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Constructions of eigenfunctions for the Sturm-Liouville operator by comparison method

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§ 1. Introduction

This paper is concerned with constructions of eigenfunctions for the Sturm-Liouville operator $L=-\frac{d^2}{dx^2}+q(x)$ in $(-\infty, \infty)$. Here we assume that the real valued function $q(x)$ satisfies the following conditions:

$$\text{(C)} \quad \begin{cases} q(x) \text{ is piecewise continuous and has the minimum value at } x=x_0, \\ m=q(x_0)=\inf\limits_{-\infty<x<\infty} q(x)<M=\varliminf\limits_{x\to\infty} q(x), \quad (M=\infty \text{ is included.}) \end{cases}$$

Especially we consider concrete constructions of eigenfunctions corresponding to eigenvalues in (m, M), relying upon comparison theorems which assure the existence of bounded solutions $u_+(x, \lambda)$ and $u_-(x, \lambda)$ of $\frac{d^2}{dx^2}u=(q(x)-\lambda)u$ in neighbourhoods of $+\infty$ and $-\infty$ respectively. Namely we try to consider the Sturm's method of comparison even in the case of infinite demain $(-\infty, \infty)$. As we see later, this consideration motivates originally comparison theorems of type stated in Section 2, which are generalized in [2] and [3]. Incidentally we show that there exists a continuous monotone increasing function $\Phi(\lambda)$ satisfying $-\pi<\Phi(m)<0$ such that λ is eigenvalue if and only if $\Phi(\lambda)=(n-1)\pi$, $(n=1, 2, 3,\dots)$. In order to see that appearance of eigenvalues more precisely we need some estimates for $\Phi(\lambda)$. For this purpose we write

$$\Phi(\lambda)=\int_{\Omega(\lambda)}(\lambda-q(x))^{1/2}dx+R(\lambda),$$

where $\Omega(\lambda)=\{x;\, \lambda-q(x)>0\}$ and obtain a suitable estimate for $R(\lambda)$, to show that $R(\lambda)$ is a remainder term as compared with the first term. In many books of physics (for example [1], [5] etc.) we find the following type of formula: $\int_{\Omega(\lambda_n)}(\lambda_n-q(x))^{1/2}dx=\left(n-\frac{1}{2}\right)\pi$, which was explained by the so-called W. K. B. method. As for mathematics Tichmarsh [6] showed that there exists a constant C such that

$|R(\lambda_n)|<C$ for all λ_n if $q(x)$ is convex. And many authors treated the related problems under various assumptions on $q(x)$, $q'(x)$ and $q''(x)$, (for example see [4]). Here we use comparison theorems related with only the value of $q(x)$ and exhibit an estimate for $R(\lambda)$ of type $\underline{E}(\lambda)\leq R(\lambda)\leq \bar{E}(\lambda)$, where $\underline{E}(\lambda)$ and $\bar{E}(\lambda)$ concern the logarithmic order of variations of $q(x)$ in $\Omega(\lambda)$. The results are stated in Section 2 more precisely. In Section 3, to make our argument smooth we verify that all the eigenvalues of L in L^∞ space are real, and then we prove some simple lemmas concerning the comparison theorems which will be used later. Theorems 1 and 2 are proved in Section 4 and 5. In the last section we generalize the estimate for eigenvalues to the case where $\Omega(\lambda)$ is not necessarily one interval, thus clarifying some properties of solutions in the case of tunnel effects.

§2. Statements of results

First we mention the definition of eigenvalues and eigenfunctions of $L=-\frac{d^2}{dx^2}+q(x)$ in the space L^∞ of bounded measurable functions defined in $(-\infty, \infty)$. We say λ and $u(x)$ eigenvalue and eigenfunction of L in L^∞ respectively, if there exists $(\lambda, u(x))$ belonging to $C\times L^\infty$ such that $Lu=\lambda u$ in $(-\infty, \infty)$, $u(x)$ not being identically zero. We have

Lemma 1. *Suppose the conditions* (C) *in Introduction. Then all the eigenvalues of L in L^∞ are real and contained in (m, ∞).*

A simple and direct proof of Lemma 1 will be given in next section. In view of Lemma 1 we suppose that the parameter λ is real and larger than m. Especially we restrict λ in (m, M). As we will see later, eigenfunctions in L^∞ and those in L^2 coincide for eigenvalues in (m, M). So we do not mention function spaces hereafter. Now we put for $\lambda\in[m, M)$

$$\begin{aligned} x_+(\lambda)&=\inf\{x_1;\ q(x)-\lambda>0 \quad \text{in} \quad (x_1, \infty)\},\\ x_-(\lambda)&=\sup\{x_1;\ q(x)-\lambda>0 \quad \text{in} \quad (-\infty, x_1)\}. \end{aligned} \tag{2.1}$$

Then from (2.1), $x_0\leq x_+(\lambda_1)\leq x_+(\lambda_2)$ and $x_-(\lambda_2)\leq x_-(\lambda_1)\leq x_0$ if $m\leq\lambda_1\leq\lambda_2<M$. We have

