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On the algebraic K-group of lens spaces and its applications

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§1. Introduction

In this note we compute the algebraic K-group $\tilde{K}_{F_q}(BZ_p^{(n)})$ over a finite field F_q where $BZ_p^{(n)}$ is a lens space or its skeleton and p is an odd prime. We restrict ourself to the case that $q=l^a$ is a generator of $\hat{Z}_p^*$ the group of p-adic units. In this case $\tilde{K}_{F_q}(BZ_p^{(n)})$ turns out to be cyclic for n even, but if q is not a generator of $\hat{Z}_p^*$ then $\tilde{K}_{F_q}(BZ_p^{(n)})$ contains another summands which are not essential for our applications.

In [9], Tornehave has shown that $\tilde{K}_{F_q}(X)_{(p)}$ is a direct summand of the multiplicative group $1+\tilde{\pi}^0(X)_{(p)}$. We regard the Kahn-Priddy map $h: BZ_p \to QS^0$, [2], as an element of $\tilde{\pi}^0(BZ_p)$ or $\tilde{\pi}^0(BZ_p^{(n)})$. Then from the computation of $\tilde{K}_{F_q}(BZ_p^{(n)})$ one can determine the order of $h \in \tilde{\pi}^0(BZ_p^{(n)})$. Let P be a class of finite abelian groups. Suppose that there is a positive integer m such that $mA=0$ for all $A \in P$, then the exponent of P (written as $\exp(P)$) is defined as the smallest one of such integers. As an application of the Kahn-Priddy theorem we determine the exponent of groups $\tilde{\pi}^0(X)_{(p)}$, $\dim(X) \leq n$ and groups $\tilde{K}_{F_q}(X)_{(p)}$, $\dim(X) < n$ for given n, and we see that they are equal.

§2. Algebraic K-groups of lens spaces

Let l be a prime number and let F_q be a field with $q=l^a$ elements. Let $BGL(F_q)^+$ be the plus construction of Quillen [6] of $BGL(F_q)=\lim_{\to n} BGL(n, F_q)$. For a CW-complex X the algebraic K-group over F_q is defined by

$$\tilde{K}_{F_q}(X)=[X, BGL(F_q)^+].$$

Quillen has shown that the sequence

$$BGL(F_q)^+ \xrightarrow{b} BU \xrightarrow{\psi^q-1} BU$$

is a homotopy fibration where b is a map induced from the Brouer lifting. Therefore we have an exact sequence

(2.1) $$\tilde{K}^{-1}(X) \xrightarrow{\psi^q-1} \tilde{K}^{-1}(X) \xrightarrow{\delta} \tilde{K}_{F_q}(X) \xrightarrow{b_*} \tilde{K}(X) \xrightarrow{\psi^q-1} \tilde{K}(X).$$

Let p be an odd prime such that $(p, q)=1$. Let $BZ_p^{(2n+1)}$ denote the standard lens space S^{2n+1}/Z_p and $BZ_p^{(2n)}$ is the $2n$-skeleton. $BZ_p=\lim BZ_p^{(n)}$ is the classifying space for Z_p. Let λ: $Z_p \to GL(n, F_q)$ be a representation of Z_p over F_q. There is a canonical map $BGL(n, F_q) \to BGL(F_q)^+ \times Z$ and hence $B\lambda$: $BZ_p \to BGL(n, F_q)$ determine an element of $K_{F_q}(BZ_p)$ and we obtain a homomorphism

