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Some basic results on canonical modules

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Introduction

The notion of a canonical module of a (noetherian) local ring is due to Grothendieck, who called it a module of dualizing differentials (cf. [5, p. 94]). The term "a canonical module" was first adopted in [6]. In [2] a canonical module is called a dualizing module, which seems to be an apposite terminology (cf. [2, p. 203]), but in [5] a dualizing module means an injective envelope of the residue class field of a local ring. In the case that rings are Cohen-Macaulay, a canonical module is a Gorenstein module of rank one in the sense of Sharp [10]. Grothendieck defined a module of dualizing differentials for a complete local ring ([5, Definition on p. 94]) and proved some theorems on local duality for complete local rings (cf. [5, p. 95, Theorem 6.7 and Theorem 6.8]) and some properties of a module of dualizing differentials (cf. [5, Proposition 6.4 and Proposition 6.6]). Subsequently, in their seminar note [6], Herzog, Kunz et al. defined the notion of a canonical module for general local rings ([6, Definition 5.6]) and made a systematic study of the theory of canonical modules. They first established some elementary properties of canonical modules over general local rings including an important and useful existence theorem [6, Satz 5.12] which we recall in section 1 ([6, 5 Vortrag]) and made a detailed study of the structure of canonical modules over Cohen-Macaulay local rings ([6, 6 Vortrag]). They reconstructed the theory of Gorenstein rings from this point of view and studied the type of Cohen-Macaulay modules (see [6, Definition 1.20] for the definition of the type), especially gave important formula and inequality on the type ([6, Satz 6.10 and Satz 6.16]). These are excellent results of the theory of canonical modules. Furthermore some persons study canonical modules over general local rings and various results were given. But there are unknowns about fundamentals of canonical modules over general local rings, for instance, existence theorems, structure theorems including characterizations of a canonical module, behavior under localization or flat base change and so on. In the case that rings are Cohen-Macaulay, there is an existence theorem due to Foxby and Reiten, that is, a Cohen-Macaulay local ring has a canonical module if and only if it is a homomorphic image of a Gorenstein local ring, but relations between the existence of

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a canonical module and formal fibres are not known except the case of one dimensional Cohen-Macaulay local rings (cf. [2, Theorem (5.3)], see also [6, Satz 6.14] and Corollary 4.3). Main purposes of this paper are to consider the trivial extension by a canonical module (section 2), to show some elementary properties of the endomorphism ring of a canonical module (section 3) and to prove that the localization of a canonical module is a canonical module of the localization ring, which was known only for local rings with dualizing complexes (section 4).

Section 1 consists of preliminaries. In section 2 we give a condition for the trivial extension to be a quasi-Gorenstein ring. Let A be a local ring and T an A-module. We prove that the trivial extension $A \ltimes T$ is a quasi-Gorenstein ring if and only if the completion of A is (S_2) and T is a canonical module of A. This is a generalization of a result of Foxby [3, §4] and Reiten [9]. As a corollary, we have a proof of Corollary 4.3 in the case that the completion is (S_2), which is used in section 3. We also give a result on quasi-Gorenstein rings under flat base change. In section 3 we study the endomorphism ring of a canonical module. Let A be a local ring with canonical module K, $U=\operatorname{ann}(K)$ and $H=\operatorname{End}_A(K)$. We show that H is a (commutative noetherian) semi-local ring which is a finite (S_2) over-ring of A/U contained in the total quotient ring of A/U and has a canonical module K in the sense that $K_{\mathfrak{p}}$ is a canonical module of $H_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of H. The commutativity of H has the status of folklore but no proof is found in the literature to the writer's knowledge and our proof is based on a suggestion of S. Goto. In section 4 we study canonical modules under flat base change. Let A be a local ring, T an A-module and B a faithfully flat local A-algebra. We prove that, if $T \otimes_A B$ is a canonical module of B, then T is a canonical module of A. As a corollary, we have the following: Let A be a local ring with canonical module K and let $\mathfrak{p}$ be a prime ideal in $\operatorname{Supp}_A(K)$. Then $K_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ and, for every minimal prime ideal $\mathfrak{q}$ of $\mathfrak{p}\hat{A}$, $\hat{A}_{\mathfrak{q}}/\mathfrak{p}\hat{A}_{\mathfrak{q}}$ is a Gorenstein ring where $\hat{A}$ is the completion of A.

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1. Preliminaries

In this paper a semi-local ring means a commutative noetherian ring with a finite number of maximal ideals and a local ring is a semi-local ring with unique maximal ideal. We denote by ^ the Jacobson radical adic completion over a semi-local ring. Let R be a commutative noetherian ring, M a finitely generated R-module and N a submodule of M. We denote by $\operatorname{Min}_R(M)$ the set of minimal elements in $\operatorname{Supp}_R(M)$ and put $U_M(N)=\cap Q$ where Q runs through all the primary components of N in M such that $\dim M/Q=\dim M/N$. Let T be an R-module and $\mathfrak{a}$ an ideal of R. We denote by $E_R(T)$ an injective envelope of T and $H_{\mathfrak{a}}^i(T)$ denotes the i-th local cohomology module of T with respect to $\mathfrak{a}$ (cf. [5] and [6, 4 Vortrag]). First we state the definition of a canonical module after [6].