Proposition 1. *Assume* (C). *Then there exists a continuous function $\theta_+(x_1, \lambda)$, (resp. $\theta_-(x_1, \lambda)$) defined on $(-\infty, \infty)\times[m, M)$ which has the following properties: (I) The solution of $u''=(q(x)-\lambda)u$ satisfying $u(x_1)=u_0$ and $u'(x_1)=u_1$ is bounded in (x_1, ∞), (resp. $(-\infty, x_1)$) if and only if $\frac{u_1}{u_0}=\tan\theta_+(x_1, \lambda)$, (resp. $\frac{u_1}{u_0}=\tan\cdot\theta_-(x_1, \lambda)$), (II) $\theta_\pm(x, \lambda)$ satisfies*

$$\begin{cases} -\frac{\pi}{2}<\theta_+(x, \lambda)<0 & \text{for} \quad x\in(x_+(\lambda), \infty),\\ 0<\theta_-(x, \lambda)<\frac{\pi}{2} & \text{for} \quad x\in(-\infty, x_-(\lambda)), \end{cases} \tag{2.2}$$

and the following differential equation respectively

$$\frac{d\theta}{dx}=\frac{(q(x)-\lambda)-\tan^2\theta}{1+\tan^2\theta}, \qquad \text{for all} \quad \lambda\in[m,\ M), \tag{2.3}$$

(*III*) $\theta_+(x,\lambda)$, (*resp.* $\theta_-(x,\lambda)$) *is monotone increasing*, (*resp. decreasing*) *in* λ *at every point* $x\in(-\infty,\infty)$.

Remark 2.1. From the above property (II) and the uniqueness of the solution of (2.3), $\theta_+(x,\lambda)=\theta_-(x,\lambda)+k\pi$ in $(-\infty,\infty)$, (k, integer), if $\theta_+(x_1,\lambda)=\theta_-(x_1,\lambda)+k\pi$ at a point $x_1\in(-\infty,\infty)$.

Now we put

$$\Phi(x,\lambda)=\theta_+(x,\lambda)-\theta_-(x,\lambda) \tag{2.4}$$

Then we have the following theorem.

Theorem 1. *Suppose* (C). *Then there exists a continuous function* $\Phi(x,\lambda)$ *defined on* $(-\infty,\infty)\times[m,M)$, *which satisfies* (1) $\Phi(x,\lambda)$ *is monotone increasing in* λ *at every* $x\in(-\infty,\infty)$, (2) $-\pi<\Phi(x,m)<0$, *and* $\Phi(x,\lambda_n)=(n-1)\pi$ *in* $(-\infty,\infty)$ *if* $\Phi(x_1,\lambda_n)=(n-1)\pi$ *at a point* $x_1\in(-\infty,\infty)$. *At that time* λ_n, $(n=1,2,\ldots,)$ *are eigenvalues if and only if* $\Phi(x,\lambda_n)=(n-1)\pi$. *Corresponding eigenfunctions* $u_n(x)$ *are* equal to *the solutions of* $u''=(q(x)-\lambda_n)u$ *with* $u(x_+(\lambda_n))=1$ *and* $u'(x_+(\lambda_n))=\tan\theta_+(x_+(\lambda_n),\lambda_n)$. *Incidentally* u_n *has* $(n-1)$ *roots. Moreover* $u'_n(x)$ *has just* n *zeros if* $q(x)$ *is assumed to be monotone in* $(-\infty,x_0)$ *and* (x_0,∞), *where* x_0 *is a point satisfying* $q(x_0)=\min\limits_{-\infty<x<\infty} q(x)$.

Now we put

$$R(x,\lambda)=\Phi(x,\lambda)-\int_{-\infty}^{\infty} Q(s,\lambda)ds, \text{ where} \tag{2.5}$$

$$Q(x,\lambda)=\begin{cases}(\lambda-q(x))^{1/2}, & \text{if } \ 0\le\lambda-q(x),\\ 0, & \text{if } \ \lambda-q(x)<0.\end{cases}$$

Remark that $R(x,\lambda_n)$ is constant: $(n-1)\pi-\int_{-\infty}^{\infty}Q(s,\lambda_n)ds$. Put

$$R(\lambda)=R(x_0,\lambda). \tag{2.6}$$

We show the estimates for $R(x,\lambda)$ and $R(\lambda_n)$, in order to assure the actual appearance of λ_n in each given case.

Theorem 2. *Suppose* (C) *and that* $q(x)$ *is monotone in* (x_0,∞) *and* $(-\infty,x_0)$. *Then we have the following estimates for* $\lambda\in[m,M)$:

$$\underline{E}(\lambda)-2\pi<R(x,\lambda)<\bar{E}(\lambda)+\pi,$$

$$\underline{E}(\lambda_n)-\pi<R(\lambda_n)<\bar{E}(\lambda_n), \quad (\lambda_n,\ \textit{eigenvalues of } L)$$

where $\underline{E}(\lambda)=\int_{-\infty}^{\infty}\underline{E}(s,\lambda)ds$, $\bar{E}(\lambda)=\int_{-\infty}^{\infty}\bar{E}(s,\lambda)ds$ *and*

$$\underline{E}(x,\lambda)=\begin{cases}-\frac{1}{2}|(\log Q(x,\lambda))'| & \text{if } 1\leq\lambda-q(x),\\ -Q(x,\lambda)(1-Q(x,\lambda)) & \text{if } 0\leq\lambda-q(x)<1,\\ 0 & \text{if } \lambda-q(x)<0,\end{cases}$$

$$\bar{E}(x,\lambda)=\begin{cases}\frac{1}{2}|(\log Q(x,\lambda))'| & \text{if } 1\leq\lambda-q(x),\\ 1-Q(x,\lambda) & \text{if } 0\leq\lambda-q(x)<1,\\ 0 & \text{if } \lambda-q(x)<0.\end{cases}$$