$$\alpha: R_{F_q}(Z_p) \longrightarrow K_{F_q}(BZ_p).$$

Similarly for complex representations we have a homomorphism α: $R(Z_p) \to K(BZ_p) = \varprojlim K(BZ_p^{(n)})$. Let $\chi \in R_{F_q}(Z_p)$, then $\alpha(\chi) \in K_{F_q}(BZ_p)$ and its restriction over $BZ_p^{(n)}$ are also denoted by χ, and similarly for $R(Z_p)$. Let $r_q=F_q[Z_p]$ be the regular representation and let λ: $Z_p \to C^*$ be the canonical complex representation. Put $\gamma=r_q-p \in \tilde{K}_{F_q}(BZ_p^{(n)})$, $\xi=\lambda-1 \in \tilde{K}(BZ_p^{(n)})$ and $\sigma=\lambda+\cdots+\lambda^{p-1}-(p-1) \in K(BZ_p^{(n)})$. In the rest of the paper we suppose that $q=l^a$ is a generator of $\hat{Z}_p^*$, namely, $q^{p-1} \not\equiv 1 \bmod p^2$.

Theorem 2.2. *Let $n=s(p-1)+r$, $0 \leq r < p-1$. Then we have $b_*(\gamma)=\sigma$ and*

i) $\tilde{K}_{F_q}(BZ_p^{(2n)}) \simeq Z_{p^s}\{\gamma\}$.

ii) $\tilde{K}_{F_q}(BZ_p^{(2n+1)}) \simeq \tilde{K}_{F_q}(BZ_p^{(2n)}) \oplus \tilde{K}_{F_q}(S^{2n+1})$ if $n+1 \not\equiv 0 \bmod p-1$.

iii) *Suppose that $n+1 \equiv 0 \bmod p-1$. Then there is a short exact sequence*

$$0 \longrightarrow \tilde{K}_{F_q}(S^{2n+1}) \xrightarrow{\pi^*} \tilde{K}_{F_q}(BZ_p^{(2n+1)}) \xrightarrow{i^*} \tilde{K}_{F_q}(BZ_p^{(2n)}) \longrightarrow 0$$

and there is an element $\alpha \in \tilde{K}_{F_q}(S^{2n+1})$ of order p such that $\pi^*\alpha=p^s\gamma$.

The theorem follows from the following propositions.

Proposition 2.3. (*Kambe* [3]). *Let $n=s(p-1)+r$, $0 \leq r < p-1$. Then there is an isomorphism*

$$\tilde{K}(BZ_p^{(2n)}) \simeq \tilde{K}(BZ_p^{(2n+1)}) \simeq (Z_{p^{s+1}})^r \oplus (Z_{p^s})^{p-r-1}$$

where the first r summands are generated by $\xi, \dots, \xi^r$ and the rest are generated by $\xi^{r+1}, \dots, \xi^{p-1}$. Moreover there is a relation $\xi^p = -\sum_{1 \leq i \leq p-1} \binom{p}{i} \xi^i$.

Note that the order of ξ^{p-1} is p^s, and $p^s\xi^i=0$ for all i if and only if $n \equiv 0 \bmod p-1$. Let I_{2n} denote $\mathrm{Ker}\,[\psi^q-1: \tilde{K}(BZ_p^{(2n)}) \to \tilde{K}(BZ_p^{(2n)})]$.

Proposition 2.4. i) *Let $0<r<p-1$. Then $a\xi^r \in I_{2n}$, $a \in Z$ if and only if $a\xi^r=0$.*

ii) *Let $n \not\equiv 0 \bmod p-1$. Then $a\xi^{p-1} \in I_{2n}$ if and only if $a\xi^{p-1}=0$. Let $n \equiv 0 \bmod p-1$. Then $a\xi^{p-1} \in I_{2n}$ if and only if $a \equiv 0 \bmod p^{s-1}$.*