Definition 1.1 ([6, Definition 5.6]). Let A be a local ring with maximal ideal $\mathfrak{m}$ and of dimension d. An A-module K is a *canonical module* of A if $K\otimes_A\hat{A}\cong \mathrm{Hom}_A(H^d_{\mathfrak{m}}(A), E_A(A/\mathfrak{m}))$.

When A is complete, a canonical module K of A exists and is a module which represents the functor $\mathrm{Hom}_A(H^d_{\mathfrak{m}}(\ \ ), E_A(A/\mathfrak{m}))$, that is, $\mathrm{Hom}_A(H^d_{\mathfrak{m}}(M), E_A(A/\mathfrak{m}))\cong \mathrm{Hom}_A(M, K)$ (functional) for any A-module M ([6, Satz 5.2]). A local ring with dualizing complex has a canonical module and there is a local ring which does not have a canonical module. Here we recall an important and useful existence theorem given in [6].

Theorem 1.2 ([6, Satz 5.12]). *Let A be a local ring with canonical module K, C a commutative A-algebra which is finitely generated as an A-module and $B=C_{\mathfrak{n}}$ where $\mathfrak{n}$ is a maximal ideal of C. Assume that $H^i_{\mathfrak{m}}(A)=0$ for $\dim B\le i<\dim A$ where $\mathfrak{m}$ is the maximal ideal of A* (cf. [5, Theorem 6.7]). *Then a canonical module of B exists and is given by $(\mathrm{Ext}^r_A(C, K))_{\mathfrak{n}}$ where $r=\dim A-\dim B$.*

Definition 1.3 ([7, p. 82 and p. 124]). A semi-local ring A is said to be *unmixed* if $\dim \hat{A}/\mathfrak{p}=\dim A$ for every $\mathfrak{p}$ in $\mathrm{Ass}(\hat{A})$ and *quasi-unmixed* if $\dim \hat{A}/\mathfrak{p}=\dim A$ for every $\mathfrak{p}$ in $\mathrm{Min}(\hat{A})$.

Definition 1.4. Let R be a commutative noetherian ring, M a finitely generated R-module and t a positive integer. We say that M is (S_t) if $\operatorname{depth} M_{\mathfrak{p}}\ge\min\{t, \dim M_{\mathfrak{p}}\}$ for every $\mathfrak{p}$ in $\mathrm{Supp}_R(M)$.

Now we summarize some known results on canonical modules.

Let A be a local ring of dimension d and with canonical module K.

(1.5) K is unique up to isomorphisms and a finitely generated A-module of dimension d. (cf. [6, 5 Vortrag])

(1.6) If A is a homomorphic image of a Gorenstein local ring B, then $K\cong \mathrm{Ext}^r_B(A, B)$ where $r=\dim B-d$, the least integer i such that $\mathrm{Ext}^i_B(A, B)\ne 0$, and therefore, for every $\mathfrak{p}$ in $\mathrm{Supp}_A(K)$, $K_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$. (cf. [6, 5 Vortrag])

(1.7) $\mathrm{Ass}_A(K)=\{\mathfrak{p}\in\mathrm{Spec}(A)\mid \dim A/\mathfrak{p}=d\}$. (cf. [5, Proposition 6.6(5)]) Consequently the following are equivalent: (a) $\mathrm{Supp}_A(K)=\mathrm{Spec}(A)$. (b) For every $\mathfrak{p}$ in $\mathrm{Min}(A)$, $\dim A/\mathfrak{p}=d$. (c) A is quasi-unmixed.

(1.8) $\mathrm{ann}_A(K)=U_A(0)$ and K, as an $A/U_A(0)$-module, is a canonical module of $A/U_A(0)$. (cf. [5, Proposition 6.6(7)] and Theorem 1.2) Consequently the following are equivalent: (a) K is faithful. (b) For every $\mathfrak{p}$ in $\mathrm{Ass}(A)$, $\dim A/\mathfrak{p}=d$. (c) A is unmixed.

