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On the characters of the averaged discrete series representations of semisimple Lie groups

Dedicated to Professor Hisaaki Yoshizawa on his 60th birthday

By

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Introduction.

Let G be a real connected semisimple Lie group with finite center. Suppose that G is acceptable and has a compact Cartan subgroup B. We denote the Lie algebras of G and B by $\mathfrak{g}$ and $\mathfrak{b}$ respectively.

Let $\mathfrak{b}_B^*$ be a lattice in $\sqrt{-1}\mathfrak{b}^*$ consisting of λ such that the mapping $\xi_\lambda(\exp X)=e^{\lambda(X)}(X\in\mathfrak{b})$ defines a unitary character of B. Then for each regular λ, Harish-Chandra proved that there exists uniquely a certain kind of tempered invariant eigendistribution Θ_λ, which is the character of a discrete series representation up to a known multiplicative sign ± 1.

Let $\Phi(\mathfrak{g}_c, \mathfrak{b}_c)$ be the root system of $(\mathfrak{g}_c, \mathfrak{b}_c)$ and $W(\mathfrak{g}_c, \mathfrak{b}_c)$ be its Weyl group. Here we take $\lambda\in\mathfrak{b}_B^*$ such that $w\lambda$ also belongs to $\mathfrak{b}_B^*$ for any $w\in W(\mathfrak{g}_c, \mathfrak{b}_c)$. For such a λ, we consider the explicit expression of the sum

$$\Theta_\lambda^* = \sum_{w\in W(\mathfrak{g}_c, \mathfrak{b}_c)} \operatorname{sgn}(w)\Theta_{w\lambda}.$$

Up to a well-determined sign ± 1, this is the sum of characters of discrete series representations with the same infinitesimal character. It also represents the contribution of these discrete series in the Plancherel formula for G.

Let $\mathfrak{h}$ be a Cartan subalgebra and H be the Cartan subgroup of G corresponding to it. Harish-Chandra showed that there exist constants $c(w\lambda, H^+)$ such htat

$$\Delta^{\mathfrak{h}}\Theta_\lambda^* = \sum_{w\in W(\mathfrak{g}_c, \mathfrak{b}_c)} \operatorname{sgn}(w)c(w\lambda, H^+)\xi_{w,\lambda} \qquad \text{on} \quad H^+,$$

where H^+ is a connected component of $H'(R)$, and $\Delta^{\mathfrak{h}}$ a well-known function on H, and $\xi_{w,\lambda}$ a character of H. (For their exact definitions, see §1). R. Herb determined the constants $c(w\lambda, H^+)$ for G_2 in [2], and for the case that the root system $\Phi(\mathfrak{g}_c, \mathfrak{b}_c)$ contains no simple components of exceptional type in [3].

In this note, we remove this restriction and determine the constants $c(w\lambda, H^+)$ for general cases. Their explicit expression is given in §2. This is our main theorem. After R. Herb, our method is based on an induction on the ranks of root systems.

For this purpose, we use two key lemmas (Lemmas 1 and 2). Main theorem and Lemma 1 are proved in §3. Two sections, §4 and §5, are devoted to proving Lemma 2. To be more precise, let Φ be a simple component of $\Phi(\mathfrak{g}_c, \mathfrak{b}_c)$. Lemma 2 states the relation between the constants corresponding to Φ and those corresponding to a root subsystem of Φ. In §4, we treat the case that Φ is a (not necessarily exceptional) root system of class I. In §5, we treat the case that Φ is of type F_4, which is the unique simple exceptional root system of class II.

In conclusion, these formulas could be obtained by summing up Hirai's general character formula, but our method is direct and much easier in total especially when Φ is of class II.

§1. Preliminaries

We first establish some notations here. Let K be a maximal compact subgroup of G and denote θ the corresponding Cartan involution. Then $\mathfrak{g}$ has the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K. For any θ-stable Cartan subalgebra $\mathfrak{h}$ and the corresponding Cartan subgroup H, we have $\mathfrak{h}=\mathfrak{h}_{\mathfrak{k}}+\mathfrak{h}_{\mathfrak{p}}$ (direct sum) and $H=H_K H_{\mathfrak{p}}$ (direct product). Here $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{h}\cap\mathfrak{k}$, $\mathfrak{h}_{\mathfrak{p}}=\mathfrak{h}\cap\mathfrak{p}$, $H_K=H\cap K$ and $H_{\mathfrak{p}}=\exp\mathfrak{h}_{\mathfrak{p}}$. We denote by $\Phi(\mathfrak{g}_c, \mathfrak{h}_c)$ and $P(\mathfrak{g}_c, \mathfrak{h}_c)$ the root system of $(\mathfrak{g}_c, \mathfrak{h}_c)$ and a set of positive roots in $\Phi(\mathfrak{g}_c, \mathfrak{h}_c)$ respectively. Let G_c be a complexification of G and H_c denote the Cartan subgroup of G_c associated with the Cartan subalgebra $\mathfrak{h}_c$ (the complexification of $\mathfrak{h}$). Then $\phi(H)\subset H_c$, where ϕ is the homomorphism of G into G_c which corresponds to the canonical injection of $\mathfrak{g}$ into $\mathfrak{g}_c$. For $\lambda\in\mathfrak{h}_c^*$ (the dual space of $\mathfrak{h}_c$), we define ξ_λ on H_c by $\xi_\lambda(\exp H)=e^{\lambda(H)}$ $(H\in\mathfrak{h}_c)$ whenever this gives a well-defined character of H_c. Then $\xi_\lambda\circ\phi$ is a character of H and we write ξ_λ instead of $\xi_\lambda\circ\phi$ for brevity. We say that G is acceptable when there exists a complexification G_c with the property that ξ_ρ is well-defined. Here ρ is half the sum of $\alpha\in P(\mathfrak{g}_c, \mathfrak{h}_c)$. Put

$$\Delta^{\mathfrak{h}}(h)=\xi_\rho(h)\prod_{\alpha\in P(\mathfrak{g}_c,\mathfrak{h}_c)}(1-\xi_\alpha(h)^{-1}) \qquad (h\in H).$$