Remark. Theorem 2 will be generalized in Section 6 to the case where $q(x)$ is not necessarily monotone in $(-\infty, x_0)$ and (x_0, ∞).

The proofs of Proposition 1 and Theorems 1 and 2 rely upon the following comparison theorems which we state as lemmas.

Lemma 2. *Let $q(x)$ and $\tilde{q}(x)$ be piecewise continuous functions such that $q(x)>\tilde{q}(x)$ on $[0, \infty)$. Suppose that there exists a solution $v(x)$ of $v''=\tilde{q}(x)v$ in $(0, \infty)$ satisfying $v(0)=1$ and $0<v(x)$ in $(0, \infty)$. Then $u''=q(x)u$ has a solution $u(x)$ satisfying $0<u(x)<v(x)$ in $(0, \infty)$ and $u(0)=1$. It holds $u'(0)<v'(0)$.*

Lemma 3. *Assume the same conditions as in Lemma 2. Moreover suppose that $v(x)$ is a unique bounded solution in $(0, \infty)$ of $v''=\tilde{q}(x)v$ satisfying $v(0)=1$. Then the bounded solution in $(0, \infty)$ of $u''=q(x)u$ with $u(0)=1$ is unique.*

Now we introduce a notation.

Notation By $t(q)$ we denote $u'(0)$, when $u''=q(x)u$ has a unique positive bounded solution $u(x)$ in $(0, \infty)$ satisfying $u(0)=1$.

Lemma 4. *Let $\{q_\mu(x)\}_{0<\mu<1}$ be a family of functions satisfying the condition (C). Assume that q_μ converges to q_{μ_0} uniformly in each compact set of $[0, \infty)$ as μ tends to $\mu_0\in(0, 1)$. Suppose that conditions in Lemmas 2 and 3 are fulfilled replacing q by q_μ for all $\mu\in(0, 1)$. Then $t(q_\mu)$ converges to $t(q_{\mu_0})$ as μ tends to μ_0.*

Finally we point out some examples.

Example 1. If $q(x)>0$ in $(0, \infty)$, it follows that $\tilde{q}(x)\equiv 0$ satisfies all the conditions in Lemmas 2 and 3. Then applying Lemmas 2 and 3 in (x_1, ∞) for all $x_1\in[0, \infty)$, we see that $u''=q(x)u$ has a bounded solution $u(x)$ satisfying $\frac{u'}{u}<0$ in $(0, \infty)$. Similarly $q(x)>\tilde{q}(x)>0$ in $(0, \infty)$ implies $\frac{u'}{u}<\frac{v'}{v}$ in $(0, \infty)$, where u and v are bounded solutions $u''=q(x)u$ and $v''=\tilde{q}(x)v$ respectively.

Remark. Let us confine ourselves to simple cases: $q>\tilde{q}$ or $q<\tilde{q}$, where

$q>\tilde{q}$ means $q(x)>\tilde{q}(x)$ on $[0, \infty]$. Then in Example 1, $0<\tilde{q}<q$ equals to $\frac{u'}{u}<\frac{v'}{v}<0$. $\frac{u'}{u}$ converges to $\frac{v'}{v}$ if and only if q converges to $\tilde{q}$ in the sense of Lemma 4.

Example 2. As an application of the above Remark, we consider the harmonic oscillator: $q(x)=x^2$. Put $v=\exp(-ax^k)$, then $\tilde{q}(x)=\frac{v''}{v}=(ak)^2x^{2k-2}-ak(k-1)x^{k-2}$. Taking $k=2$ and $a=\frac{1}{2}$ we see $\lambda_1=1$ and $u_1=\exp\left(-\frac{1}{2}x^2\right)$. Next put $v=x^k\exp\left(-\frac{1}{2}x^2\right)$, then similarly $\lambda_2=3$ and $v_2=x\exp\left(-\frac{1}{2}x^2\right)$ follow. Step by step we have $\lambda_n=2n-1$ and $u_n=\varphi_n(x)\exp\left(-\frac{1}{2}x^2\right)$ where $\varphi_n(x)$ is a polynominal of degree $n-1$ with $n-1$ real roots. Thus we approach to a motivation of Hermite polynomials.

Finally we comment on Lemmas stated above.

Remark. In Lemmas 2, 3, 4 and Example 1, we have the same results even if we replace $q(x)>\tilde{q}(x)$ by $q(x)\geq\tilde{q}(x)$, where $q(x)$ is not identically equal to $\tilde{q}(x)$. The proof is accomplished in the same way as in next section.