(1.9) Every maximal chain of prime ideals in $\mathrm{Supp}_A(K)$ is of length d and $(U_A(0))_{\mathfrak{p}}=U_{A_{\mathfrak{p}}}(0)$ for every $\mathfrak{p}$ in $\mathrm{Supp}_A(K)$. For a prime ideal $\mathfrak{p}$ of A, $\mathfrak{p}$ is in $\mathrm{Supp}_A(K)$ if and only if $\dim A_{\mathfrak{p}}+\dim A/\mathfrak{p}=d$. (cf. [7, p. 125])

(1.10) For every $\mathfrak{p}$ in $\mathrm{Supp}_A(K)$ and for every subsystem of parameters $\underline{x}$ of length ≤ 2 for $A_{\mathfrak{p}}$, $\underline{x}$ is a $K_{\mathfrak{p}}$-regular sequence. In particular K is (S_2). (cf. [2, Proof of

Lemma 2.4])
(1.11) Let M be a finitely generated A-module and let h_M: $M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, K), K)$ be the natural map. Then:
(1.11.1) $\mathrm{Ker}\, h_M = \{x \in M \mid \dim Ax < d\}$. (cf. [5, Proposition 6.6(8)]) Therefore, if $\dim M = d$, $\mathrm{Ker}\, h_M = U_M(0)$ and $U_M(0) \otimes_A \hat{A} = U_{\hat{M}}(0)$.
(1.11.2) The map h_M is an isomorphism if and only if $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p}$ in $\mathrm{Min}_A(M)$ and $\hat{M}$ is (S_2). (cf. [1, Proof of Proposition 2])

The following (1.12) is also known but no literature can be cited, so we give a proof. In (1.12) we do not assume the existence of a canonical module of A.

(1.12) Let A be a local ring. Assume that $A/U_A(0)$ has a canonical module K. Then K, as an A-module, is a canonical module of A.

Proof. By (1.8) K is a faithful $A/U_A(0)$-module and $A/U_A(0)$ is unmixed. Hence we have $U_A(0)\hat{A} = U_{\hat{A}}(0)$. Let C be a canonical module of $\hat{A}$. Then C is a canonical module of $\hat{A}/U_{\hat{A}}(0)$ by (1.8). Hence, by (1.5), we have $\hat{K} \cong C$, a canonical module of $\hat{A}$, namely K is a canonical module of A. q.e.d.

2. The trivial extension

To state the main result of this section, we need the notion of a quasi-Gorenstein ring due to Platte and Storch.

Definition 2.1 ([8, § 3]). A local ring A is said to be a *quasi-Gorenstein* ring if a canonical module of A exists and is a free A-module (of rank one). This is equivalent to say that $H^d_{\mathfrak{m}}(A) \cong E_A(A/\mathfrak{m})$ where $d = \dim A$ and $\mathfrak{m}$ is the maximal ideal of A.

It is obvious that a local ring A is quasi-Gorenstein if and only if so is $\hat{A}$. A local ring is a Gorenstein ring if and only if it is a quasi-Gorenstein Cohen-Macaulay ring.

The following Lemma 2.2 is a corollary to [4, Theorem 1], so we omit the proof.

Lemma 2.2. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and B a faithfully flat local A-algebra with maximal ideal $\mathfrak{n}$. Then*:
(1) *If $B/\mathfrak{m}B$ is an artinian Gorenstein ring, then $E_A(A/\mathfrak{m}) \otimes_A B \cong E_B(B/\mathfrak{n})$.* (cf. [6, Lemma 6.15])
(2) *If T is an A-module and $T \otimes_A B \cong E_B(B/\mathfrak{n})$, then $T \cong E_A(A/\mathfrak{m})$ and $B/\mathfrak{m}B$ is an artinian Gorenstein ring.*

Theorem 2.3. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and B a local ring with maximal ideal $\mathfrak{n}$. Assume that B is a flat A-algebra and $\mathfrak{m}B$ is an $\mathfrak{n}$-primary ideal. Then B is a quasi-Gorenstein ring if and only if A is a quasi-Gorenstein ring and $B/\mathfrak{m}B$ is a Gorenstein ring.*

Proof. First we note that, for an A-module M and for $i \geq 0$, $H^i_{\mathfrak{m}}(M) \otimes_A B \cong (\varinjlim_t \mathrm{Ext}^i_A(A/\mathfrak{m}^t, M)) \otimes_A B \cong \varinjlim_t (\mathrm{Ext}^i_A(A/\mathfrak{m}^t, M) \otimes_A B) \cong \varinjlim_t \mathrm{Ext}^i_B(B/\mathfrak{m}^t B, M \otimes_A B) \cong$

$H_{\mathfrak{n}}^{i}(M \otimes_A B)$ because B is flat over A and $\mathfrak{m}B$ is $\mathfrak{n}$-primary (cf. [5, Theorem 2.8] or [6, Bemerkung 4.5]). A (resp. B) is a quasi-Gorenstein ring if and only if $H_{\mathfrak{m}}^{d}(A) \cong E_A(A/\mathfrak{m})$ (resp. $H_{\mathfrak{n}}^{d}(B) \cong E_B(B/\mathfrak{n})$) where $d = \dim A = \dim B$. Since $H_{\mathfrak{m}}^{d}(A) \otimes_A B \cong H_{\mathfrak{n}}^{d}(B)$, the assertion follows from Lemma 2.2. q.e.d.