For a pair (G, H) as above, we define $W_G(H)$ as $W_G(H)=N_G(H)/H$, where $N_G(H)$ is the normalizer of H in G. Then we can regard $W_G(H)$ as a subgroup of $W(\mathfrak{g}_c, \mathfrak{h}_c)$, the Weyl group of the root system $\Phi(\mathfrak{g}_c, \mathfrak{h}_c)$. Since two Cartan subalgebras $\mathfrak{h}_c$ and $\mathfrak{b}_c$ are conjugate in $\mathfrak{g}_c$, there exists an element $y\in G_c$ such that $\mathrm{Ad}(y)\mathfrak{b}_c=\mathfrak{h}_c$. For $\lambda\in\mathfrak{b}_c^*$, we define an element $\lambda^y\in\mathfrak{h}_c^*$ as $\lambda^y(H)=\lambda(\mathrm{Ad}(y^{-1})H)$. Then obviously

$$\Phi(\mathfrak{g}_c, \mathfrak{h}_c)=\{\alpha^y;\ \alpha\in\Phi(\mathfrak{g}_c, \mathfrak{b}_c)\}.$$

In the following, we fix y and a positive system $P(\mathfrak{g}_c, \mathfrak{b}_c)$ in $\Phi(\mathfrak{g}_c, \mathfrak{b}_c)$ and set $P(\mathfrak{g}_c, \mathfrak{h}_c)=\{\alpha^y;\ \alpha\in P(\mathfrak{g}_c, \mathfrak{b}_c)\}$.

Let $H'(R)$ be the totality of elements h in H such that $\xi_\alpha(h)\neq 1$ for any real root $\alpha\in\Phi(\mathfrak{g}_c, \mathfrak{h}_c)$, and H^+ be a connected component of $H'(R)$. Here we call a root α real if $\alpha(\mathfrak{h})\subset\boldsymbol{R}$. Put $H_K^+=H_K\cap H^+$ and $\mathfrak{z}=\{X\in\mathfrak{g};\ \mathrm{Ad}(h)X=X \text{ for any } h\in H_K^+\}$.

We can assume that $H_K^+ \subset B$ without loss of generality. Then $\mathfrak{z}$ is a reductive subalgebra which contains $\mathfrak{b}$ and $\mathfrak{h}$. Put $\Phi = \Phi(\mathfrak{z}_c, \mathfrak{b}_c)$. Then we can consider Φ as a root subsystem of $\Phi(\mathfrak{g}_c, \mathfrak{b}_c)$ and for any $\alpha \in \Phi$, α^ν is a real root of $(\mathfrak{z}_c, \mathfrak{h}_c)$. Furthermore, $W(\Phi)$, the Weyl group of Φ, can be considered as a subgroup of $W(\mathfrak{g}_c, \mathfrak{b}_c)$. Put

$$\Phi^+ = \{\alpha \in \Phi;\ \alpha^\nu(\log h) > 0 \quad \text{for an element } h \text{ in } H^+\}.$$

Then Φ^+ is a positive system of Φ and it depends only on H^+ which contains h but not on h.

Now we state the result of Harish-Chandra about the character of averaged discrete series ([1]). It tells that there exist constants $c(w\lambda, H^+)$ such that for $h = h_K h_p \in H^+$,

$$\Delta^{\mathfrak{h}}(h) \sum_{w \in W(\mathfrak{g}_c, \mathfrak{b}_c)} \operatorname{sgn}(w) \Theta_{w\lambda}(h) = \sum_{w \in W(\mathfrak{g}_c, \mathfrak{b}_c)} c(w\lambda, H^+) \xi_{w,\lambda}(h), \qquad \cdots\cdots\cdots(1)$$

where $h_K \in H_K^+$, $h_p \in H^+ \cap H_{\mathfrak{p}}$ and $\xi_{w,\lambda}(h) = \xi_{w\lambda}(h_K) \exp((w\lambda)^\nu(\log(h_p)))$. After R. Herb, put $c(w\lambda, \Phi^+) = c(w\lambda, H^+)/|W_G(B)|$, where $|W_G(B)|$ denotes the order of the group $W_G(B)$. Our task is to determine these constants explicitly which will be done in the following sections.

We note here some properties of the constants $c(\lambda, \Phi^+)$ given in [3].

1) For any $s \in W(\Phi)$, it holds that $c(s\lambda, s\Phi^+) = c(\lambda, \Phi^+)$.

2) Let $\Psi = \{\gamma_1, \ldots, \gamma_l\}$ be the fundamental system of Φ^+ and $\{\omega_1, \ldots, \omega_l\}$ be the set of fundamental weights with respect to Ψ. In other words, each ω_i belongs to $\mathfrak{b}_c^*$ and $\langle \omega_i, \gamma_j \rangle = \delta_{ij}$, where $\langle\,,\rangle$ denotes the inner product in $\mathfrak{b}_c^*$ induced by the Killing form. Then $c(\lambda, \Phi^+) = 0$ if $\langle \lambda, \omega_i \rangle > 0$ for all $1 \le i \le l$. Temperedness of Θ's leads this equality.

