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On the deformation of Riemann surfaces and differentials by quasiconformal mappings

By

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Introduction

On the Teichmüller space of a compact Riemann surface, Ahlfors [2] first showed the continuity of Dirichlet norms of Abelian differentials with prescribed A-periods. Recently, this result has been extended to some classes of open Riemann surfaces (cf. Kusunoki-Taniguchi [8], Shiga [16]).

On the other hand, Minda [12] proved that a quasiconformal mapping of Riemann surfaces induces isomorphisms between the Hilbert spaces of square integrable differentials with specific properties, and these isomorphisms are quasi-isometric (cf. Proposition 2.2).

To generalize these results, we shall define here the notion of the "*deformation family of Hilbert spaces*" and investigate the variation of reproducing kernels for bounded linear functionals (Sec. 1).

The subspaces of square integrable harmonic differentials and the isomorphisms induced by quasiconformal mappings whose maximal dilatations converge to one are typical examples of our deformation family.

In Sec. 2, we shall prove the variational formulae of the period reproducing differentials for subspaces of square integrable harmonic differentials by using the results of Sec. 1 (e.g. Theorem 2.3). Further we shall show the continuity of norms of reproducing differentials for a fixed Jordan arc on a surface, which gives an extension of the author's previous result [17].

We shall use freely the concepts in Ahlfors-Sario [4] (or Kusunoki [7]), especially notations and basic facts for the square integrable differentials on Riemann surfaces.

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§1. The deformation family of Hilbert spaces

1.1. Let H_1, H_2 be Hilbert spaces and A be an isomorphism of H_1 to H_2.

One can consider the *adjoint* mapping A_* of A such that

(1.1) $(A(x), y)_2=(x, A_*(y))_1$ for any $x\in H_1$ and $y\in H_2$, where $(\ ,\ )_i$ are are the inner products in H_i $(i=1, 2)$.

Lemma 1.1. *The adjoint mapping A_* has the following properties;*

(1) *A_* is an isomorphism of H_2 onto H_1 and $\|A_*\|=\|A\|$,*

(2) *$A_{**}=A$, and*

(3) *Let S_1 be a closed subspace of H_1 and $S_2=A(S_1)$. Put $\tilde{A}_*=(A|_{S_1})_*$, then $\tilde{A}_*\circ P_2=P_1\circ A_*$, where P_i are orthogonal projections of H_i onto S_i $(i=1, 2)$.*

Proof. Since (1) and (2) are classical, we prove (3) only. For any $x\in S_1$ and any $y\in H_2$, $(x, \tilde{A}_*\circ P_2(y))_1=(A(x), P_2(y))_2=(A(x), y)_2=(x, A_*(y))_1=(x, P_1\circ A_*(y))_1$, that is, $A_*\circ P_2=P_1\circ A_*$.

1.2. Let H_t $(t\geqq 0)$ be Hilbert spaces and A_t $(t>0)$ be isomorphisms of H_0 onto H_t. We call $\{(H_t, A_t)\}_{t>0}$ the *deformation family* of H_0 if there exist constants K_t $(\geqq 1)$ such that $\lim_{t\to 0} K_t=1$ and for any $x\in H_0$

$$K_t^{-1/2}\|x\|_0\leqq\|A_t(x)\|_t\leqq K_t^{1/2}\|x\|_0, \tag{1.2}$$

where $\|\cdot\|_t$ denotes the norms in H_t.

Considering a bounded linear functional L_t on H_t for each t, we denote by b_t the reproducing kernel for L_t on H_t, that is, $L_t(x)=(x, b_t)_t$ for any $x\in H_t$, where $(\ ,\ )_t$ is the inner product in H_t.

