *for* $p\geqslant 2$, $q\geqslant 1$ *or* $p\geqslant 1$, $q\geqslant 2$. *Hence*

$$\phi\colon \pi_{p+1,q}(\Sigma^{p,q})\longrightarrow\pi_q(S^q)$$

is injective.

Proof. As for $\pi_{2,1}(\Sigma^{1,1})$ the proof is entirely parallel to the computation of $\pi_{p,q}(\Sigma^{p,q})$ and will be left to reders.

For $p\geqslant 2$ we consider the composition

$$\mathring{\eta}_{p,q}\colon \Sigma^{p,q+1}\xrightarrow{\delta_2}\Sigma^{p-2,q+2}S_+^{2,0}\xrightarrow{\omega_2}\Sigma^{p,q},$$

where δ_2 is the coboundary map in the τ-cofibre sequence $S_+^{2,0}\to B_+^{2,0}\to\Sigma^{2,0}\to\cdots$ and ω_2 is the periodicity element ([1], §3). Then we have $\psi(\mathring{\eta}_{p,q})=\eta_{p+q}$ and $\phi(\mathring{\eta}_{p,q})=0$ by [1], Theorem 3.5. Since $\psi(\mathring{\eta}_{p,q})=\eta_{p+q}$ if $p\geqslant 2$ and $\psi\Sigma_*^{p,0}(\eta_q)=\eta_{p+q}$ if $q\geqslant 2$, the forgetful exact sequence gives the isomorphism

$$\chi\colon \pi_{p+1,q}(\Sigma^{p,q})\cong Z\cdot(1+\rho)\iota_{p,q}$$

for $p\geqslant 2$, $q\geqslant 1$ or $p\geqslant 1$, $q\geqslant 2$, which implies the proposition.

4. Proof of Theorem 1.2.

By parallel arguments to the non-equivariant case (cf. [7], 3) we obtain

Proposition 4.1. *Let X be a Hopf G-space which is a G-connected G-complex (i.e. X^H is connected for all subgroups H of G), or which has an equivariantly homotopy associative and commutative multiplication. Then the set of equivariant homotopy classes of multiplications of X is in one-to-one correspondence with the equivariant homotopy set $[X \wedge X, X]^G$.*

Proof of Theorem 1.2. i) Since the canonical multiplication μ_1 of $S^{1,1} \approx_\tau \Sigma^{1,0}$ is equivariantly associative and commutative, we have only to show that $\pi_{2,0}(\Sigma^{1,0})=0$ by the above proposition. But it is easily proved by the routine arguments.

ii) Note that $S^{2,2} \approx_\tau \Sigma^{2,1}$ and $S^{4,4} \approx_\tau \Sigma^{4,3}$. By the above proposition as to the second statement of Theorem 1.2 it is sufficient to prove that

$$\pi_{4,2}(\Sigma^{2,1})_{(0)}=Q \quad \text{and} \quad \pi_{8,6}(\Sigma^{4,3})_{(0)}=Q,$$

where an index (0) denotes the localization at the prime ideal (0).

The forgetful exact sequence (3.1) and Propositions 3.2 and 3.3 give the isomorphisms

$$\pi_{4,2}(\Sigma^{2,1})_{(0)} \overset{\chi^3}{\underset{\to}{\approx}} \pi_{1,2}(\Sigma^{2,1})_{(0)}=Q,$$

and

$$\pi_{8,6}(\Sigma^{4,3})_{(0)} \overset{\chi^7}{\underset{\to}{\approx}} \pi_{1,6}(\Sigma^{4,3})_{(0)}=Q.$$

5. Equivariant James' exact sequence

Here we discuss the equivariant version of the James' exact sequence (cf. [8], (2.11)).

Denote by $(\Sigma^{r,s})_\infty$, $s>0$, the reduced product complex of $\Sigma^{r,s}$. $(\Sigma^{r,s})_\infty$ is a τ-complex and a free semi-group with the set $\Sigma^{r,s}-*$ of generators and the unit $*$. Each point of $(\Sigma^{r,s})_\infty$ is represented by a product $x_1 \cdots x_k$ of $x_1,\dots, x_k \in \Sigma^{r,s}$. For fixed positive integer k, $(\Sigma^{r,s})_k$ denotes the τ-subcomplex consisting of all elements $x_1 \cdots x_k$.