3) For any $\gamma \in \Psi$, put $\Phi_0 = \{\beta \in \Phi;\ \langle \beta, \gamma \rangle = 0\}$ and $\Phi_0^+ = \Phi_0 \cap \Phi^+$. Then

$$c(\lambda, \Phi^+) + c(s_\gamma \lambda, \Phi^+) = 2c(\lambda, \Phi_0^+), \qquad \cdots\cdots\cdots(2)$$

where s_γ denotes the reflection corresponding to γ. This equality follows from the "patching conditions" for $\Delta^{\mathfrak{h}}\Theta_\lambda$ and $\Delta^{\mathfrak{h}^\gamma}\Theta_\lambda$ on H_γ^+. Here $H_\gamma^+ = \{h \in Cl(H^+);\ \xi_\gamma(h) = 1\}$ and $\mathfrak{h}^\gamma = \boldsymbol{R}(X_\gamma - X_{-\gamma}) + \{H \in \mathfrak{h};\ \gamma(H) = 0\}$, where $Cl(E)$ denotes the closure of a set E and X_γ and $X_{-\gamma}$ are root vectors of γ and $-\gamma$ respectively which satisfies certain conditions. (See §2 in [4].) In other words, put $\nu_\gamma = \exp\sqrt{-1}\pi \operatorname{ad}(X_\gamma + X_{-\gamma})/4$, then $\mathfrak{h}^\gamma = \nu_\gamma \mathfrak{h}_c \cap \mathfrak{g}$, and it is a Cartan subalgebra of $\mathfrak{g}$ not conjugate to $\mathfrak{h}$ in $\mathfrak{g}$.

4) For two regular elements λ_1 and λ_2 in $\mathfrak{b}_B^*$, suppose that the signs of both $\langle \lambda_1, \alpha \rangle$ and $\langle \lambda_2, \alpha \rangle$ are the same for any $\alpha \in \Phi^+$. Then $c(\lambda_1, \Phi^+) = c(\lambda_2, \Phi^+)$.

5) When Φ is not simple, let $\Phi = \Phi_1 \cup \cdots \cup \Phi_s$ be a decomposition of Φ into its simple components. Put $\Phi_i^+ = \Phi_i \cap \Phi^+$, then

$$c(\lambda, \Phi^+) = \prod_{i=1}^{s} c(\lambda, \Phi_i^+). \qquad \cdots\cdots\cdots(3)$$

§ 2. Explicit expression of constants $c(\lambda, \Phi^+)$ (Main Theorem)

Owing to the property 5) in §1, we only have to determine the constants $c(w\lambda, \Phi^+)$ for the case that Φ is simple. According as Φ is a root system of class I or class II, we define the function $P(\lambda; \Phi^+)$ separatedly.

Case 1: Φ is a simple root system of class I (that is, of type A_1, D_{2n}, E_7, E_8 or G_2).

Let $F=\{\alpha_1,\dots, \alpha_l\}$ be the standard maximal positive orthogonal system in Φ^+. A maximal positive orthogonal system $F'=\{\alpha'_1,\dots, \alpha'_l\}$ is called standard if it satisfies the following condition: α'_1 is the highest root in Φ^+ and for $2\leq i\leq l$, α'_i is the highest root in $\Phi'_{i-1}=\{\beta\in\Phi; \langle\beta, \alpha'_j\rangle=0\ (j=1,\dots, i-1)\}$ with respect to $\Phi'_{i-1}\cap\Phi^+$. In this case, F is uniquely determined by the positive system Φ^+. Then we define a subset S' of $W(\Phi)$ as follows:

$$S'=\{w\in W(\Phi);\quad s\alpha_i>0 \qquad \text{for} \quad i=1,\dots, l\}.$$

We say that two elements s_1 and s_2 in S' are equivalent when $s_1F=s_2F$ as sets, where $sF=\{s\alpha_1,\dots, s\alpha_l\}$. We take S as a complete system of representatives of S'. By Lemma 1.2 in [4], we know that for two equivalent elements s_1 and s_2, $\mathrm{sgn}\,(s_1)=\mathrm{sgn}\,(s_2)$. Then we define the function $P(\lambda; \Phi_+)$ as

$$P(\lambda; \Phi^+)=\sum_{\sigma\in S}\mathrm{sgn}\,(\sigma)\prod_{i=1}^{l} c_1((\sigma^{-1}\lambda)_i). \qquad \cdots\cdots\cdots(4)$$

(This does not depend on the choice of S in S'). Here $(\sigma^{-1}\lambda)_i=(\sigma^{-1}\lambda)(\alpha_i)$ and c_1 is the function defined as

$$c_1(X)=2 \quad \text{if} \quad X>0 \quad \text{and} \quad c_1(X)=0 \quad \text{if} \quad X<0.$$

Case 2: Φ is a simple root system of class II (that is, of type B_n, C_n or F_4).

Let $E=(\alpha_1,\dots, \alpha_n)$ be a standard maximal orthogonal positive *ordered* system in Φ^+ (for detailed definition for a class II root system Φ, see §2 in [4] or §1 in [6]) which satisfies the following condition: *For type of B_n and F_4, E contains at most one short root as its components and for type C_n, E contains at most one long root as its components.*

For example, let

$\Phi=\{\pm e_i\pm e_j\ (1\leq i<j\leq n),\ \pm e_i\ (1\leq i\leq n)\}$ for type B_n,

$\Phi=\{\pm e_i\pm e_j\ (1\leq i<j\leq n),\ \pm 2e_i\ (1\leq i\leq n)\}$ for type C_n,

$\Phi=\{\pm e_i\ (1\leq i\leq 4),\ \pm e_i\pm e_j\ (1\leq i<j\leq 4),\ (\pm e_1\pm e_2\pm e_3\pm e_4)/2\}$ for type F_4.