Lemma 1.2. *Let $\{(H_t, A_t)\}_{t>0}$ be a deformation family of H_0. Suppose that there exist bounded linear functionals L_t on H_t $(t\geqq 0)$ such that $\lim_{t\to 0} L_t(A_t(x))=L_0(x)$ for any $x\in H_0$. Then*

$$\varliminf_{t\to 0}\|b_t\|_t\geqq\|b_0\|_0.$$

$$\lim_{t\to 0}\|b_t\|_t=\|b_0\|_0 \textit{ if and only if } \lim_{t\to 0}\|(A_t)_*(b_t)-b_0\|_0=0.$$

Proof. From the above assumption we have

$$(x, b_0)_0=L_0(x)=\lim_{t\to 0} L_t(A_t(x))=\lim_{t\to 0}(A(x), b)_t=\lim_{t\to 0}(x, (A_t)_*(b_t))_0.$$

Namely, $\{(A_t)_*(b_t)\}_{t>0}$ converge to b_0 *weakly* in H_0 as $t\searrow 0$. Therefore, $\varliminf_{t\to 0}\|(A_t)_*(b_t)\|_0\geqq\|b_0\|_0$.

On the other hand, from (1.2) and Lemma 1.1 (1) we have

$$\varliminf_{t\to 0}\|(A_t)_*(b_t)\|_0=\varliminf_{t\to 0}\|b_t\|_t.$$

The last statement is easily obtained.

Theorem 1.3. *Let $\{(H_t, A_t)\}_{t>0}$ be a deformation family of H_0 and L_t $(t\geqq 0)$*

be linear functionals with reproducing kernels b_t *on* H_t *satisfying* $\lim_{t\to 0} L_t(A_t(x)) = L_0(x)$ *for any* $x \in H_0$. *For a fixed closed subspace* S_0 *of* H_0, *put* $S_t = A_t(S_0)$ *and denote by* c_t *the reproducing kernel for* $L_t|_{S_t}$ *on* S_t. *If* $\lim_{t\to 0} \|b_t\|_t = \|b_0\|_0$, *then* $\lim_{t\to 0} \|c_t\|_t = \|c_0\|_0$.

Proof. Recall that $c_t = P_t(b_t)$ where P_t are orthogonal projections of H_t onto S_t ($t \geqq 0$). Putting $\widetilde{(A_t)}_* = (A_t|_{S_0})_*$, we have $\widetilde{(A_t)}_*(c_t) = P_0 \circ (A_t)_*(b_t)$ from Lemma 1.1 (3). So, we have from Lemma 1.2

$$\lim_{t\to 0} \|\widetilde{(A_t)}_*(c_t) - c_0\|_0 = \lim_{t\to 0} \|P_0((A_t)_*(b_t) - b_0)\|_0 \leqq \lim_{t\to 0} \|(A_t)_*(b_t) - b_0\|_0 = 0.$$

Thus, from Lemma 1.2 again, $\lim_{t\to 0} \|c_t\|_t = \|c_0\|_0$.

§2. The quasiconformal deformation of Riemann surfaces

2.1. Suppose that $f\colon R \to R'$ is a K-quasiconformal mapping of a Riemann surface R onto a Riemann surface R'. Then f induces an isomorphism $f^{\#}\colon \Gamma(R') \to \Gamma(R)$ as follows (cf. [12]).

For each $\omega' = a(w)dw + b(w)d\bar{w} \in \Gamma(R')$ in terms of a local parameter w in a neighbourhood of $p' = f(p)$, $f^{\#}(\omega') \in \Gamma(R)$ is defined by

$$f^{\#}(\omega') = \{a(f)f_z + b(f)(\bar{f})_z\}dz + \{a(f)f_{\bar{z}} + b(f)(\bar{f})_{\bar{z}}\}d\bar{z}$$

in terms of a local parameter z in a neighbourhood of p.

Furthermore, we define $f_h^{\#}$ as $P_h \circ f^{\#}$, where P_h is the orthogonal projection from Γ onto Γ_h.

Proposition 2.1 ([12]) (i) $f^{\#}$ *is an isomorphism of* $\Gamma_x(R')$ *onto* $\Gamma_x(R)$, *where* $\Gamma_x = \Gamma_c, \Gamma_{se}, \Gamma_e, \Gamma_{c0}, \Gamma_{e0}$, *and for any* $\omega' \in \Gamma(R')$

$$K^{-1/2}\|\omega'\|_{R'} \leqq \|f^{\#}(\omega')\|_R \leqq K^{1/2}\|\omega'\|_{R'}. \tag{2.1}$$