The canonical injection $i\colon \Sigma^{r,s} \to \Omega\Sigma^{r,s+1}$ is extended to the whole of $(\Sigma^{r,s})_\infty$, which is also denoted by $i\colon (\Sigma^{r,s})_\infty \to \Omega\Sigma^{r,s+1}$. Then i is a Hopf τ-map between Hopf τ-spaces.

Proposition 5.1. i: $(\Sigma^{r,s})_\infty \to \Omega\Sigma^{r,s+1}$ *for* $s>0$ *is a τ-homotopy equivalence.*

Proof. $\psi(\mathrm{i})\colon (S^{r+s})_\infty \to \Omega S^{r+s+1}$ and $\phi(\mathrm{i})\colon (S^s)_\infty \to \Omega S^{s+1}$ are homotopy equivalences by [8], Lemma 2.1. By [2], Corollary 2.6, $\Omega\Sigma^{r,s+1}$ has a homotopy type of τ-complex. Thus i is a τ-homotopy equivalence.

Define an isomorphism Ω_1 by

(5.2) $$\Omega_1 = i_*^{-1} \circ \Omega_0 : [\Sigma X, \Sigma^{r,s+1}]^\tau \cong [X, (\Sigma^{r,s})_\infty]^\tau,$$

where Ω_0: $[\Sigma X, \Sigma^{r,s+1}]^\tau \cong [X, \Omega \Sigma^{r,s+1}]^\tau$ is the usual isomorphism. Then for the inclution i': $\Sigma^{r,s} \subset (\Sigma^{r,s})_\infty$, the following diagram is commutative:

(5.3) $$\begin{array}{ccc} [X, \Sigma^{r,s}]^\tau & \xrightarrow{\Sigma_*} & [\Sigma X, \Sigma^{r,s+1}]^\tau \\ & {\scriptstyle i'_*} \searrow \quad \swarrow {\scriptstyle \Omega_1} & \\ & [X, (\Sigma^{r,s})_\infty]^\tau . & \end{array}$$

Let

(5.4) $$h_{r,s} : ((\Sigma^{r,s})_\infty, \Sigma^{r,s}) \longrightarrow ((\Sigma^{r,s} \wedge \Sigma^{r,s})_{\infty}, *)$$

be the combinatorial extension of $h'_{r,s}$: $((\Sigma^{r,s})_2, \Sigma^{r,s}) \rightarrow ((\Sigma^{r,s} \wedge \Sigma^{r,s})_\infty, *)$ which collapses $\Sigma^{r,s}$ to the base point $*$. Then $h_{r,s}$ is a τ-map.

Define a generalized τ-Hopf invariant H_τ by the formula:

(5.5) $$H_\tau = \Omega_1^{-1} \circ (T_\infty)_* \circ (h_{r,s})_* \circ \Omega_1 : [\Sigma X, \Sigma^{r,s+1}]^\tau \longrightarrow [\Sigma X, \Sigma^{2r,2s+1}]^\tau$$

where T_∞: $(\Sigma^{r,s} \wedge \Sigma^{r,s})_\infty \rightarrow (\Sigma^{2r,2s})_\infty$ is the τ-map induced by the switching map T: $\Sigma^{r,s} \wedge \Sigma^{r,s} \rightarrow \Sigma^{2r,2s}$. In particular, we have homomorphisms

$$H_\tau : \pi_{p,q+1}(\Sigma^{r,s+1}) \longrightarrow \pi_{p,q+1}(\Sigma^{2r,2s+1}) .$$

Proposition 5.6. *We have that*

$$(h_{r,s})_* : \pi_{p,q}((\Sigma^{r,s})_\infty, \Sigma^{r,s}) \longrightarrow \pi_{p,q}((\Sigma^{r,s} \wedge \Sigma^{r,s})_\infty)$$

are isomorphic for $p+q<3(r+s)-1$ *and* $1 \leqslant q < 3s-1$. *If* $r+s$ *and* s *are odd, then* $(h_{r,s})_*$ *are isomorphic for all* p *and* q.

Proof. By use of [3], Theorem 11.6, we have that $(h_{r,s})_*$ are isomorphic for $p+q<3(r+s)-1$ and $1 \leqslant q < 3s-1$.

If $r+s$ and s are odd, we see that $(h_{r,s})_*$ are isomorphic for all p and q by [3], Theorem 11.4 and [8], Theorem 2.4.