Choose Φ^+ as

$\Phi^+=\{e_i\pm e_j\ (1\leq i<j\leq n),\ e_i\ (1\leq i\leq n)\}$ for type B_n,

$\Phi^+ = \{e_i \pm e_j\ (1 \leq i < j \leq n),\ 2e_i\ (1 \leq i \leq n)\}$ for type C_n,

$\Phi^+ = \{e_i (1 \leq i \leq 4),\ e_i \pm e_j\ (1 \leq i < j \leq 4),\ (e_1 \pm e_2 \pm e_3 \pm e_4)/2\}$ for type F_4.

Then E is uniquely given as

$E = (e_1 + e_2,\ e_1 - e_2, \dots,\ e_{2m-1} - e_{2m},\ [e_n])$ for type B_n

$E = (e_1 + e_2,\ e_1 - e_2, \dots,\ e_{2m-1} - e_{2m},\ [2e_n])$ for type C_n,

$E = (e_1 + e_2,\ e_1 - e_2,\ e_3 + e_4,\ e_3 - e_4)$ for type F_4,

where $[n/2] = m$ and the notation $[e_n]$ (or $[2e_n]$) means that the element e_n (or $2e_n$) appears only when n is odd. (In order to treat the three cases uniformly, the short roots are ahead of long one for type C_n against the definitions in [4] or [6].)

Now we define a subset S of $W(\Phi)$ as follows:

$$S = \left\{ \begin{array}{c} \sigma \in W(\Phi);\ \sigma(\alpha_i) > 0\ (1 \leq i \leq n),\ \sigma(\alpha_{2i-1}) > \sigma(\alpha_{2i})\ (1 \leq i \leq m) \\ \text{and } \sigma(\alpha_1) > \sigma(\alpha_3) > \cdots > \sigma(\alpha_{2m-1}) \end{array} \right\}.$$

We put

$$P(\lambda;\ \Phi^+) = \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} c_2((\sigma^{-1}\lambda)'_i) \qquad \text{when } n = 2m,$$

$$= \sum_{\sigma \in S} \operatorname{sgn}(\sigma) \prod_{i=1}^{m} c_2((\sigma^{-1}\lambda)'_i) c_1((\sigma^{-1}\lambda)_n) \qquad \text{when } n = 2m+1, \quad \cdots\cdots(5)$$

where $(\sigma^{-1}\lambda)'_i = ((\sigma^{-1}\lambda)(\alpha_{2i-1}),\ (\sigma^{-1}\lambda)(\alpha_{2i})) \in \boldsymbol{R}^2$ and that function $c_2(X, Y)$ is defined as follows:

$$c_2(X, Y) = \begin{cases} 4 & \text{if } 0 > X > Y \ \text{ or } \ 0 > -Y > X, \\ 0 & \text{otherwise.} \end{cases}$$

After these preparations, we state our main theorem.

Theorem. *Let Φ be a simple root system of class I or class II. Then the constants $c(\lambda, \Phi^+)$ is given for a regular element λ in $\mathfrak{b}_B^*$ as*

$$c(\lambda,\ \Phi^+) = P(\lambda;\ \Phi^+),$$

where $P(\lambda;\ \Phi^+)$ is the function defined above ((4), (5)).

Note. In [3], R. Herb proved the above formula for Φ a simple root system of classical type. Our theorem assirts that the analogous formula holds in general.

§ 3. Proof of main theorem

In this section, we prepare the following two lemmas and prove our main theorem with the aid of them just as in [3].

Lemma 1. *Let $\lambda \in \mathfrak{b}_B^*$ be a regular element such that $\langle \lambda, \gamma_i \rangle > 0$ for all $\gamma_i \in \Psi$. Then*

$$P(\lambda; \Phi^+)=0 \quad and \quad c(\lambda, \Phi^+)=0.$$

Proof. Since any fundamental weight ω_i is a positive combination of positive roots, $\langle \lambda, \omega_i \rangle > 0$ for all $1 \leq i \leq l$. Then $c(\lambda, \Phi^+)=0$ by 2) in §1. On the other hand, for any $\sigma \in S$, $\sigma(\alpha_i)>0$, so $(\sigma^{-1}\lambda)(\alpha_i)>0$. Therefore by the definition of $P(\lambda; \Phi^+)$, we have $P(\lambda; \Phi_+)=0$. Q.E.D.

Lemma 2. *Let the rank of Φ be n, and assume that the theorem is true for root systems of rank less than n. Then for $\gamma \in \Psi$,*

$$P(\lambda; \Phi^+)+P(s_\gamma\lambda; \Phi^+)=2c(\lambda, \Phi_0^+), \qquad \cdots\cdots\cdots(6)$$

where $\Phi_0^+=\{\alpha \in \Phi^+ ; \langle \alpha, \gamma \rangle =0\}$.

We will prove this lemma in the following two sections.

Assuming these two lemmas as granted, we give here a proof of the main theorem. For a regular $\lambda \in \mathfrak{b}_B^*$ such that $\langle \lambda, \gamma_i \rangle > 0$ for all $\gamma_i \in \Psi$, we have

$$\begin{aligned} P(s_\gamma\lambda; \Phi^+) &= P(s_\gamma\lambda; \Phi^+)+P(\lambda; \Phi^+) && \text{(by Lemma 1)} \\ &= 2c(\lambda, \Phi_0^+) && \text{(by Lemma 2)} \\ &= c(\lambda, \Phi^+)+c(s_\gamma\lambda, \Phi^+) && \text{(by 3) in §1)} \\ &= c(s_\gamma\lambda, \Phi^+) && \text{(by Lemma 1),} \end{aligned}$$

for $\gamma \in \Psi$. Then using Lemma 2 repeatedly, we have for any $w \in W(\Phi)$, $P(w\lambda; \Phi^+)= c(w\lambda, \Phi^+)$. Together with the property 4) in §1, we have $c(\lambda, \Phi^+)=P(\lambda, \Phi^+)$ for any regular element λ in $\mathfrak{b}_B^*$. Hence we get the theorem.