Corollary 2.4. *Let A be a quasi-Gorenstein local ring and $\mathfrak{p}$ a prime ideal of A. Then $A_{\mathfrak{p}}$ is a quasi-Gorenstein ring and, for every minimal prime ideal $\mathfrak{q}$ of $\mathfrak{p}\hat{A}$, $\hat{A}_{\mathfrak{q}}/\mathfrak{p}\hat{A}_{\mathfrak{q}}$ is a Gorenstein ring.*

Proof. First we assume that A is complete. Since A is a canonical module of A, $A_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ by (1.6), that is, $A_{\mathfrak{p}}$ is a quasi-Gorenstein ring. If A is not complete, we obtain the result, applying Theorem 2.3 to the flat local homomorphism $A_{\mathfrak{p}} \to \hat{A}_{\mathfrak{q}}$. q.e.d.

Here we recall the notion of the trivial extension introduced by Nagata under the terminology "the principle of idealization". Let R be a commutative ring and M an R-module.

Definition 2.5 ([7, p. 2]). In the cartesian product $R \times M$, we introduce addition by componentwise and multiplication by $(r, m)(s, n) = (rs, rn + sm)$. These operations give a structure of a commutative ring to $R \times M$. This ring is called the *trivial extension* of R by M and denoted by $R \ltimes M$.

We summarize some simple properties of the trivial extension.

(2.6) $\operatorname{Spec}(R \ltimes M) = \{\mathfrak{p} \times M \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$ and height $\mathfrak{p} \times M =$ height $\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of R.

(2.7) If S is a multiplicatively closed set in R, then $S \times M$ is a multiplicatively closed set in $R \ltimes M$ and $(r, m)/(s, n) \mapsto (r/s, (sm - rn)/s^2)$ defines an isomorphism $(S \times M)^{-1}(R \ltimes M) \xrightarrow{\sim} (S^{-1}R) \ltimes (S^{-1}M)$. In particular, $(R \ltimes M)_{(\mathfrak{p} \times M)} \cong R_{\mathfrak{p}} \ltimes M_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$ of R.

(2.8) $R \ltimes M$ is noetherian if and only if R is noetherian and M is finitely generated.

(2.9) If R is a local ring with maximal ideal $\mathfrak{m}$ and M is finitely generated, then $R \ltimes M$ is a local ring with maximal ideal $\mathfrak{m} \times M$ and $\widehat{R \ltimes M} \cong \hat{R} \ltimes \hat{M}$.

(2.10) If R is a local ring and M is finitely generated, then depth $R \ltimes M = \min \{\operatorname{depth} R, \operatorname{depth}_R M\}$.

Theorem 2.11. *Let A be a local ring and K a non-zero A-module. Then $A \ltimes K$ is a quasi-Gorenstein ring if and only if $\hat{A}$ is (S_2) and K is a canonical module of A.*

Proof. The "only if" part: $\widehat{A \ltimes K} \cong \hat{A} \ltimes \hat{K}$ is a quasi-Gorenstein ring. Since a canonical module is (S_2) by (1.10), so does $\hat{A} \ltimes \hat{K}$. Hence $\hat{A}$ is (S_2) by (2.7) and (2.10). By Theorem 1.2, $\operatorname{Hom}_{A \ltimes K}(A, A \ltimes K) \cong \operatorname{ann}(K) \oplus K$ is a canonical module of A. Since $\hat{A}$ is (S_2), the endomorphism ring of a canonical module of A is isomorphic to A by [1, Proposition 2]. Since A is an indecomposable A-module, we have that $\operatorname{ann}(K) = 0$ and K is a canonical module of A. The "if" part: $A \ltimes K$ is a finitely generated A-module when it is regarded as an A-module by means of q:

$A \ni a \mapsto (a, 0) \in A \ltimes K$. By virtue of Theorem 1.2, a canonical module of $A \ltimes K$ exists and is isomorphic to $\mathrm{Hom}_A(A \ltimes K, K)$. Let F be the map $\mathrm{Hom}_A(A \ltimes K, K) \ni f \mapsto (fq, fj) \in \mathrm{Hom}_A(A, K) \oplus \mathrm{Hom}_A(K, K)$ where j is the map $K \ni x \mapsto (0, x) \in A \ltimes K$. Then F is an A-isomorphism. $\mathrm{Hom}_A(A \ltimes K, K)$ is an $A \ltimes K$-module by the usual way and $\mathrm{Hom}_A(A, K) \oplus \mathrm{Hom}_A(K, K)$ is an $A \ltimes K$-module by the following operation: $(a, x)(g, h) = (ag + xh, ah)$ for $a \in A$, $x \in K$, $g \in \mathrm{Hom}_A(A, K)$ and $h \in \mathrm{Hom}_A(K, K)$. It is not difficult to see that F is an $A \ltimes K$-homomorphism. Because $\hat{A}$ is (S_2), we have $\mathrm{Hom}_A(A, K) \oplus \mathrm{Hom}_A(K, K) \cong K \oplus A$ by [1, Proposition 2]. Hence a canonical module of $A \ltimes K$ is isomorphic to $A \ltimes K$, that is, $A \ltimes K$ is a quasi-Gorenstein ring. q.e.d.