§3. Proofs of Lemmas

Proof of Lemma 1. Let $\chi(x)$ be a real valued C^1 function satisfying $\chi(x)>0$ in $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\chi(x)=0$ in $\left(-\infty, -\frac{1}{2}\right)\cup\left(\frac{1}{2}, \infty\right)$, $\chi(x)=\chi(-x)$, $\chi(x_1)\geq(x_2)$ for $0\leq x_1\leq x_2$ and

$$\chi'(x)^2\leq C\chi(x) \quad \text{in} \quad (-\infty, \infty), \quad \text{where} \quad C>0. \tag{3.1}$$

For example define $\chi(x)$ by $\left(x+\frac{1}{2}\right)^2 H\left(x+\frac{1}{2}\right)$ in $\left(-\infty, -\frac{1}{3}\right)$, $\left(-x+\frac{1}{2}\right)^2\cdot H\left(-x+\frac{1}{2}\right)$ in $\left(\frac{1}{3}, \infty\right)$ and a suitable positive function in $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $H(x)=1$ for $x>0$ and $H(x)=0$ for $x\leq 0$. Denote $(u, v)=\int_{-\infty}^{\infty}u(x)\overline{v(x)}dx$ and

$$(u, v)_\varepsilon=(\chi(\varepsilon x)u, v) \quad \text{and} \quad L_\varepsilon=\chi(\varepsilon x)L. \tag{3.2}$$

Let u be eigenfunction in L^∞ corresponding to eigenvalue λ:

$$\begin{cases} (L_\varepsilon u, u)=\lambda(u, u)_\varepsilon \\ \sup|u|\leq C_1. \end{cases} \tag{3.3}$$

On the other hand the integration by parts yields

$$(L_\varepsilon u, u)=(u', u')_\varepsilon+(qu, u)_\varepsilon+\varepsilon\int_{-\infty}^{\infty}\chi'(\varepsilon x)u'(x)\overline{u(x)}dx. \tag{3.4}$$

From Schwartz inequality and (3.1) it follows

$$(3.5)\qquad \left|\int \chi'(\varepsilon x)u'(x)\overline{u(x)}dx\right| \leq C_1\varepsilon^{-1/2}\left(\int\left|\chi'(\varepsilon x)u'(x)\right|^2 dx\right)^{1/2}$$
$$\leq CC_1\varepsilon^{-1/2}(u', u')_\varepsilon^{1/2}$$

From (3.3) and (3.4) we have

$$(3.6)\qquad |\mathrm{Im}\,\lambda| \leq CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{1/2}/(u, u)_\varepsilon \equiv J_\varepsilon(u),$$

$$(3.7)\qquad \mathrm{Re}\,\lambda - \{(u', u')_\varepsilon + (qu, u)_\varepsilon\}/(u, u)_\varepsilon \geq -J_\varepsilon(u).$$

Since u' is not identically zero it follows

$$(3.8)\qquad \lim_{\varepsilon\to 0} \varepsilon^{1/2}(u', u')_\varepsilon^{-1/2} = 0.$$

Therefore (3.7) makes

$$(3.9)\qquad \mathrm{Re}\,\lambda - m \geq \{(1 - CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{-1/2})(u', u')_\varepsilon + ((q-m)u, u)_\varepsilon\}/(u, u)_\varepsilon.$$

Now making ε sufficiently small we have $\mathrm{Re}\,\lambda - m > 0$. Then (3.6) and (3.9) give

$$(3.10)\qquad \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda - m} \leq \frac{CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{-1/2}}{1 - CC_1\varepsilon^{1/2}(u', u')_\varepsilon^{-1/2}}.$$

Tending ε to zero we have $\mathrm{Im}\,\lambda = 0$.

Remark. The results of Lemma 1 and the outline of the above proof are also valid even if we replace $-\frac{d^2}{dx^2} + q(x)$ in R by $-\sum \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + q(x)$ in R^n. We will discuss it elsewhere.

Proof of Lemma 2. Let $u(x; t)$ be solution of $u'' = q(x)u$, $u(0) = 1$ and $u'(0) = t$. Denote

$$U(q) = \{t \in (-\infty, v'(0)),\ {}^\exists x_1 > 0,\ 0 < u(x; t) < v(x) \quad \text{in} \quad (0, x_1), \quad u(x_1; t) = v(x_1)\}$$

$$D(q) = \{t \in (-\infty, v'(0)),\ {}^\exists x_1 > 0,\ 0 < u(x; t) < v(x) \quad \text{in} \quad (0, x_1), \quad u(x_1; t) = 0\}$$

Then $U(q)$ and $D(q)$ are open as follows. If $t \in U(q)$, then $u'(x_1; t) \geq v'(x_1)$ follows. Now suppose $u'(x_1; t) = v'(x_1)$, then $u''(x_1; t) = q(x_1)u(x_1; t) > \tilde{q}(x_1)\cdot v(x_1) = v''(x_1)$ contradicts $0 < u(x; t) < v(x)$ in $(0, x_1)$. Hence $u'(x_1; t) > v'(x_1)$. Therefore $U(q)$ is open. The similar argument is valid for $D(q)$. Now let us show that $U(q)$ contains an open interval $(v'(0) - \delta, v'(0))$. We have $u(x; v'(0)) > v(x)$ holds in a neighbourhood of $x = 0$, because of $u''(0; v'(0)) = q(0) > \tilde{q}(0)$. Therefore the above assertion follows from the continuity of $u(x; t)$ in t. On the other hand an interval $(-\infty, -t_0 + \varepsilon)$ belongs to $D(q)$ for sufficient large t_0. Hence the connected interval $(-t_0, v'(0))$ involves at least a point α such that $\alpha \notin U(q) \cup D(q)$. Namely we have $0 < u(x; \alpha) < v(x)$ in $(0, \infty)$.