§4. Proof of Lemma 2 (Case of class I)

In this section, we prove Lemma 2 for a simple root system Φ of class I. Fix $\gamma \in \Psi$ and divide S into the following two subsets S_1 and S_2:

$$S_1=\{\sigma \in S; \gamma \notin \sigma F\} \quad \text{and} \quad S_2=\{\sigma \in S; \gamma \in \sigma F\}.$$

Then for $\sigma \in S_1$, $s_\gamma\sigma\alpha_j>0$ for all j, so $s_\gamma\sigma$ also belongs to S'. Comparing the signs of $s_\gamma\sigma$ and σ, we get that $s_\gamma\sigma$ is not equivalent to σ. Therefore we can renumber the elements S as $S=\{\sigma_1, \sigma_2,\dots, \sigma_{2m}\}$, $\sigma_{i+m} \sim s_\gamma\sigma_i$ for $i=1,\dots, m$.

Next, we consider $\sigma \in S_2$. Then $\sigma F \backslash \{\gamma\}$ is a maximal orthogonal positive system for Φ_0^+. Here again we quote a lemma, Lemma 8.2 in [4], about the simple root system of class I. It says that there exists such an element w in $W(\Phi_0)$ that $wF_0= \sigma F \backslash \{\gamma\}$, where $F_0=\{\beta_1,\dots, \beta_{l-1}\}$ is the standard maximal orthogonal positive system of Φ_0^+. Furthermore $\mathrm{sgn}(w)=\mathrm{sgn}(\sigma)$.

Put $S_0'=\{s \in W(\Phi_0); s\beta_j>0$ for all $\beta_j \in F_0\}$. Let S_0 be the totality of $w \in W(\Phi_0)$ corresponding to $\sigma \in S_2$. Then S_0 is a complete system of representatives of S_0'

under the similar equivalence relation in S_0' as in S'. Obviously the correspondence $\sigma \in S_2 \to w \in S_0$ is bijective.

Now we calculate the left hand side of the equality in Lemma 2. At first we get for S_1 that

$$\sum_{\sigma\in S_1} \operatorname{sgn}(\sigma) \prod_{i=1}^{l} c_1((\sigma^{-1}\lambda)_i) + \sum_{\sigma\in S_1} \operatorname{sgn}(\sigma) \prod_{i=1}^{l} c_1((\sigma^{-1}s_\gamma\lambda)_i)$$

$$= \sum_{j=1}^{m} \operatorname{sgn}(\sigma_j) \prod_{i=1}^{l} c_1((\sigma_j^{-1}\lambda)_i) + \sum_{j=1}^{m} \operatorname{sgn}(s_\gamma\sigma_j) \prod_{i=1}^{l} c_1(((s_\gamma\sigma_j)^{-1}\lambda)_i)$$

$$+ \sum_{j=1}^{m} \operatorname{sgn}(\sigma_j) \prod_{i=1}^{l} c_1((\sigma_j^{-1}s_\gamma\lambda)_i) \sum_{j=1}^{m} \operatorname{sgn}(s_\gamma\sigma_j) \prod_{i=1}^{l} c_1((s_\gamma\sigma_j)^{-1}s_\gamma\lambda)_i)$$

$$=0.$$

Next, for S_2,

$$\sum_{\sigma\in S_2} \operatorname{sgn}(\sigma) \prod_{i=1}^{l} c_1((\sigma^{-1}\lambda)_i) + \sum_{\sigma\in S_2} \operatorname{sgn}(\sigma) \prod_{i=1}^{l} c_1((\sigma^{-1}s_\gamma\lambda)_i)$$

$$= \sum_{w\in S_0} \operatorname{sgn}(w) \prod_{i=1}^{l-1} c_1(\lambda(w\beta_i))c_1(\lambda(\gamma))$$

$$+ \sum_{w\in S_0} \operatorname{sgn}(w) \prod_{i=1}^{l-1} c_1(\lambda(s_\gamma w\beta_i))c_1(\lambda(s_\gamma\gamma))$$

$$=\{ \sum_{w\in S_0} \operatorname{sgn}(w) \prod_{i=1}^{l-1} c_1(\lambda(w\beta_i))\} \{c_1(\lambda(\gamma))+c_1(-\lambda(\gamma))\}$$

$$=2 \sum_{w\in S_0} \operatorname{sgn}(w) \prod_{i=1}^{l-1} c_1(\lambda(w\beta_i)).$$

Therefore, by induction hypothesis applied on Φ_0^+, we get that

$$P(\lambda;\ \Phi^+)+P(s_\gamma\lambda;\ \Phi^+)=2c(\lambda,\ \Phi_0^+).$$

Q. E. D.

§5. Proof of Lemma 2 (Case of class II)

5.1 In this section, we prove Lemma 2 for the root system Φ of type F_4. We assume that Φ and Φ^+ are represented as in §2. Then $\Psi=\{e_2-e_3,\ e_3-e_4,\ e_4,\ (e_1-e_2-e_3-e_4)/2\}$ is the fundamental system of Φ. Put $\alpha_1=e_1+e_2$, $\alpha_2=e_1-e_2$, $\alpha_3=e_3+e_4$ and $\alpha_4=e_3-e_4$, then $E=(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the standard maximal orthogonal positive ordered system. As is easily seen, for $\gamma=e_2-e_3$, or e_3-e_4, Φ_0 is of type C_3. For $\gamma=e_4$ or $(e_1-e_2-e_3-e_4)/2$, Φ_0 is of type B_3. In both cases, Φ_0 is a simple root system of classical type.