Corollary 2.12 ([3, §4] and [9]). *Let A be a local ring and K a non-zero A-module. Then $A \ltimes K$ is a Gorenstein ring if and only if A is a Cohen-Macaulay ring and K is a canonical module of A.*

Corollary 2.13. *Let A be a local ring with canonical module K and $\mathfrak{p}$ a prime ideal of A. Assume that $\hat{A}$ is (S_2). Then $K_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ and, for every minimal prime ideal $\mathfrak{q}$ of $\mathfrak{p}\hat{A}$, $\hat{A}_{\mathfrak{q}}/\mathfrak{p}\hat{A}_{\mathfrak{q}}$ is a Gorenstein ring.*

Proof. Since $\hat{A}$ is (S_2), we have $\mathrm{Hom}_A(K, K) \cong A$ by [1, Proposition 2]. Hence $K_{\mathfrak{p}}$ is a non-zero $A_{\mathfrak{p}}$-module. Since $A \ltimes K$ is a quasi-Gorenstein ring by Theorem 2.11, the assertion follows from Corollary 2.4 and Theorem 2.11. q.e.d.

3. The endomorphism ring

Let A be a local ring of dimension d and with canonical module K. We set $U = U_A(0) = \mathrm{ann}_A(K)$ (cf. (1.8)) and $H = \mathrm{End}_A(K)$. For a commutative ring R, $Q(R)$ denotes the total quotient ring of R.

Lemma 3.1. *Let $\mathfrak{p}$ be a prime ideal of A with $\dim A/\mathfrak{p} = d$. Then $K_{\mathfrak{p}} \cong E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$.*

Proof. By (1.7) we have $\mathfrak{p} \in \mathrm{Ass}_A(K)$. Let $\mathfrak{q}$ be an element of $\mathrm{Ass}_{\hat{A}}(\hat{A}/\mathfrak{p}\hat{A})$. Then $\mathfrak{q}$ is in $\mathrm{Ass}_{\hat{A}}(\hat{K})$ and $\dim \hat{A}/\mathfrak{q} = d$ by (1.7). $\hat{K}_{\mathfrak{q}}$ is a canonical module of $\hat{A}_{\mathfrak{q}}$ by (1.6) and $\hat{A}_{\mathfrak{q}}$ is artinian. Hence we have $\hat{K}_{\mathfrak{q}} \cong E_{\hat{A}_{\mathfrak{q}}}(\hat{A}_{\mathfrak{q}}/\mathfrak{q}\hat{A}_{\mathfrak{q}})$ by Definition 1.1. Because $K_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \hat{A}_{\mathfrak{q}} \cong \hat{K}_{\mathfrak{q}}$, the assertion follows from Lemma 2.2(2). q.e.d.

Theorem 3.2. *The following statements hold for H:*

(1) *H is a semi-local ring which is finitely generated as an A-module and $A/U \subseteq H \subseteq Q(A/U)$.*

(2) *Every maximal chain of prime ideals in H is of length d.*

(3) *$\hat{H}$ is (S_2).*

(4) *For every prime ideal $\mathfrak{p}$ of H, $K_{\mathfrak{p}}$ is a canonical module of $H_{\mathfrak{p}}$. (K is an H-module by the usual way.)*

Proof. Since $H = \mathrm{End}_A(K) = \mathrm{End}_{A/U}(K)$ and K is a canonical module of A/U by (1.8), we may assume that A is unmixed.

(1) Let $\mathrm{Ass}(A)=\{\mathfrak{p}_1,\dots,\mathfrak{p}_t\}$ and $S=A\setminus\bigcup_{i=1}^{t}\mathfrak{p}_i$, the set of non-zero-divisors of A. Since K is torsion free by (1.7), H is also torsion free and the natural map $H\to S^{-1}H$ is injective. Since $S^{-1}K\cong\bigoplus_{i=1}^{t}E_A(A/\mathfrak{p}_i)$ by Lemma 3.1, we have $S^{-1}H\cong\mathrm{Hom}_A(S^{-1}K, S^{-1}K)\cong\bigoplus_{i=1}^{t}A_{\mathfrak{p}_i}\cong S^{-1}A=Q(A)$ because $\mathrm{Hom}_A(E_A(A/\mathfrak{p}_i), E_A(A/\mathfrak{p}_j))=0$ for $i\neq j$ and $\mathrm{Hom}_A(E_A(A/\mathfrak{p}_i), E_A(A/\mathfrak{p}_i))\cong A_{\mathfrak{p}_i}$. Hence we have that H is commutative and $A\subseteq H\subseteq Q(A)$. Other assertions are obvious.
(2) Let $\mathfrak{p}$ be an element of $\mathrm{Ass}_H(H)$. Then we have $\mathfrak{p}\cap S=\emptyset$, therefore $\mathfrak{p}\cap A=\mathfrak{p}_i$ for some i. Hence we have $\dim H/\mathfrak{p}=d$ because $H/\mathfrak{p}$ is an integral extension of $A/\mathfrak{p}_i$. Since A is unmixed and H is an integral extension of A, the assertion follows from [7, (34.6)].
(3) Since $\hat{H}\cong H\otimes_A\hat{A}\cong\mathrm{Hom}_{\hat{A}}(\hat{K},\hat{K})$ and $\hat{K}$ is (S_2) by (1.10), we have that $\hat{H}$ is (S_2).
(4) Because K satisfies the condition stated in (1.11.2), the natural map $h_K\colon K\to\mathrm{Hom}_A(H, K)$ is an isomorphism and it is easy to see that h_K is an H-homomorphism. Hence, for a maximal ideal $\mathfrak{n}$ of H, $K_{\mathfrak{n}}$ is a canonical module of $H_{\mathfrak{n}}$ by virtue of (2) and Theorem 1.2, and the assertion follows from (3) and Corollary 2.13. q. e. d.