(ii) $f_h^{\#}$ *is an isomorphism of* $\Gamma_x(R')$, *where* $\Gamma_x = \Gamma_h, \Gamma_{hse}, \Gamma_{he}, \Gamma_{h0}, \Gamma_{hm}$, *and for any* $\omega' \in \Gamma_h(R')$

$$K^{-1/2}\|\omega'\|_{R'} \leqq \|f_h^{\#}(\omega')\|_R \leqq K^{1/2}\|\omega'\|_{R'}. \tag{2.2}$$

To use the results of the preceding section, we shall consider a bounded linear functional $L\colon \omega \to \int_c \omega$ on $\Gamma_h(R)$ where c is a 1-cycle on R. Now, we note the following;

Proposition 2.2 (*cf.* [12] *Theorem* 4). *Let* $f\colon R \to R'$ *be a* K-*quasiconformal mapping. For every* 1-*cycle* c *on* R *and* 1-*cycle* c' *determined by* $f(c)$,

$$\int_{c'} \omega = \int_c f_h^{\#}(\omega) \qquad \omega \in \Gamma_h(R'). \tag{2.3}$$

2.2. Let f_t ($t > 0$) be K_t-quasiconformal mappings from R_t onto R_0 with

$\lim_{t\to 0} K_t=1$. Then from Proposition 2.1 (ii), $\{(\Gamma_h(R_t), (f_t)_h^\sharp)\}_{t>0}$ is a deformation family of $\Gamma_h(R_0)$.

For each closed subspace $X(t)$ of $\Gamma_h(R_t)$ and every 1-cycle c_t on R_t, one can find the $X(t)$-**reproducing differential** $\sigma(X(t), c_t)$, that is, it is the differential in $X(t)$ satisfying $\int_{c_t} \omega=(\omega, \sigma(X(t), c_t))_t$ for any $\omega\in X(t)$ where $(\ ,\ )_t$ is the inner product on $\Gamma_h(R_t)$.

Theorem 2.3. *For each closed subspace $X(0)$ of $\Gamma_h(R_0)$ and a 1-cycle c_0 on R_0, we put $\sigma(X(0))=\sigma(X(0), c_0)$ and $\sigma(f_t; X(0))=\sigma((f_t)_h^\sharp(X(0)), f_t^{-1}(c_0))$ respectively. Then*

$$\|\sigma(f_t; X(0))-(f_t)_h^\sharp(\sigma(X(0))\|_t\leqq\sqrt{2(K_t-1)}\|\sigma(X(0))\|_0, \tag{2.4}$$ *and*

$$K_t^{-1/2}\|\sigma(X(0))\|_0\leqq\|\sigma(f_t; X(0))\|_t\leqq K_t^{1/2}\|\sigma(X(0))\|_0, \tag{2.5}$$

where $\|\cdot\|_t$ is the Dirichlet norm on $\Gamma(R_t)$.

Proof. From Proposition 2.2, when $X(0)=\Gamma_h(R_0)$, $\sigma(\Gamma_h(R_0))=(f_t)_{h*}^\sharp(\sigma(\Gamma_h(R_t), f_t^{-1}(c_0)))$ where $(f_t)_{h*}^\sharp$ are adjoint mappings of $(f_t)_h^\sharp$ $(t>0)$ (cf. 1.1). Hence from Lemma 1.1 (3) we have

$$\begin{aligned}\sigma(X(0))&=P_{0X}(\sigma(\Gamma_h(R_0)))\\&=P_{0X}\circ(f_t)_{h*}^\sharp(\sigma(\Gamma_h(R_t), f_t^{-1}(c_0)))\\&=\widetilde{(f_t)_{h*}^\sharp}\circ P_{tX}(\sigma(\Gamma_h(R_t), f_t^{-1}(c_0)))\\&=\widetilde{(f_t)_{h*}^\sharp}(\sigma(f_t; X(0)))\,,\end{aligned}$$

where P_{0X} (resp. P_{tX}) is the orthogonal projection of $\Gamma_h(R_0)$ onto $X(0)$ (resp. $\Gamma_h(R_t)$ onto $(f_t)_h^\sharp(X(0))$) and $\widetilde{(f_t)_{h*}^\sharp}=((f_t)_h^\sharp|_{X(0)})_*$. Therefore, we conclude

$$\begin{aligned}\|\sigma(X(0))\|_0&=\|\widetilde{(f_t)_{h*}^\sharp}(\sigma(f_t; X(0)))\|_0\\&\leqq\|\widetilde{(f_t)_{h*}^\sharp}\|\,\|\sigma(f_t; X(0))\|_t\leqq\|(f_t)_h^\sharp\|\,\|\sigma(f_t; X(0))\|_t\\&\leqq K_t^{1/2}\|\sigma(f_t; X(0))\|_t.\end{aligned}$$