Next consider the exact sequence for the pair $((\Sigma^{r,s})_\infty, \Sigma^{r,s})$:

(5.7) $$\cdots \longrightarrow \pi_{p,q}(\Sigma^{r,s}) \longrightarrow \pi_{p,q}((\Sigma^{r,s})_\infty) \longrightarrow \pi_{p,q}((\Sigma^{r,s})_\infty, \Sigma^{r,s}) \xrightarrow{\partial} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow \cdots .$$

Define a homomorphism Δ_τ by

(5.8) $$\Delta_\tau = \partial \circ (T_\infty \circ h_{r,s})_*^{-1} \circ \Omega_1 : \pi_{p,q+1}(\Sigma^{2r,2s+1}) \longrightarrow \pi_{p,q-1}(\Sigma^{r,s}) / \partial \ker (h_{r,s})_*$$

where $\partial \ker (h_{r,s})_*$ is trivial if $r+s$ and s are odd or if $p+q<3(r+s)-1$ and $1 \leqslant q < 3s-1$.

Thus we have the exact sequence

(5.9) $$\cdots\longrightarrow \pi_{p,q}(\Sigma^{r,s}) \xrightarrow{\Sigma_*} \pi_{p,q+1}(\Sigma^{r,s+1}) \xrightarrow{H_\tau} \pi_{p,q+1}(\Sigma^{2r,2s+1}) \xrightarrow{\Delta_\tau} \pi_{p,q-1}(\Sigma^{r,s}) \longrightarrow\cdots$$

for $r+s$ and s are odd or $p+q<3(r+s)-1$ and $1\leqslant q<3s-1$. If $p>0$, H_τ and Δ_τ commute with the involution ρ, i.e.

(5.10) $$H_\tau\circ\rho=\rho\circ H_\tau \quad\text{and}\quad \Delta_\tau\circ\rho=\rho\circ\Delta_\tau$$

Moreover we have that

(5.11) $$\phi\circ H_\tau=H\circ\phi \quad\text{and}\quad \phi\circ\Delta_\tau=\Delta\circ\phi$$

and that

(5.12) $$\psi\circ H_\tau=H\circ\psi \quad\text{and}\quad \psi\circ\Delta_\tau=(-1)^p\Delta\circ\psi$$

if r is even.

6. Proof of Theorem 1.3.

Proof of Theorem 1.3.

i) By Proposition 3.2, we see that

$$\psi\circ\delta^*(\alpha)=2\alpha \quad\text{for}\quad \alpha\in\pi_6(S^3)$$

where $\delta^*\colon \pi_6(S^3)\to\pi_{4,2}(\Sigma^{2,1})$.

By Proposition 3.6, ii), the equivariant Toda bracket

$$\{\hat{\eta}_{2,1}, (1-\rho)\iota_{3,1}, \hat{\eta}_{3,1}\}^\tau\subset\pi_{4,2}(\Sigma^{2,1})$$

is well-defined and

$$\psi(\{\hat{\eta}_{2,1}, (1-\rho)\iota_{3,1}, \hat{\eta}_{3,1}\}^\tau)=\{\eta_3, 2\iota_4, \eta_4\}=\{\nu', -\nu'\}.$$

Thus $\psi\colon \pi_{4,2}(\Sigma^{2,1})\to\pi_6(S^3)$ is surjective.

ii) By definition of sq in [1], §4, for $\alpha\in\pi_6(S^3)$ we have

$$\tilde{\alpha}=\Sigma_*^{1,0}sq(\alpha)\circ\hat{\eta}_{7,6}\in\pi_{8,6}(\Sigma^{4,3}).$$

Then we have $\psi(\tilde{\alpha})=0$ and $\phi(\tilde{\alpha})=2\alpha$.

Since

$$\hat{\nu}_{4,3}\circ\hat{\eta}_{6,4}\in\pi_{7,4}(\Sigma^{4,3})\cong\pi_{3,1}^S \quad \text{(by [3], Theorem 11.12)}$$
$$=0 \quad \text{(by [1], Theorem 11.3)}$$

and

$$\hat{\eta}_{6,4}\circ\hat{\nu}_{7,4}\in\pi_{9,5}(\Sigma^{6,4})\cong\pi_{3,1}^S=0,$$

the equivariant Toda bracket

$$\{\hat{\nu}_{4,3}, \hat{\eta}_{6,4}, \hat{\nu}_{7,4}\}^\tau\subset\pi_{9,6}(\Sigma^{4,3})$$

is well-defined. We have

$$\chi\{\hat{\nu}_{4,3}, \hat{\eta}_{6,4}, \hat{\nu}_{7,4}\}^{\tau} \subset \pi_{8,6}(\Sigma^{4,3}),$$