Professor Goto told me the relation between H and the global transform and showed the coincidence of them for certain local rings, e.g. Buchsbaum rings of positive depth. But I here state the simplest case only.

Example 3.3 (Goto). Let k be a field, $P=k[\![X, Y, Z, W]\!]$ a formal power series ring of four variables over k, $R=P/(XZ, YW)$ and $A=P/(X, Y)\cap(Z, W)$. Then K, a canonical module of A, $\cong\mathrm{Hom}_R(A, R)\cong(XZ, YW)\colon(XZ, XW, YZ, YW)/(XZ, YW)=(XY, ZW, XZ, YW)/(XZ, YW)=(ZW, XZ, YW)/(XZ, YW)\oplus(XY, XZ, YW)/(XZ, YW)\cong P/(X, Y)\oplus P/(Z, W)$ and $H=\mathrm{End}_A(K)\cong P/(X, Y)\oplus P/(Z, W)$. On the other hand, $Q(A)\cong Q(k[\![Z, W]\!])\oplus Q(k[\![X, Y]\!])$ and A^g, the global transform of A, $=\{a\in Q(A)\mid\mathfrak{m}^s a\subseteq A$ for some s ($\mathfrak{m}$ is the maximal ideal of A)$\}\cong k[\![Z, W]\!]\oplus k[\![X, Y]\!]$. This example shows that H is not necessarily a local ring.

4. The flat base change

Proposition 4.1. *Let A be a local ring with maximal ideal $\mathfrak{m}$ and canonical module K and let B be a local ring with maximal ideal $\mathfrak{n}$. Assume that B is a flat A-algebra and $\mathfrak{m}B$ is an $\mathfrak{n}$-primary ideal. Then $K\otimes_A B$ is a canonical module of B if and only if $B/\mathfrak{m}B$ is a Gorenstein ring.* (cf. [6, Satz 6.14])

Proof. We may assume that A and B are both complete. Hence B has a canonical module, say L. Let $d=\dim A=\dim B$, $I=E_A(A/\mathfrak{m})$ and $J=E_B(B/\mathfrak{n})$. We note that, for an A-module M and for $i\geq 0$, $H^i_{\mathfrak{m}}(M)\otimes_A B\cong H^i_{\mathfrak{n}}(M\otimes_A B)$ by the assumption (cf. Proof of Theorem 2.3). The "if" part: We have $I\otimes_A B\cong J$ by Lemma 2.2(1). Since $H^d_{\mathfrak{n}}(B)\cong H^d_{\mathfrak{m}}(A)\otimes_A B\cong\mathrm{Hom}_A(K, I)\otimes_A B\cong\mathrm{Hom}_B(K\otimes_A B, I\otimes_A B)\cong\mathrm{Hom}_B(K\otimes_A B, J)$, we have $K\otimes_A B\cong\mathrm{Hom}_B(\mathrm{Hom}_B(K\otimes_A B, J), J)\cong\mathrm{Hom}_B(H^d_{\mathfrak{n}}(B), J)\cong L$. The "only if" part: Let $\mathfrak{a}=U_A(0)=\mathrm{ann}_A(K)$ (cf. (1.8)). Since $\mathfrak{a}\subseteq\mathfrak{m}$, we have $U_B(0)=\mathrm{ann}_B(L)=\mathfrak{a}B\subseteq\mathfrak{m}B$. By (1.8), K (resp. L) is a canonical module of $A/\mathfrak{a}$ (resp. $B/\mathfrak{a}B$). Hence we may assume that K and L are both faithful, considering $A/\mathfrak{a}$ (resp. $B/\mathfrak{a}B$) instead of A (resp. B). We note that $I\otimes_A B$ is an artinian B-module and $\mathrm{Hom}_B(I\otimes_A B, J)$ is finitely generated. Since $H^d_{\mathfrak{n}}(B)\cong \mathrm{Hom}_B(K\otimes_A B, I\otimes_A B)\cong \mathrm{Hom}_B(L, I\otimes_A B)$, we have $L\cong \mathrm{Hom}_B(H^d_{\mathfrak{n}}(B), J)\cong \mathrm{Hom}_B(\mathrm{Hom}_B(L, I\otimes_A B), J)\cong L\otimes_B \mathrm{Hom}_B(I\otimes_A B, J)$ because L is finitely generated and J is injective. Comparing the number of minimal generators, we have that $\mathrm{Hom}_B(I\otimes_A B, J)$ is cyclic. Because L is faithful, we have $\mathrm{Hom}_B(I\otimes_A B, J)\cong B$. Hence we have $I\otimes_A B\cong J$, therefore $B/\mathfrak{m}B$ is a Gorenstein ring by Lemma 2.2(2). q.e.d.