Proof of Lemma 3. It sufficies to prove that the solution $u_1(x)$ of $u_1'' = q(x)u_1$, $u_1(0) = 0$ and $u_1'(0) = 1$ is not bounded in $(0, \infty)$. First we show that $v_1(x)$ satisfying $v_1'' = \tilde{q}(x)v_1$, $v_1(0) = 0$ and $v_1'(0) = 1$ is positive and unbounded. In fact, if $v_1(a) = 0$ and

$0<v_1(x)$ in $(0, a)$ for some $a>0$, then we can find positive numbers $b\in(0, a)$ and c such that $0<v_1(x)<cv(x)$ in $(0, b)$, $v_1(b)=cv(b)$ and $v_1'(b)=cv'(b)$. However this contradicts the uniqueness theorem on the solution of $v''=\tilde{q}(x)v$. Since the bounded solution $v(x)$ satisfying $v''=\tilde{q}(x)v$ in $(0, \infty)$ and $v(0)=1$ is unique, $v_1(x)$ is not bounded in $(0, \infty)$. Let us prove $u_1(x)\geq v_1(x)$ in $(0, \infty)$, which equals to $u_1(x)> sv_1(x)$ in $(0, \infty)$ for any $s\in(0, 1)$. Suppose $u_1(a)=sv_1(a)$ for some $a>0$, and $u_1(x)> sv_1(x)$ in $(0, a)$. Then there exist $b\in(0, a)$ and $c>0$ such that $u_1(x)<sv_1(x)+cv(x)$ in $(0, b)$, $u_1(b)=sv_1(b)+cv(b)$ and $u_1'(b)=sv_1'(b)+cv'(b)$. However this is a contradiction since it holds $u_1''(b)=q(b)u_1(b)>\tilde{q}(b)(sv_1(b)+cv(b))=sv_1''(b)+cv_1''(b)$ from $q(x)>\tilde{q}(x)$. Hence we have $u_1(x)>sv_1(x)$ in $(0, \infty)$. Therefore $u_1(x)$ is positive and unbounded in $(0, \infty)$. This means the uniqueness of the bounded solution of $u''=q(x)u$ satisfying $u(0)=1$.

Proof of Lemma 4. From Lemma 2 it holds $t(q_\mu)<t(\tilde{q})$ for all $\mu\in(0, 1)$. Let $\varepsilon_0=t(\tilde{q})-t(q_{\mu_0})$. For given positive number ε less than ε_0, from Lemma 3 we have $t(q_{\mu_0})+\varepsilon\in U(q_{\mu_0})$ and $t(q_{\mu_0})-\varepsilon\in D(q_{\mu_0})$. Let us recall the property of $U(q_{\mu_0})$: if $t\in U(q_{\mu_0})$, then there exists a positive number a such that the solution $u(x)$ of $u''= q_{\mu_0}(x)u$ with $u(0)=1$ and $u'(0)=t$ satisfies $0<u(x)<v(x)$ in $(0, a)$, $u(a)=v(a)$ and $u'(a) >v'(a)$. Then from the continuity of solutions of $u''=q(x)u$ for $q(x)$, there exists a positive number δ which is smaller than $\min\{\mu_0, 1-\mu_0\}$, such that $t(q_{\mu_0})+\varepsilon\in U(q_\mu)$ and $t(q_{\mu_0})-\varepsilon\in D(q_\mu)$ for all μ belonging to $(\mu_0-\delta, \mu_0+\delta)$. Therefore from Lemma 3 we have $t(q_\mu)\in(t(q_{\mu_0})-\varepsilon, t(q_{\mu_0})+\varepsilon)$ for all $\mu\in(\mu_0-\delta, \mu_0+\delta)$. This completes the proof of Lemma 4.

§4. Proof of Theorem 1

First we state a preliminary lemma for the proof of Proposition 1.

Lemma 4.1. *Let $f(x, y)$ and $g(x, y)$ be continuous in R^2 and satisfy the Lipschitz condition in y. Assume $f(x, y)\geq g(x, y)$ in R^2. Then $u(a)>v(a)$, (reps. $u(a)<v(a)$) implies $u(x)>v(x)$ in (a, ∞), (resp. $u(x)<v(x)$ in $(-\infty, a)$).*

Proof. In the case of $f(x, y)>g(x, y)$ in R^2, the result is easily verified by method of contradiction. We prove the general case in the following way. Suppose $u(x_1)\leq v(x_1)$ at $x=x_1\in(a, \infty)$. Let $u_\varepsilon(x)$ be solution of $u_\varepsilon''=f(x, u_\varepsilon)+\varepsilon$ with $u_\varepsilon(x_1)=u(x_1)-\varepsilon$. Then form the result in the above case we have $u_\varepsilon(x)< v(x)$ in $(-\infty, x_1)$. Therefore $u(a)=\lim_{\varepsilon\to 0} u_\varepsilon(a)\leq v(a)$ contradicts the assumption.