iii) *$\sigma \in I_{2n}$, and $\sigma \equiv \xi^{p-1} \bmod p$.*

Proof. From the relation $\xi^p = -\sum \binom{p}{i} \xi^i$ we easily see that $\xi^{p+t} \equiv 0 \bmod p$ for $t \geq 0$, and if we express $\xi^{p+t} = b_1\xi + \cdots + b_{p-1}\xi^{p-1}$, then $b_1 \equiv 0 \bmod p^2$ if $t>0$. Note that $\psi^q\xi^r=((1+\xi)^q-1)^r=(q\xi+\cdots)^r$. Put $\psi^q\xi^r=c_1\xi+\cdots+c_{p-1}\xi^{p-1}$. Then from the above fact we see that $c_r\equiv q^r \bmod p$. First suppose that $0<r<p-1$. Then $q^r\not\equiv 1 \bmod p$ and hence $a(\psi^q\xi^r-\xi^r)=0$ if and only if $a\xi^r=0$. Next suppose that $r=p-1$. Then we can easily check that $c_1\equiv kp \bmod p^2$ where $k=(p-1)q^{p-2}\binom{q}{2}$, $c_i\equiv 0 \bmod p$, $1<i<p-1$ and $c_{p-1}\equiv q^{p-1} \bmod p$. Since $q-1\not\equiv 0 \bmod p$, $k\not\equiv 0 \bmod p$. Suppose that $a\xi^{p-1}\in I_{2n}$, namely, $a(\psi^q\xi^{p-1}-\xi^{p-1})=0$. Then $akp\equiv 0 \bmod p^s$ if $n\equiv 0 \bmod p-1$. In the first case $a\equiv 0 \bmod p^s$ and we see that $a\xi^{p-1}\in I_{2n}$ if and only if $a\xi^{p-1}=0$. Now suppose that $n\equiv 0 \bmod p-1$. Then $p^s\xi^i=0$ for all i. Since $c_i\equiv 0 \bmod p$ for $i<p-1$ and $c_{p-1}\equiv q^{p-1}\equiv 1 \bmod p$, we see that $p^{s-1}(\psi^q\xi^{p-1}-\xi^{p-1})=0$. This shows i) and ii).

Next we prove iii). In the proof of Theorem 2.2, we shall show that $b_*(\gamma)=\sigma$. Then it is clear that $\sigma\in I_{2n}$. Recall that $\sigma=1+\cdots+\lambda^{p-1}-p$. Then we easily see that $\sigma\xi=\sigma(\lambda-1)=-p\xi$. We express $\sigma=\sum a_i\xi^i$. Note that $a_{p-1}=1$. Then

$$-p\xi=\sigma\xi=a_1\xi^2+\cdots+a_{p-2}\xi^{p-1}+a_{p-1}\left(-\sum\binom{p}{i}\xi^i\right).$$

Therefore we have relations $\left(a_i-\binom{p}{i+1}\right)\xi^{i+1}=0,\ 0<i<p-1$. Suppose that $n<p-1$. Then the order of $\xi,\ldots,\xi^n$ are p and $\xi^{n+1}=\cdots=\xi^{p-1}=0$. Then clearly we have $a_1\xi^2+\cdots+a_{n-1}\xi^n=0$ and hence $\sigma=a_n\xi^n$. Since $\sigma\in I_{2n}$, $a_n\xi^n=0$ or $n=p-1$ by i) and ii). Therefore $\sigma\equiv\xi^{p-1} \bmod p$. Next suppose that $n\geq p$. Then $\left(a_i-\binom{p}{i+1}\right)\xi^{i+1}=0$ implies that $a_i\equiv 0 \bmod p$ for $i<p-1$. Hence $\sigma\equiv\xi^{p-1} \bmod p$. This completes the proof.