Just like the previous section, we calculate the left hand side of the equality in Lemma 2. To do it, we prepare the elementary lemma about the functions c_1 and c_2 in §2.

Lemma 3. *For two non-zero integers m and n, we have*

1) $c_2(n, m)+c_2(n, -m)=2c_1(n)$,
2) $c_2(m, n)=c_2(m+n, m-n)$.

Proof. We can easily check these relations.

5.2 First we consider the case that $\gamma\in\Psi$ is a long root. In this case we divide S into two subsets S_1 and S_2, where

$S_1=\{s\in S;$ the set $\{s\alpha_i\}_{i=1,\dots,4}$ does not contain $\gamma\}$, and $S_2=S\setminus S_1$.

Then for each $s\in S_1$, $s(e_i)$ $(1\le i\le 4)$ and $s((e_1-e_2-e_3-e_4)/2)$ are positive roots. So $s_\gamma s(e_i)$ and $s_\gamma s((e_1-e_2-e_3-e_4)/2)$ are positive. Therefore $s_\gamma s$ also belongs to S. Hence we can renumber the elements of S_1 as $S_1=\{\sigma_1,\dots,\sigma_6\}$, $\sigma_{i+3}=s_\gamma\sigma_i$ $(i=1, 2, 3)$ (See also [7]). Just as in §4, it follows from this that

$$\sum_{j=1}^{6}\operatorname{sgn}(\sigma_j)\prod_{i=1}^{2}c_2((\sigma^{-1}\lambda)'_i)+\sum_{j=1}^{6}\operatorname{sgn}(\sigma_j)\prod_{i=1}^{2}c_2((\sigma^{-1}s_\gamma\lambda)'_i)$$

$$=\sum_{j=1}^{3}\operatorname{sgn}(\sigma)\prod_{i=1}^{2}c_2((\sigma_j^{-1}\lambda)'_i)+\sum_{j=1}^{3}\operatorname{sgn}(s_\gamma\sigma_j)\prod_{i=1}^{2}c_2(((s_\gamma\sigma_j)^{-1}\lambda)'_i)$$

$$+\sum_{j=1}^{3}\operatorname{sgn}(\sigma_j)\prod_{i=1}^{2}c_2((\sigma^{-1}s_\gamma\lambda)'_i)+\sum_{j=1}^{3}\operatorname{sgn}(s_\gamma\sigma_j)\prod_{i=1}^{2}c_2(((s_\gamma\sigma_j)^{-1}s_\gamma\lambda)'_i)$$

$$=0.$$

Now the number of the elements in S_2 is 3. Take $\gamma=e_2-e_3$. Then S_2 consists of the following three elements σ_7, σ_8, σ_9:

$$\sigma_7E=(\sigma_7\alpha_1, \sigma_7\alpha_2, \sigma_7\alpha_3, \sigma_7\alpha_4)=(e_1+e_4, e_1-e_4, e_2+e_3, e_2-e_3),$$

$$\sigma_8E=(e_1+e_4, e_2+e_3, e_1-e_4, e_2-e_3)$$

$$\sigma_9E=(e_1+e_4, e_2-e_3, e_1-e_4, e_2+e_3).$$

Then $\operatorname{sgn}\sigma_7=\operatorname{sgn}\sigma_9=-\operatorname{sgn}\sigma_8=1$. On the other hand,

$$\prod_{i=1}^{2}c_2((\sigma_7^{-1}\lambda)'_i)+\prod_{i=1}^{2}c_2((\sigma_7^{-1}s_\gamma\lambda)'_i)$$

$$=c_2(\lambda(e_1+e_4), \lambda(e_1-e_4))c_2(\lambda(e_2+e_3), \lambda(e_2-e_3))$$

$$+c_2(\lambda(e_1+e_4), \lambda(e_1-e_4))c_2(\lambda(e_2+e_3), \lambda(e_3-e_2))$$

$$=2c_2(\lambda(e_1+e_4), \lambda(e_1-e_4))c_1(\lambda(e_2+e_3)) \qquad \text{(by Lemma 3 (1))}$$

$$=2c_2(\lambda(e_1), \lambda(e_4))c_1(\lambda(e_2+e_3)) \qquad \text{(by Lemma 3 (2))}.$$

Similarly for σ_8 and σ_9, we have

$$\prod_{i=1}^{2}c_2((\sigma_8^{-1}\lambda)'_i)+\prod_{i=1}^{2}c_2((\sigma_8^{-1}s_\gamma\lambda)'_i)$$

$$=2c_2(\lambda((e_1+e_2+e_3+e_4)/2), \lambda((e_1+e_4-e_2-e_3)/2))c_1(\lambda(e_1-e_4)),$$

$$\prod_{i=1}^{2} c_2((\sigma_9^{-1}\lambda)_i') + \prod_{i=1}^{2} c_2((\sigma_9^{-1}s_\gamma\lambda)_i')$$
$$= 2c_2(\lambda((e_1+e_2+e_3-e_4)/2),\ \lambda(e_1-e_2-e_3-e_4)/2)c_1(\lambda(e_1+e_4)).$$

Therefore, by (6),

$$P(\lambda;\ \Phi^+) + P(s_\gamma\lambda;\ \Phi^+) =$$
$$2\{c_2(\lambda(e_1),\ \lambda(e_4))c_1(\lambda(e_2+e_3)) - c_2(\lambda((e_1+e_2+e_3+e_4)/2,\ \lambda(e_1-e_2-e_3+e_4)/2)$$
$$\times c_1(\lambda(e_1-e_4)) + c_2(\lambda((e_1+e_2+e_3-e_4)/2),\ \lambda(e_1-e_2-e_3-e_4)/2))c_1(\lambda(e_1+e_4))\}.$$