Collary of Lemma 4.1. *In the statements of Lemma 4.1 we can replace $u(a)>v(a)$, $u(x)>v(x)$ etc. by $u(a)\geq v(a)$, $u(x)\geq v(x)$ etc. respectively, by virtue of the continuity of solutions.*

Proof of Proposition 1. From the definition of $x_+(\lambda)$, it holds

(4.1) $$q(x)-\lambda>0 \text{ in } (x_1, \infty), \text{ where } (x_1, \lambda)\in(x_+(\lambda), \infty)\times[m, M).$$

Take $\tilde{q}(x)\equiv 0$ and apply Lemma 2 replaced $(0, \infty)$ and $q(x)$ by (x_1, ∞) and $q(x)-\lambda$

respectively. Then we know that for each $\lambda \in [m, M)$, there exists a negative valued function $I_+(x_1, \lambda)$ defined in $(x_+(\lambda), \infty)$, such that the non-zero solution $u(x)$ of $u'' = (q(x)-\lambda)u$ is bounded in (x_1, ∞) if and only if $u'(x_1)/u(x_1) = I_+(x_1, \lambda)$. From Lemma 3, $I_+(x, \lambda)$ is continuous in $(x_+(\lambda), \infty)$ for each $\lambda \in [m, M)$. Put

$$\theta_+(x, \lambda) = \mathrm{Tan}^{-1} I_+(x, \lambda), \tag{4.2}$$

then $\theta_+(x, \lambda)$ satisfies $-\frac{\pi}{2} < \theta_+(x, \lambda) < 0$ in $(x_+(\lambda), \infty)$ and

$$\theta' = \frac{(q(x)-\lambda) - \tan^2\theta}{1+\tan^2\theta} \tag{4.3}$$

in $(x_+(\lambda), \infty)$. The equation (4.3) and

$$u'' = (q(x)-\lambda)u \tag{4.4}$$

are translated each other by putting $u(x) = u(x_0)\exp\int_{x_0}^{x} \tan\theta(s, \lambda)ds$ or $\theta = \tan^{-1}\frac{u'}{u}$, so long as $u \neq 0$ or $\theta \neq \left(n+\frac{1}{2}\right)\pi$, ($n$: integers). Since the right hand side of (4.3) is bounded in any compact set in $R^2 \ni (x, \theta)$ and that of (4.4) is linear, definition domains of solutions θ and u can be extended to $(-\infty, \infty)$ so that the relation $\theta = \tan^{-1}\frac{u'}{u}$ holds even if u takes zero. In this way we define $\theta_+(x, \lambda)$ on $(-\infty, \infty) \times [m, M)$. Then $\theta_+(x, \lambda)$ is continuous in $\lambda \in [m, M)$ at every fixed $x \in (-\infty, \infty)$. In fact from Lemma 4 $\theta_+(x, \lambda)$ is continuous in λ at every x in $(x_+(M-\delta), \infty)$ if λ is restricted in $[m, M-\delta)$, where δ is an arbitrary small constant. Hence we have the desired result from the continuity of solutions of (4.3) for initial data. Moreover from Lemma 2 and Lemma 4.1 $\theta_+(x, \lambda)$ is a monotone increasing function in $\lambda \in [m, M)$ at every $x \in (-\infty, \infty)$. From this monotony follows the continuity of $\theta_+(x, \lambda)$in (x, λ). Similarly we can define $\theta_-(x, \lambda)$ and the corresponding properties.

Proof of Theorem 1. From (2.2) it follows $0 < \theta_-(x_1, m) < \frac{\pi}{2}$ for $x_1 \in (-\infty, x_-(m))$. Now we compare $\theta_-(x, m)$ with $\tilde{\theta} \equiv 0$ which is a solution of $\tilde{\theta}' = \frac{-\tan^2\tilde{\theta}}{1+\tan^2\tilde{\theta}}$, applying Lemma 4.1 with $u = \theta_+(x, m)$, $v \equiv 0$, $f(x, y) = \frac{(q(x)-m) - \tan^2 y}{1+\tan^2 y}$ and $g(x, y) = \frac{-\tan^2 y}{1+\tan^2 y}$. Then $0 < \theta_-(x_1, m)$ implies $0 < \theta_-(x, m)$ in (x_1, ∞). Thus we have $0 < \theta_-(x, m)$ in $(-\infty, \infty)$. Now we remark that $\theta_-(x, \lambda)$ is decreasing when $\theta_-(x, \lambda) = \frac{\pi}{2} \pmod{\pi}$, because $\theta'_- = -1$ when $\tan^2\theta_- = \infty$ in (2.3). Therefore we have $0 < \theta_-(x, m) < \frac{\pi}{2}$ in $(-\infty, \infty)$. Similarly follows $-\frac{\pi}{2} < \theta_+(x, \lambda) < 0$ in $(-\infty, \infty)$. Hence by definition $-\pi < \Phi(x_0, m) < 0$. And in the same time $-\pi < \Phi(x, m) < 0$ in $(-\infty, \infty)$. For simplicity we consider $\Phi(x, \lambda)$ at $x = x_0$. From Proposition 1, $\Phi(x_0, \lambda)$ is increasing continuous function. If $\lim_{\lambda\to M}\Phi(x_0, \lambda) > (n-1)\pi$ we define λ_n by $\Phi(x_0, \lambda_n) = (n-1)\pi$, $(n = 1, 2, \dots)$. We have