Proof of Theorem 2.2 First we show that $b^*(\gamma)=\sigma$. It is enough to show the relation for representations. Let k be the algebraic closure of F_q and let $\phi\colon k^\times\to C^\times$ be an imbedding. It is clear that $r_{F_q}\otimes k=1+\bar\lambda+\cdots+\bar\lambda^{p-1}$ for some 1-dim representation $\bar\lambda\colon Z_p\to k^\times$ and $b(\bar\lambda)=\lambda$. This shows that $b(r_{F_q}-p)=\sigma$. Now to prove i) it is enough to show that $I_{2n}=Z_{p^s}\{\sigma\}$ for $\tilde K^{-1}(BZ_p^{(2n)})=0$. By Proposition 2.4, i) we easily see that the order of σ is p^s. By induction on n we show that I_{2n} is generated by σ and ξ^{p-1}. For $n=0$ it is clear. Let $i\colon BZ_p^{(2n)}\to BZ_p^{(2n+2)}$ be the inclusion. Then we have the induced homomorphism $i^*\colon I_{2n+2}\to I_{2n}$. Let $x\in I_{2n+2}$. By the assumption of induction we have $i^*x=a\sigma+b\xi^{p-1}$. Then clearly $x=a\sigma+b\xi^{p-1}+cp^s\xi^{r+1}$. If $r+1=p-1$ then there is nothing to prove. If $r+1<p-1$, then i^* is isomorphic on summands generated by ξ^{p-1} and hence $b\xi^{p-1}\in I_{2n+2}$. Hence $cp^s\xi^{r+1}\in I_{2n+2}$ and by Proposition 2.4, iii), we have $cp^s\xi^{r+1}=0$. This completes the induction. Now if $n\not\equiv 0 \bmod p-1$ then $b\xi^{p-1}=0$ by Proposition 2.4, ii). Suppose that $n\equiv 0 \bmod p-1$. Then by Proposition 2.4, i) and ii), we have $p^{s-1}\xi^{p-1}=qp^{s-1}\sigma$ for some q and $b\equiv 0 \bmod p^{s-1}$. Hence $a\sigma+b\xi^{p-1}=k\sigma$ for some k.

Next to prove ii) and iii) consider the cofibration $BZ_p^{(2n)}\xrightarrow{i}BZ_p^{(2n+1)}\xrightarrow{\pi}S^{2n+1}$ and the induced commutative diagram

$$\begin{array}{ccccc}
\tilde{K}^{-1}(S^{2n+1}) & \xrightarrow{\psi^q-1} & \tilde{K}^{-1}(S^{2n+1}) & \xrightarrow{\delta} & K_{F_q}(S^{2n+1}) \\
\pi^*\downarrow & & \pi^*\downarrow & & \downarrow\pi^* \\
\tilde{K}^{-1}(BZ_p^{(2n+1)}) & \longrightarrow & \tilde{K}^{-1}(BZ_p^{(2n+1)}) & \longrightarrow & \tilde{K}_{F_q}(BZ_p^{(2n+1)}) \\
i^*\downarrow & & i^*\downarrow & & \downarrow i^* \\
\tilde{K}^{-1}(BZ_p^{(2n)}) & \longrightarrow & \tilde{K}^{-1}(BZ_p^{(2n)}) & \longrightarrow & \tilde{K}_{F_q}(BZ_p^{(2n)}).
\end{array}$$

It is clear that π^*: $\tilde{K}^{-1}(S^{2n+1})=\tilde{K}(BZ_q^{(2n+1)})$. By the result of Quillen [6], $\tilde{K}_{F_q}(S^{2n})=0$. Hence we have a short exact sequence

$$0\longrightarrow\tilde{K}_{F_q}(S^{2n+1})\xrightarrow{\pi^*}\tilde{K}_{F_q}(BZ_p^{(2n+1)})\xrightarrow{i^*}\tilde{K}_{F_q}(BZ_p^{(2n)})\longrightarrow 0.$$

Similarly we have a monomorphism i^*: $\tilde{K}_{F_q}(BZ_p^{(2n+2)})\to\tilde{K}_{F_q}(BZ_p^{(2n+1)})$. Therefore the order of γ in $\tilde{K}_{F_q}(BZ_p^{(2n+1)})$ is p^{s+1} if $n+1\equiv 0 \bmod p-1$, and p^s if $n+1\not\equiv 0 \bmod p-1$ by the result of i). Then ii) and iii) follow from the above exact sequence.