Note that the standard maximal orthogonal positive ordered system of Φ_0^+ consisting of two short roots is $E_0 = (e_1,\ e_4,\ e_2+e_3)$. Identify an element w in $W(\Phi_0)$ with an ordered system $wE_0 = (w(e_1),\ w(e_4),\ w(e_2+e_3))$. Then we get that

$$S(\Phi_0^+) = \left\{ \begin{array}{l} w_1 = 1,\ w_2 = ((e_1+e_2+e_3+e_4)/2,\ (e_1-e_2-e_3+e_4)/2,\ e_1-e_4) \\ w_3 = ((e_1+e_2+e_3-e_4)/2,\ (e_1-e_2-e_3-e_4)/2,\ e_1+e_4) \end{array} \right\}.$$

Moreover $\mathrm{sgn}\,(w_1) = \mathrm{sgn}\,(w_3) = -\mathrm{sgn}\,(w_2) = 1$.

Thus we have $P(\lambda;\ \Phi^+) + P(s_\gamma\lambda;\ \Phi^+) = 2c(\lambda,\ \Phi_0^+)$ for $\gamma = e_2 - e_3$. Now take $\gamma = e_3 - e_4$. Then the calculation is completely parallel.

5.3 Next, we consider the case $\gamma = e_4$, a short root. Then,

$$\Phi_0 = \{\pm e_i \pm e_j\ (1 \le i < j \le 3),\quad \pm e_i\ (1 \le i \le 3)\},$$
$$\Phi_0^+ = \{e_i \pm e_j\ (1 \le i < j \le 3),\ e_i\ (1 \le i \le 3)\}.$$

We see that Φ_0 is a root system of type B_3. As in §4 in [7], identify an element w in $W(\Phi)$ with an ordered system wE. We divide S into the following three subsets:

$$W(E)^{(0)} = \left\{ \begin{array}{c} \sigma_j^{(0)} = (e_1+e_j,\ e_1-e_j,\ e_i+e_k,\ e_i-e_k), \\ \{i,\ j,\ k\} = \{2,\ 3,\ 4\},\quad i<k \end{array} \right\},$$

$$W(E)^{(1)} = \left\{ \begin{array}{c} \sigma_j^{(1)} = (e_1+e_j,\ e_i+e_k,\ e_1-e_j,\ e_i-e_k), \\ \{i,\ j,\ k\} = \{2,\ 3,\ 4\},\quad i<k \end{array} \right\} \text{ and}$$

$$W(E)^{(2)} = \left\{ \begin{array}{c} \sigma_j^{(2)} = (e_1+e_j,\ e_i-e_k,\ e_1-e_j,\ e_i+e_k), \\ \{i,\ j,\ k\} = \{2,\ 3,\ 4\},\quad i<k \end{array} \right\}.$$

Since $s_\gamma\sigma_j^{(1)} = \sigma_j^{(2)}$ for $j = 2,\ 3$, we get that

$$\sum_{k=1}^{2}\sum_{j=2}^{3} \mathrm{sgn}\,(\sigma_j^{(k)}) \prod_{i=1}^{2} c_2((\sigma_j^{(k)^{-1}}\lambda)_i') + \sum_{k=1}^{2}\sum_{j=2}^{3} \mathrm{sgn}\,(\sigma_j^{(k)}) \prod_{i=1}^{2} c_2((\sigma_j^{(k)^{-1}}s_\gamma\lambda)_i') = 0.$$

For $\sigma_4^{(1)}$ and $\sigma_4^{(2)}$,

$$\sum_{k=1}^{2} \operatorname{sgn}(\sigma_4^{(k)}) \prod_{i=1}^{2} c_2(\sigma_4^{(k)^{-1}}\lambda)_i') + \sum_{k=1}^{2} \operatorname{sgn}(\sigma_4^{(k)}) \prod_{i=1}^{2} c_2((\sigma_4^{(k)^{-1}} s_\gamma \lambda)_i')$$

$$= c_2(\lambda(e_1+e_4),\ \lambda(e_2+e_3))c_2(\lambda(e_1-e_4),\ \lambda(e_2-e_3))$$

$$- c_2(\lambda(e_1+e_4),\ \lambda(e_2-e_3))c_2(\lambda(e_1-e_4),\ \lambda(e_2+e_3))$$

$$+ c_2(\lambda(e_1+e_4),\ \lambda(e_2-e_3))c_2(\lambda(e_1-e_4),\ \lambda(e_2+e_3))$$

$$- c_2(\lambda(e_1+e_4),\ \lambda(e_2+e_3))c_2(\lambda(e_1-e_4),\ \lambda(e_2-e_3))$$

$$=0.$$

Therefore by (3),

$$P(\lambda;\ \Phi^+) + P(s_\gamma\lambda;\ \Phi^+)$$

$$= \sum_{j=2}^{4} \operatorname{sgn}(\sigma_j^{(0)}) \prod_{i=1}^{2} c_2((\sigma_j^{(0)^{-1}}\lambda)_i') + \sum_{j=2}^{4} \operatorname{sgn}(\sigma_j^{(0)}) \prod_{i=1}^{2} c_2((\sigma_j^{(0)^{-1}} s_\gamma\lambda)_i').$$