$$m < \lambda_1 < \lambda_2, \dots, (< M).$$

Let us put $u_n^+ = \exp\int_{x_+(\lambda_n)}^{x} \theta_+(s, \lambda_n)ds$ in $(x_+(\lambda_n), \infty)$ and $u_n^- = \exp\int_{x_-(\lambda_n)}^{x} \theta_-(s, \lambda_n)ds$

in $(-\infty, x_-(\lambda_n))$. Then u_n^+ satisfies $(u_n^+)''=(q(x)-\lambda_n)u_n^+$. We extend u_n^+ as solutions of this equation. Then $\tan\theta_\pm(x, \lambda_n)=\dfrac{u_n^{\pm\prime}(x, \lambda_n)}{u_n^\pm(x, \lambda_n)}$. $\Phi(x_0, \lambda_n)=(n-1)\pi$ equals $\theta_+(x, \lambda_n)=\theta_-(x, \lambda_n)+(n-1)\pi$, i.e. $(\log u_n^+)'=(\log u_n^-)'$. Thus we have $u_n^+(x)=Cu_n^-(x)$. Put $u_n(x)=u_n^-(x)$. Then $u_n(x)$ is bounded in $(-\infty, \infty)$ and satisfies $Lu_n=\lambda_n u_n$. From (2.2) and $\Phi(x, \lambda_n)=(n-1)\pi$ in $(-\infty, \infty)$ we see that $u_n(x)$ has $(n-1)$ zero in $(-\infty, \infty)$ since $\theta'_\pm=-1$ if $\tan^2\theta_\pm=\infty$. Moreover, if we assume that $q(x)$ is monotone in $(-\infty, x_0)$ and in (x_0, ∞), then u'_n has n zero in $(-\infty, \infty)$, because $\theta'_-(x, \lambda)$ is decreasing in $(x_-(\lambda), x_+(\lambda))$.

§5. Proof of Theorem 2

From the definition of $R(x, \lambda)$ in (2.5) it follows

$$(5.1)\quad R(x, \lambda)=-\int_{x_-(\lambda)}^{x}(\theta'_-(s, \lambda)+Q(s, \lambda))ds-\int_{x}^{x_+(\lambda)}(\theta'_+(s, \lambda)+Q(s, \lambda))ds$$
$$+\{\theta_+(x_+(\lambda), \lambda)-\theta_-(x_-(\lambda), \lambda)\}.$$

Remark that the third term is estimated from (2.2), (2.3) and Lemma 4.1 as follows.

$$(5.2)\qquad -\pi<\{\theta_+(x_+(\lambda), \lambda)-\theta_-(x_-(\lambda), \lambda)\}<0.$$

However the integrations of $|\theta'_\pm+Q|$ are not small in general. Therefore we define a modification of $\theta_\pm$; $\tilde\theta_\pm$ in the following way such that the integrations of $|\tilde\theta'_\pm+Q|$ become smaller:

$$(5.3)\qquad \tan\tilde\theta_\pm(x, \lambda)=\tan(\theta_\pm(x, \lambda)/\hat{Q}(x, \lambda)),$$
$$\sup|\tilde\theta_\pm(x, \lambda)-\theta_\pm(x, \lambda)|<\pi/2 \qquad \text{for} \quad (x, \lambda)\in(-\infty, \infty)\times[m, M),$$

where
$$\tilde{Q}(x, \lambda)=\begin{cases} Q(x, \lambda) & \text{if} \quad q(x)-\lambda\le -1,\\ 1 & \text{if} \quad -1<q(x)-\lambda.\end{cases}$$

First we notice that $\tilde\theta_+(x, \lambda)$ satisfies

$$(5.4)\quad \begin{aligned}
&\text{(i)} \quad |\tilde\theta_\pm(x, \lambda)-\theta_\pm(x, \lambda)|<\frac{\pi}{2},\\
&\text{(ii)} \quad \tilde\theta_\pm(x, \lambda)=\frac{k\pi}{2} \qquad \text{if} \quad \theta_\pm(x, \lambda)=\frac{k\pi}{2}, \quad (k\text{: integers}),\\
&\text{(iii)} \quad \tilde\theta_\pm(x, \lambda)=\theta_\pm(x, \lambda) \qquad \text{if} \quad -1\le q(x)-\lambda,\\
&\text{(iv)} \quad \tilde\theta_+(x, \lambda_n)-\tilde\theta_-(x, \lambda_n)=\theta_+(x, \lambda_n)-\theta(x, \lambda_n)=(n-1)\pi.
\end{aligned}$$