§3. Order of the Kahn-Priddy map

Let $A(Z_p)$ be the Burnside ring of Z_p. Let $\pi^0(X)=[X_+, Q(S_0)]$ be the (non-reduced) stable cohomotopy group. Let S be a finite Z_p-set represented by a homomorphism λ: $Z_p\to\Sigma_m$. By the Barratt-Priddy-Quillen theorem [1] there is a map ω: $B\Sigma_n\to Q(S^0)$. Then the composite

$$BZ_p\xrightarrow{B\lambda}B\Sigma_m\subset B\Sigma_n\longrightarrow Q(S^0)$$

determines an element of $\pi^0(BZ_p)$ and we obtain a homomorphism α: $A(Z_p)\to\pi^0(BZ_p)$. We regard Z_p as a left Z_p-set and put $h=Z_p-p\in A(Z_p)$. Then we obtain a map $\alpha(h)$: $(BZ_p)_+\to Q(S^0)$ and its restriction $\alpha(h)$: $(BZ_p^{(n)})_+\to Q(S^0)$. For simplicity those maps and their adjoints $\boldsymbol{S}(BZ_p^{(n)})_+\to\boldsymbol{S}$ are all denoted by h and called Kahn-Priddy maps.

Let X be a based connected CW-complex and let $\tilde{\pi}^0(X)$ be the reduced group. For an abelian group M, the localization at p of M is denoted by $M_{(p)}$. Then $\pi^0(X)_{(p)}\supset 1+\tilde{\pi}^0(X)_{(p)}$ is a group under multiplication.

Theorem 3.1. *Let $n=s(p-1)+r$, $0\le r<p-1$. Then in $\tilde{\pi}^0(BZ_p^{(2n)})_{(p)}$ and $\tilde{\pi}^0(BZ_p^{(2n-1)})_{(p)}$ the order of h is p^s. Similarly in the associated multiplicative groups the order of $1+h$ is p^s.*

Corollary 3.2. *Let $n=s(p-1)+r$, $0\le r<p-1$ and let $m=2n$ or $2n-1$. Then* $\exp(\tilde{\pi}^0(X)_{(p)};\ \dim(X)\le m)=p^s$.

Proof. Let $\tilde{\pi}^0(X)_{(p)}\ni x$ be an element represented by a stable map f: $\boldsymbol{S}X\to\boldsymbol{S}$. By the Kahn-Priddy theorem there is a stable map g: $\boldsymbol{S}X\to\boldsymbol{S}BZ_p$ such that $f\sim h\circ g$. Let $m=2n$ or $2n-1$. If $\dim(X)\le m$ then g is factored through $\boldsymbol{S}BZ_p^{(m)}$. Then by Theorem 3.1 we easily see that $p^sx=0$. Now let $X=BZ_p^{(m)}$. Then by Theorem 3.1, $p^{s-1}h\ne 0$. This completes the proof.

In order to prove Theorem 3.1, we need a result of Tornehave [9]. The canonical inclusion $\Sigma_n \to GL(n, F_q)$ induces a map $B\Sigma_n \to BGL(n, F_q)$ and a stable natural ring homomorphism

$$e: \pi^0(X) \longrightarrow K_{F_q}(X).$$

Next regarding the vector space $(F_q)^n$ as a finite $GL(n, F_q)$-set we obtain a map μ_n: $BGL(n, F_q) \to B\Sigma_{q^n}$. Given a map $f: X \to BGL(n, F_q)$, the composite $\rho(f)$

$$X \xrightarrow{f} BGL(n, F_q) \xrightarrow{\mu_n} B\Sigma_{q^n} \subset B\Sigma_i \xrightarrow{\omega} QS^0 \longrightarrow (QS^0)_{(p)}$$

is an element of $\pi^0(X)_{(p)}$. Let $x \in [X, BGL(n, F_q)]$ be represented by f. Let $\varepsilon(x) = \dim(f) = n$. Let $j(x) = \rho(x)/q^{\varepsilon(x)} \in 1 + \tilde{\pi}^0(X)_{(p)}$. It is clear that $j(x+y) = j(x)j(y)$. The group valued functor $1 + \tilde{\pi}^0(X)_{(p)}$ is represented by $Q_1(S^0)_{(p)} = SF_{(p)}$. Then by the group completion theorem (see e.g., [7]), we have a natural homomorphism