Note that $\operatorname{sgn}(\sigma_2^{(0)}) = \operatorname{sgn}(\sigma_4^{(0)}) = -\operatorname{sgn}(\sigma_3^{(0)}) = 1$, and that for $j=2, 3$,

$$\operatorname{sgn}(\sigma_j^{(0)})c_2(\lambda(e_1+e_j),\ \lambda(e_1-e_j))c_2(\lambda(e_i+e_4),\ \lambda(e_i-e_4))$$

$$+\operatorname{sgn}(\sigma_j^{(0)})c_2(\lambda(e_1+e_j),\ \lambda(e_1-e_j))c_2(\lambda(e_i-e_4),\ \lambda(e_i+e_4))$$

$$=\operatorname{sgn}(\sigma_j^{(0)})c_2(\lambda(e_1+e_j),\ \lambda(e_1-e_j))c_2(\lambda(e_i),\ \lambda(e_4))$$

$$+\operatorname{sgn}(\sigma_j^{(0)})c_2(\lambda(e_1+e_j),\ \lambda(e_1-e_j))c_2(\lambda(e_i),\ -\lambda(e_4)) \quad \text{(by Lemma 3 (1))}$$

$$=2\operatorname{sgn}(\sigma_j^{(0)})c_2(\lambda(e_1+e_j),\ \lambda(e_1-e_j))c_1(\lambda(e_i)) \quad \text{(by Lemma 3 (2))}.$$

Similarly, it holds for $j=4$ that

$$\operatorname{sgn}(\sigma_4^{(0)}) \prod_{i=1}^{2} c_2((\sigma_4^{(0)^{-1}}\lambda)_i') + \operatorname{sgn}(\sigma_4^{(0)}) \prod_{i=1}^{2} c_2((\sigma_4^{(0)^{-1}} s_\gamma\lambda)_i')$$

$$=2\operatorname{sgn}(\sigma_4^{(0)})c_2(\lambda(e_2+e_3),\ \lambda(e_2-e_3))c_1(\lambda(e_1)).$$

In this case, $E_0=(e_1+e_2,\ e_1-e_2,\ e_3)$ and $S(\Phi_0^+)=\{1,\ s_{e_2-e_3},\ s_{e_1-e_2}s_{e_2-e_3}\}$. So we get the desired equality:

$$P(\lambda;\ \Phi^+)+P(s_\gamma\lambda;\ \Phi^+)=2c(\lambda,\ \Phi_0^+) \qquad \text{for} \quad \gamma=e_4.$$

5.4 Finally, we check the case $\gamma=(e_1-e_2-e_3-e_4)/2$ (a short root). In this case, the relations $\sigma_j^{(2)^{-1}} s_\gamma \sigma_j^{(2)} = s_{e_4}$ $(j=2, 3, 4)$ leads us that

$$(\sigma_j^{(2)^{-1}} s_\gamma\lambda)_1' = (s_{e_4}\sigma_j^{(2)^{-1}}\lambda)_1' = (\sigma_j^{(2)^{-1}}\lambda)_1' \qquad \text{and}$$

$$(\sigma_j^{(2)^{-1}} s_\gamma\lambda)(e_3+e_4) = (\sigma_j^{(2)^{-1}}\lambda)(e_3-e_4).$$

Therefore,

$$\operatorname{sgn}(\sigma_j^{(2)}) \prod_{i=1}^{2} c_2((\sigma_j^{(2)^{-1}}\lambda)'_1) + \operatorname{sgn}(\sigma_j^{(2)}) \prod_{i=1}^{2} c_2((\sigma_j^{(2)^{-1}} s_\gamma \lambda)'_i)$$
$$= \operatorname{sgn}(\sigma_j^{(2)}) c_2(\sigma_j^{(2)^{-1}}\lambda)'_1)\{c_2((\sigma_j^{(2)^{-1}}\lambda)'_2) + c_2((s_{e_4}\sigma_j^{(2)^{-1}}\lambda)'_2)\}$$
$$= 2\operatorname{sgn}(\sigma_j^{(2)}) c_2(\sigma_j^{(2)^{-1}}\lambda)'_1) c_1((\sigma_j^{(2)^{-1}}\lambda)(e_3)).$$

Thus we get for γ that

$$P(\lambda;\, \Phi^+) + P(s_\gamma\lambda;\, \Phi^+)$$
$$= 2\{c_2(\lambda(e_1+e_2),\, \lambda(e_3-e_4)) c_1(\lambda((e_1-e_2+e_3+e_4)/2))$$
$$- c_2(\lambda(e_1+e_3),\, \lambda(e_2-e_4)) c_1(\lambda((e_1+e_2-e_3+e_4)/2))$$
$$+ c_2(\lambda(e_1+e_4),\, \lambda(e_2-e_3)) c_1(\lambda((e_1+e_2+e_3-e_4)/2))\}.$$

For this γ, the standard maximal orthogonal positive ordered system of Φ_0^+ is $E_0 = (e_1+e_2,\, e_3-e_4,\, (e_1-e_2+e_3+e_4)/2)$, and

$$S(\Phi_0^+) = \left\{ \begin{array}{l} \sigma_1 = 1,\ \sigma_2 = (e_1+e_3,\, e_2-e_4,\, (e_1+e_2-e_3+e_4)/2) \\ \sigma_3 = (e_1+e_4,\, e_2-e_3,\, (e_1+e_2+e_3-e_4)/2) \end{array} \right\}$$

(under the identification of an element w in $W(\Phi_0)$ with an ordered system wE_0). Furthermore, $-\operatorname{sgn}(\sigma_2) = \operatorname{sgn}(\sigma_3) = 1$.

Hence for $\gamma = (e_1-e_2-e_3-e_4)/2$, we get also the equality

$$P(\lambda;\, \Phi^+) + P(s_\gamma\lambda;\, \Phi^+) = 2c(\lambda,\, \Phi_0^+).$$

Thus we completed the proof of Lemma 2. Q. E. D.

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