Let us put

$$(5.5)\qquad \tilde\Phi(x, \lambda)=\tilde\theta_+(x, \lambda)-\tilde\theta_-(x, \lambda).$$

Then from (5.4) we have

$$(5.6)\qquad -\pi<\Phi(x, \lambda)-\tilde\Phi(x, \lambda)<\pi,$$

(5.7) $$\Phi(x, \lambda_n)=\tilde{\Phi}(x, \lambda_n)=(n-1)\pi.$$

We rewrite (5.1) as follows

(5.1)′ $$R(x, \lambda)=-\int_{x_-(\lambda)}^{x}(\tilde{\theta}'_-(s, \lambda)+Q(s, \lambda))ds-\int_{x}^{x_+(\lambda)}(\tilde{\theta}'_+(s, \lambda)+Q(s, \lambda))ds$$
$$+\{\theta_+(x_+(\lambda), \lambda)-\theta_-(x_-(\lambda), \lambda)\}+\{\Phi(x, \lambda)-\tilde{\Phi}(x, \lambda)\}.$$

From (5.3) we have

(5.8) $$\tilde{\theta}'_\pm(x, \lambda)=\begin{cases} -Q(x, \lambda)-(Q'/Q)(QI_\pm/(Q^2+I_\pm^2)), & \text{if } q(x)-\lambda\le -1, \\ (q(x)-\lambda-I_\pm^2)/(1+I_\pm^2), & \text{if } -1<q(x)-\lambda, \end{cases}$$

where $Q=Q(x, \lambda)=(\lambda-q(x))^{1/2}$ and $I_\pm=\tan\theta_\pm$.
Then the integrands in (5.1)′ are estimated as follows.

(5.9) $$|\tilde{\theta}'_\pm+Q|\le\frac{1}{2}|(\log Q)'|, \qquad \text{if } q(x)-\lambda\le -1,$$

(5.10) $$Q-1\le\theta'_\pm+Q\le Q(1-Q), \qquad \text{if } -1<q(x)-\lambda.$$

From (5.1)′, (5.2), (5.6), (5.9) and (5.10) we have Theorem 2.

§6. Generalization of Theorem 2

Here we consider Theorem 2 without the monotony of $q(x)$ in $(-\infty, x_0)$ and in (x_0, ∞). First we describe

Lemma 6.1. *Let $q(x)$ be continuous and positive in (x_1, x_2). Suppose that the solution $\theta(x)$ of $\theta'=\frac{q(x)-\tan^2}{1+\tan^2}$ satisfies $n\pi\le\theta(x_3)\le\left(n+\frac{1}{2}\right)\pi$ at $x_3\in(x_1, x_2)$. Then it follows*

$$n\pi<\theta(x)<\left(n+\frac{1}{2}\right)\pi \qquad \textit{in} \quad (x_3, x_2).$$

Proof. $F(x, \theta)=\frac{q(x)-\tan^2\theta}{1+\tan^2\theta}$ is negative if $\theta=\left(n+\frac{1}{2}\right)\pi$ and $F(x, \theta)$ is positive if $\theta=n\pi$ and $q(x)>0$. Therefore we have

$$n\pi<\theta(x)<\left(n+\frac{1}{2}\right)\pi \qquad \text{in} \quad (x_3, x_2).$$

Corollary. *Suppose $q(x)>0$ in (x_1, x_2). Then the variation of $\theta(x)$ in (x_1, x_2) is less than π.*

Now we decompose $\Omega_+(\lambda)=\{x;\ q(x)-\lambda>0\}$ as follows

(6.1) $$\Omega_+(\lambda)=(-\infty, x_-(\lambda))\cup((x_+(\lambda), \infty)\bigcup_{i=1}^{k}\omega_i(\lambda),$$

where $\omega_i(\lambda)$ are connected open finite intervals. From (5.8) we have

(6.2) $$-1\leq\theta'_{\pm}(x, \lambda)\leq q(x)-\lambda \quad \text{if} \quad 0<q(x)-\lambda.$$

From the above Corollary and (6.2) we have

Theorem 2′. *Assume* (C). *Then we have the estimates*:

$$\underline{E}(\lambda)-\underline{F}(\lambda)-2\pi<R(x, \lambda)<\bar{E}(\lambda)+\bar{F}(\lambda)+\pi,$$

$$\underline{E}(\lambda_n)-F(\lambda_n)-\pi<R(\lambda_n)<\bar{E}(\lambda_n)+F(\lambda_n),$$

where $\underline{F}(\lambda)$ *and* $\bar{F}(\lambda)$ *are given below and other notations are the same ones as in the precedent sections. Incidentally* $u'_n(x)$ *has at most* $n+2k$ *zeros.*

$$\underline{F}(\lambda)=\sum_{i=1}^{k}\min\left\{\int_{\omega_i(\lambda)}(q(x)-\lambda)dx, \pi\right\}$$

$$\bar{F}(\lambda)=\sum_{i=1}^{k}\min\left\{\int_{\omega_i(\lambda)}1\, dx, \pi\right\}.$$

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