$$j: \tilde{K}_{F_q}(X)_{(p)} \longrightarrow 1 + \tilde{\pi}^0(X)_{(p)}.$$

Theorem 3.3. (Tornehave [9]). *Let p be an odd prime and $q = l^a$ a generator of $\hat{Z}_q^*$. Then for a connected CW-complex X the composite*

$$\tilde{K}_{F_q}(X)_{(p)} \xrightarrow{j} 1 + \tilde{\pi}^0(X)_{(p)} \xrightarrow{e} 1 + \tilde{K}_{F_q}(X)_{(p)}$$

is an isomorphism.

Remark. Four our purpose $X = BZ_p^{(2n)}$ and in this case every element of $\tilde{K}_{F_q}(BZ_p^{(2n)})$ comes from a representation (Theorem 2.2). Then the theorem is directly verified. In particular

Lemma 3.4. *Let q be as in Theorem 3.3. Put $\alpha = (q^{p-1}-1)/pq^{p-1} \in \hat{Z}_p$. Then $\alpha \not\equiv 0 \bmod p$ and $j(\gamma) = 1 + \alpha h$ where $\gamma \in K_{F_q}(BZ_p^{(n)})$ and $h \in \tilde{\pi}^0(BZ_p^{(n)})$.*

Proof. By the assumption q is a generator of $\hat{Z}_p^*$. Hence $q^{p-1} \not\equiv 1 \bmod p^2$, namely $(q^{p-1}-1)/p \not\equiv 0 \bmod p$ and we have $\alpha \not\equiv 0 \bmod p$. Recall that $\gamma = r_{F_q} - p$. Hence $j(\gamma) = \rho(r_{F_q})/q^p$. Note that in the regular representation Z_p acts on $(F_q)^p$ as permutations of factors. Therefore $(F_q)^p \ni (x_1, \dots, x_p)$ is a fixed point if and only if $x_1 = \cdots = x_p$. Let $\Delta = \{(x, \dots, x); x \in F_q\}$, then $(F_q)^p - \Delta$ is a free Z_p-set. Hence $\rho(r_{F_q}) = ((q^p - q)/p)[Z_p] + q = ((q^p - q)/p)([Z_p] - p) + q^p$, and we have $j(\gamma) = 1 + \alpha h$.

Proof of Theorem 3.1. Let $m = 2n$ or $2n-1$. By Theorem 3.3, j: $\tilde{K}_{F_q}(BZ_p^{(m)})_{(p)} \to 1 + \tilde{\pi}^0(BZ_p^{(m)})_{(p)}$ is a split monomorphism. By Theorem 2.2, the order of $\gamma \in \tilde{K}_{F_q}(BZ_p^{(m)})$ is p^s and hence by lemma 3.4 the order of $1 + \alpha h \in 1 + \tilde{\pi}^0(BZ_p^{(m)})_{(p)}$ is p^s. We have $(1+\alpha h)^{p^s} = 1 + \alpha p^s h + \binom{p^s}{2}(\alpha h)^2 + \cdots \equiv 1 + \alpha p^s h \bmod (p^{s+1}h)$. Since $\alpha \not\equiv 0 \bmod p$, we see that the order of h is p^s. It is similar to see that the order of $1 + h$ is p^s. This completes the proof.

Now we state the Kahn-Priddy type theorem for algebraic K-groups.