By considering f_t^{-1} we obtain the other inequality of (2.5) similarly.
Furthermore, from the above consideration we have

$$\begin{aligned}&\|\sigma(f_t; X(0))-(f_t)_h^\sharp(\sigma(X(0)))\|_t^2\\&=\|\sigma(f_t; X(0))\|_t^2+\|(f_t)_h^\sharp(\sigma(X(0)))\|_t^2\\&\quad-2(\sigma(f_t; X(0)), (f_t)_h^\sharp(\sigma(X(0))))_t\\&\leqq\|\sigma(f_t; X(0))\|_t^2+K_t\|\sigma(X(0))\|_0^2-2(\sigma(X(0)), \sigma(X(0)))_0\\&\leqq 2(K_t-1)\|\sigma(X(0))\|_0^2.\end{aligned}$$

Thus the proof is complete.

Corollary 2.4. *We denote by μ_t $(t>0)$ the complex dilatations of f_t. Suppose that $\lim_{t\to 0}\|\mu_t\|_\infty/t=0$, where $\|\mu_t\|_\infty=$ess. sup $|\mu_t|$. Then*

$$\frac{d}{dt}\|\sigma(f_t;\ X(0))-(f_t)_h^\#(\sigma(X(0)))\|_t^2|_{t=0}=0, \text{ and}$$

$$\frac{d}{dt}\|\sigma(f_t;\ X(0))\|_t|_{t=0}=0.$$

Proof. Since $K_t=(1+\|\mu_t\|_\infty)/(1-\|\mu_t\|_\infty)$, we can easily show the statements from the above theorem.

Corollary 2.5. *Let c_0' be another 1-cycle on R_0 and c_t' be the 1-cycle determined by $f_t^{-1}(c_0')$. We define the reproducing differentials $\psi(X(0))$ and $\psi(f_t;\ X(0))$ for c_0' and c_t' respectively as in Theorem 2.3. Then*

$$\left|\int_{c_t'}\sigma(f_t;\ X(0))-\int_{c_0'}\sigma(X(0))\right|$$
$$\leqq\sqrt{2K_t(K_t-1)}\|\sigma(X(0))\|_0\|\psi(X(0))\|_0.$$

Proof. Since $\psi(X(0))=\widetilde{(f_t)}{}_{h*}^\#(\psi(f_t;\ X(0)))$,

$$\int_{c_0'}\sigma(X(0))=(\sigma(X(0)),\ \psi(X(0)))_0$$
$$=(\sigma(X(0)),\ \widetilde{(f_t)}{}_{h*}^\#(\psi(f_t;\ X(0))))_0$$
$$=((f_t)_h^\#(\sigma(X(0))),\ \psi(f_t;\ X(0)))_t.$$

Hence from (2.4) and (2.5) we have

$$\left|\int_{c_t'}\sigma(f_t;\ X(0))-\int_{c_0'}\sigma(X(0))\right|$$
$$=|(\sigma(f_t;\ X(0)),\ \psi(f_t;\ X(0)))_t-(\sigma(X(0)),\ \psi(X(0)))_0|$$
$$=|(\sigma(f_t;\ X(0))-(f_t)_h^\#(\sigma(X(0))),\ \psi(f_t;\ X(0)))_t|$$
$$\leqq\sqrt{2K_t(K_t-1)}\|\sigma(X(0))\|_0\|\psi(X(0))\|_0.$$

2.3. Remarks. 1) For a quasiconformal mapping $f\colon R\to R'$ it is known (cf. [10] Lemma 3) that for $\omega_1,\ \omega_2\in\Gamma_h(R')$

$$(f_h^\#(\omega_1)^*,\ f_h^\#(\omega_2^*))_R=(\omega_1,\ \omega_2)_{R'}.$$

Hence we have $f_{h*}^\#(\omega)=-((f_h^\#)^{-1}(\omega^*))^*$ for $\omega\in\Gamma_h(R)$.