Theorem 4.2. *Let A be a local ring, T an A-module and B a faithfully flat local A-algebra with canonical module. If $T\otimes_A B$ is a canonical module of B, then T is a canonical module of A.*

Proof. First we note that T is a finitely generated A-module. It is sufficient to show that $\hat{T}$ is a canonical module of $\hat{A}$. Hence we may assume that A and B are both complete. Let $\mathfrak{m}$ be the maximal ideal of A, $\mathfrak{q}$ a minimal prime ideal of $\mathfrak{m}B$ and L a canonical module of B. Then $L_{\mathfrak{q}}$ is a canonical module of $B_{\mathfrak{q}}$ by (1.6) because B is complete, and $T\otimes_A B_{\mathfrak{q}}\cong L_{\mathfrak{q}}$. Hence furthermore we may assume that $B/\mathfrak{m}B$ is artinian (, considering $\widehat{B_{\mathfrak{q}}}$). (I) The case that B is (S_2): First we consider the case that B is (S_2). By Theorem 2.11, $B\ltimes L$ is a quasi-Gorenstein ring. Since $B\ltimes L\cong(A\ltimes T)\otimes_A B$ is flat over $A\ltimes T$, we have that $A\ltimes T$ is a quasi-Gorenstein ring by Theorem 2.3. Hence we have that T is a canonical module of A by Theorem 2.11. (II) The general case: We put $d=\dim A=\dim B$. Since $\mathrm{Ass}_B(L)=\{\mathfrak{q}\in \mathrm{Spec}(B)\mid \dim B/\mathfrak{q}=d\}$ by (1.7) and B is flat over A, we have $\mathrm{Ass}_A(T)=\{\mathfrak{p}\in \mathrm{Spec}(A)\mid \dim A/\mathfrak{p}=d\}$. Let $\mathfrak{p}$ be in $\mathrm{Ass}_A(T)$ and $\mathfrak{q}$ in $\mathrm{Ass}_B(B/\mathfrak{p}B)$. Then $\mathfrak{q}$ is in $\mathrm{Ass}_B(L)$ and $L_{\mathfrak{q}}\cong E_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}})$ by Lemma 3.1. Since $T_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}}B_{\mathfrak{q}}\cong L_{\mathfrak{q}}$, we have $T_{\mathfrak{p}}\cong E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}})$ by Lemma 2.2(2). Let $\mathfrak{a}=\mathrm{ann}_A(T)$. Then $\mathfrak{a}A_{\mathfrak{p}}=\mathrm{ann}_{A_{\mathfrak{p}}}(T_{\mathfrak{p}})=\mathrm{ann}_{A_{\mathfrak{p}}}(E_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}))=0$ for every $\mathfrak{p}$ in $\mathrm{Ass}_A(T)$. Hence we have $\mathfrak{a}=U_A(0)$ because $\mathrm{Ass}_A(T)=\{\mathfrak{p}\in \mathrm{Spec}(A)\mid \dim A/\mathfrak{p}=d\}$. Because $U_B(0)=\mathrm{ann}_B(L)=\mathfrak{a}B$ (cf. (1.8)), we may assume that $U_A(0)=0$ and $U_B(0)=0$ by virtue of (1.8) and (1.12), considering $A/\mathfrak{a}$ (resp. $B/\mathfrak{a}B$) instead of A (resp. B). We set $R=\mathrm{End}_A(T)$. Looking at the proof of Theorem 3.2 again, it is known that R possesses the properties stated in Theorem 3.2(1) and (2) from the above argument. We set $S=\mathrm{End}_B(L)$. Then $S\cong R\otimes_A B$ is a faithfully flat R-algebra and $T\otimes_R S\cong L$. Let $\mathfrak{r}$ be a maximal ideal of R and $\mathfrak{s}$ a maximal ideal of S lying over $\mathfrak{r}$. By Theorem 3.2(4), $L_{\mathfrak{s}}$ is a canonical module of $S_{\mathfrak{s}}$. Since $T_{\mathfrak{r}}\otimes_{R_{\mathfrak{r}}}S_{\mathfrak{s}}\cong L_{\mathfrak{s}}$ and $S_{\mathfrak{s}}$ is (S_2) by Theorem 3.2(3), we have that $T_{\mathfrak{r}}$ is a canonical module of $R_{\mathfrak{r}}$ by the case (I). Since A is complete, A has a canonical module, say K. Then, for every maximal ideal $\mathfrak{r}$ of R, $\mathrm{Hom}_A(R, K)_{\mathfrak{r}}$ is a canonical module of $R_{\mathfrak{r}}$ by Theorem 1.2. Hence we have $T_{\mathfrak{r}}\cong \mathrm{Hom}_A(R, K)_{\mathfrak{r}}$ for every maximal ideal $\mathfrak{r}$ of R by (1.5). Therefore we have $T\cong \mathrm{Hom}_A(R, K)$ because R is a complete semi-local ring. We consider the exact sequence $0\to A\xrightarrow{f} R\to Z\to 0$ where f is the natural map and $Z=\mathrm{Coker} f$. Let $\mathfrak{p}$ be a prime ideal of A such that height $\mathfrak{p}\leq 1$ and let $\mathfrak{q}$ be a minimal prime ideal of $\mathfrak{p}B$. Then height $\mathfrak{q}=$ height $\mathfrak{p}\leq 1$ and $B_{\mathfrak{q}}$ is Cohen-Macaulay because of $U_B(0)=0$. Since $T_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}}B_{\mathfrak{q}}\cong L_{\mathfrak{q}}$ is a canonical module of $B_{\mathfrak{q}}$ by (1.6), $T_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ by the case (I). Hence $f_{\mathfrak{p}}$ is an isomorphism because $A_{\mathfrak{p}}$ is Cohen-Macaulay, and $Z_{\mathfrak{p}}=0$, that is, height ann$(Z)>1$ if $Z\neq 0$ and $d>1$. By virtue of (1.10), we have $\mathrm{Hom}_A(Z, K)=0$ and $\mathrm{Ext}_A^1(Z, K)=0$. Therefore, from the above exact sequence, we have $\mathrm{Hom}_A(R, K)\cong \mathrm{Hom}_A(A, K)\cong K$. Hence we obtain $T\cong K$, a canonical module of A.