Theorem 3.5. *Let $\gamma: QBZ_p \to BGL(F_q)^+_{(p)}$ be an extention of $\gamma: BZ_p \to BGL(F_q)^+_{(p)}$ as an infinite loop map (exists uniquely up to homotopy). Then for any CW-complex X, $\gamma^*: [X, QBZ_p] \to \tilde{K}_{F_q}(X)_{(p)}$ is a split epimorphism.*

Proof. Let $e_{(p)}: (Q_0S^0)_{(p)} \to BGL(F_q)^+_{(p)}$ be the localization of the map e. Clearly it is an infinite loop map. Note that $\gamma = e_{(p)} \circ h$. Then we have $\gamma = e_{(p)} \circ \tilde{h}$, where $\tilde{h}: QBZ_p \to QS^0$ is the extention of h. Then the theorem follows from the Kahn-Priddy theorem and Theorem 3.3.

Now quite similarly as for Corollary 3.2 we have

Corollary 3.6. *Let $n = s(p-1)+r$, $0 \leq r < p-1$ and let $m = 2n$ or $2n-1$. Then* $\exp(\tilde{K}_{F_q}(X)_{(p)}; \dim(X) \leq m) = p^s$.

Remark. By the Adams conjecture and the theorem of Tornehave, the J-group of suspension space $J(SX)_{(p)}$ is regarded as a subgroup of $\tilde{K}_{F_q}(X)_{(p)}$. Thus $\exp(J(SX)_{(p)}; \dim(SX) \leq m)$ divides p^s. By the result of [8], it is shown that $\exp(J(X)_{(p)}; \dim(X) \leq m) > p^s$ for certain m.

§4. Some consequences

As in §3 Let q be a gnerator of $\hat{Z}_p^*$. Let $\pi: BZ_p^{(m)} \to S^m$ be the canonical projection. Let $m = 2s(p-1)-1$ and consider the commutative diagrm

$$\begin{array}{ccccc} \tilde{K}_{F_q}(S^m)_{(p)} & \xrightarrow{\pi^*} & \tilde{K}_{F_q}(BZ_p^{(m)})_{(p)} & \xrightarrow{i^*} & \tilde{K}_{F_q}(BZ_p^{(m-1)})_{(p)} \\ \downarrow j & & \downarrow j & & \\ 1+\tilde{\pi}^0(S^m)_{(p)} & \xrightarrow{\pi^*} & 1+\tilde{\pi}^0(BZ_p^{(m)})_{(p)}. & & \end{array}$$

By Theorem 2.2 there is an element $\bar{\alpha}_s \in \tilde{K}_{F_q}(S^m)$ of order p such that $\pi^*(\bar{\alpha}_s) = p^{s-1}\gamma \in \tilde{K}_{F_q}(BZ_p^{(m)})$. Let $j(\bar{\alpha}_s) = 1 + \alpha_s$. The element α_s is essentially the element of α-series of Adams-Toda. Now from the above diagram we see that

$$\pi^*(1+\alpha_s) = j(p^{s-1}\gamma) = (1+ah)^{p^{s-1}} = 1 + ap^{s-1}h$$

where $a \not\equiv 0 \bmod p$. Thus we have

Proposition 4.1. *Let $\alpha_s \in \pi^S_{2s(p-1)-1}(S^0)_{(p)}$ be as above. Then $\alpha_s \circ \pi = p^{s-1}h$.*

Now using 4.1 and the order of J-image, one can show

Corollary 4.2. *Suppose that there is an element $\gamma \in \pi^S_{2s(p-1)-1}(S^0)_{(p)}$ with* mod p *Hopf invariant* 1. *Then* $s=1$.

Proof. By the Kahn-Priddy theorem there is a stable map $u: S^{2s(p-1)-1}$ $SBZ_p^{(2s(p-1)-1)}$ such that $\gamma = h \circ u$. As an argument in [5] we see that the degree of $u_*: H_{2s(p-1)-1}(S^{2s(p-1)-1}) \to H_{2s(p-1)-1}(SBZ_p^{(2s(p-1)-1)})$ is prime to p. Now $p^{s-1}\gamma = p^{s-1}h \circ u = \deg(u_*)\alpha_s$. But it is well-known that α_s is not divisible by p^{s-1} if $s \neq 1$.

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