$$\phi(\chi\{\hat{\nu}_{4,3}, \hat{\eta}_{6,4}, \hat{\nu}_{7,4}\}^{\tau}) = \{\eta_3, 2\iota_4, \eta_4\} = \{\nu', -\nu'\}$$

and

$$\psi(\chi\{\hat{\nu}_{4,3}, \hat{\eta}_{6,4}, \hat{\nu}_{7,4}\}^{\tau}) = 0.$$

Thus we obtain

(6.1) for each $\alpha \in \pi_6(S^3)$ there exists $\alpha' \in \pi_{8,6}(\Sigma^{4,3})$ such that $\psi(\alpha') = 0$ and $\phi(\alpha') = \alpha$.

To complete the proof, we have to prove that $\psi: \pi_{8,6}(\Sigma^{4,3}) \to \pi_{14}(S^7)$ is surjective. By Proposition 3.2, we have that

(6.2) $$\psi \circ \delta^*(\alpha) = 2\alpha \quad \text{for} \quad \alpha \in \pi_{14}(S^7),$$

where $\delta^*: \pi_{14}(S^7) \to \pi_{8,6}(\Sigma^{4,3})$. Therefore we restrict ourselves to the 2-primary component of $\pi_{14}(S^7)$ and we use Toda's notation of the 2-primary components of $\pi_{n+k}(S^n)$.

By (5.11), (5.12), [8], Proposition 2.7, we have that

$$\psi \circ H_{\tau} \circ \Delta_{\tau}(\iota_{8,9}) = 2\varepsilon_1 \iota_{15}, \quad \varepsilon_1 = \pm 1$$

and

$$\phi \circ H_{\tau} \circ \Delta_{\tau}(\iota_{8,9}) = 2\varepsilon_2 \iota_7, \quad \varepsilon_2 = \pm 1.$$

We have that $\psi \circ H_{\tau}(\hat{\sigma}_{4,4}) = \pm \iota_{15}$ and $\phi \circ H_{\tau}(\hat{\sigma}_{4,4}) = \pm \iota_7$. Hence, by replacing it with $\pm \hat{\sigma}_{4,4}$ or $\pm \rho \hat{\sigma}_{4,4}$ if necessary, there exists an element $\hat{\sigma}_{4,4} \in \pi_{8,7}(\Sigma^{4,4})$ such that $\psi \circ H_{\tau}(\hat{\sigma}_{4,4}) = \varepsilon_1 \iota_{15}$ and $\phi \circ H_{\tau}(\hat{\sigma}_{4,4}) = \varepsilon_2 \iota_7$.

Now we consider the exact sequence (5.9):

$$\cdots \longrightarrow \pi_{8,6}(\Sigma^{4,3}) \xrightarrow{\Sigma_*} \pi_{8,7}(\Sigma^{4,4}) \xrightarrow{H_{\tau}} \pi_{8,7}(\Sigma^{8,7}) \longrightarrow \cdots.$$

$2\hat{\sigma}_{4,4} - \Delta_{\tau}(\iota_{8,9}) \in \pi_{8,7}(\Sigma^{4,4})$ and $H_{\tau}(2\hat{\sigma}_{4,4} - \Delta_{\tau}(\iota_{8,9})) = 0$ since $\psi \oplus \phi: \pi_{8,7}(\Sigma^{8,7}) \to \boldsymbol{Z} \oplus \boldsymbol{Z}$ is isomorphic. Thus there exist $\alpha \in \pi_{8,6}(\Sigma^{4,3})$ such that

$$\Sigma_* \alpha = 2\hat{\sigma}_{4,4} - \Delta_{\tau}(\iota_{8,9}).$$

On the other hand

$$\psi(2\hat{\sigma}_{4,4} - \Delta_{\tau}(\iota_{8,9})) = \pm E\sigma' + 2\beta'$$

where $\beta' \in \pi_{15}(S^8)$ is a torsion element. Thus we have that

(6.3) $$\psi(\alpha) = \pm \sigma' + 2\beta$$

where $\beta \in \pi_{14}(S^7)$. By Proposition 3.2 and (6.3) there exists an element $\sigma'_{4,3} \in \pi_{8,6}(\Sigma^{4,3})$ such that

(6.4) $$\psi(\sigma'_{4,3}) = \pm \sigma',$$

which completes the proof.

DEPARTMENT OF MATHEMATICS,
OSAKA CITY UNIVERSITY

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