Finally we note that $B/\mathfrak{m}B$ is a Gorenstein ring by Proposition 4.1, which means that $B_{\mathfrak{q}}/\mathfrak{m}B_{\mathfrak{q}}$ is a Gorenstein ring for every minimal prime ideal $\mathfrak{q}$ of $\mathfrak{m}B$ in the statement of the theorem where $\mathfrak{m}$ is the maximal ideal of A. q.e.d.

Corollary 4.3. *Let A be a local ring with canonical module K and let $\mathfrak{p}$ be in* $\mathrm{Supp}_A(K)$. *Then $K_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ and, for every minimal prime ideal $\mathfrak{q}$ of $\mathfrak{p}\hat{A}$, $\hat{A}_{\mathfrak{q}}/\mathfrak{p}\hat{A}_{\mathfrak{q}}$ is a Gorenstein ring.*

Proof. Since $K_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}}\hat{A}_{\mathfrak{q}}\cong\hat{K}_{\mathfrak{q}}$ is a canonical module of $\hat{A}_{\mathfrak{q}}$ by (1.6), we have that $K_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$, applying Theorem 4.2 to the flat local homomorphism $A_{\mathfrak{p}}\to\hat{A}_{\mathfrak{q}}$. The Gorensteinness of $\hat{A}_{\mathfrak{q}}/\mathfrak{p}\hat{A}_{\mathfrak{q}}$ follows from Proposition 4.1. q.e.d.

Looking at the proof of [1, Proposition 2] again, we can remove the symbol "^" from the statement of (1.11.2) by virtue of Corollary 4.3. Hence we have the following

Proposition 4.4. *With the same notation as in* (1.11), *the following are equivalent for M with* $\dim M=d$:

(a) *The map h_M is an isomorphism.*

(b) *$\hat{M}$ is (S_2) and* $\dim A/\mathfrak{p}=d$ *for every $\mathfrak{p}$ in* $\mathrm{Min}_A(M)$.

(c) *M is (S_2) and* $\dim A/\mathfrak{p}=d$ *for every $\mathfrak{p}$ in* $\mathrm{Min}_A(M)$.

Remark 4.5. Recently Professor Ogoma of Kochi University proved that the property (S_2) implies the equidimensionality for certain local rings, e.g. local rings with canonical modules and gave a proof of Proposition 4.4 which does not need Corollary 4.3. His result and proof say that the condition of equidimensionality in Proposition 4.4(b) and (c) is superfluous in the essential case, that is, in the case that M has no direct summand of dimension $<d$. Hence we can remove the symbol "^" from the statement of Theorem 2.11 and the assumption of Corollary 2.13.

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