2) From Proposition 2.1, $f_h^\#(\Gamma_x(R'))=\Gamma_x(R)$ for $\Gamma_x=\Gamma_h$, Γ_{hse}, Γ_{he}, Γ_{h0} and Γ_{hm}. Hence Theorem 2.3 implies variational formulae of $\{\sigma(\Gamma_x(R_t), f_t^{-1}(c_0))\}$, Γ_x-period reproducing differentials.

On the other hand, it is shown in [10] and [12] that Γ_x^* is not preserved by a quasiconformal mapping generally. Hence Theorem 2.3 is not applicable to

$\Gamma_x(R_t)^*$-period reproducing differentials. But by considering the orthogonal decompositions in Γ_h, we can show the continuity of their Dirichlet norms (cf. Theorem 2.7').

3) As for the fact $\lim_{t\to 0} \|\sigma(f_t;\, X(0))\|_t = \|\sigma(X(0))\|_0$, one can give a simpler proof as follows. Let C_0 (resp. C_t) be the homology class on R_0 (resp. R_t) determined by c_0 (resp. $c_t = f_t^{-1}(c_0)$). Then it is known that

$$\|\sigma(f_t;\, \Gamma_h(R_0))\|_t^2 = \|\sigma(\Gamma_h(R_t),\, c_t)\|_t^2 = \lambda(C_t),$$

where $\lambda(C_t)$ is the extremal length of C_t(cf. [1] or [7]). By the quasiconformality of f_t, $\lim_{t\to 0} \|\sigma(f_t;\, \Gamma_h(R_0))\|_t = \lim_{t\to 0} \lambda(C_t)^{1/2} = \lambda(C_0)^{1/2} = \|\sigma(\Gamma_h(R_0))\|_0$. Thus from Theorem 1.3, we obtain $\lim_{t\to 0} \|\sigma(f_t;\, X(0))\|_t = \|\sigma(X(0))\|_0$ for every $X(0) \subset \Gamma_h(R_0)$.

4) From (2.4) and (2.5) we can easily show that

$$\|\sigma(f_t;\, X(0)) - (f_t)^*(\sigma(X(0)))\|_t \leqq M_t \|\sigma(X(0))\|_0,$$

where $M_t = \sqrt{(K-1)(3K_t+1)K_t^{-1}}$.

As for Γ_{h0}-period reproducing differentials, the similar result is known in [8]. Another result for the variation of Γ_{h0}-period reproducing differentials is given in [18].

2.4. Let p_0 and q_0 be arbitrary fixed points on R_0 and let d_0 be a fixed Jordan arc from p_0 to q_0 and $d_t = f_t^{-1}(d_0)$. Then linear functionals $L_t\colon \omega \to \int_{d_t} \omega$ are bounded in $\Gamma_h(R_t)$ $(t \geqq 0)$. For each closed subspace $X(t)$ of $\Gamma_h(R_t)$, we denote by $\varphi(X(t),\, d_t)$ the reproducing kernel for $L_t|_{X(t)}$ in $X(t)$.

For $X(t) = \Gamma_{he}(R_t)$, the following is known:

Proposition 2.6 (*cf.* [17] *Theorem* 5). *Let $f_t\colon R_t \to R_0$ $(t>0)$ be K_t-quasiconformal mappings with $\lim_{t\to 0} K_t = 1$. Then*

$$\lim_{t\to 0} \|\varphi(\Gamma_{he}(R_t),\, d_t)\|_t = \|\varphi(\Gamma_{he}(R_0),\, d_0)\|_0.$$

In this section we shall extend the above result to arbitrary subspace as follows;

Theorem 2.7. *Let R_0 be an arbitrary Riemann surface and let $f_t\colon R_t \to R_0$ $(t>0)$ be K_t-quasiconformal mappings with $\lim_{t\to 0} K_t = 1$. For each closed subspace $X(0)$ of $\Gamma_h(R_0)$, we put $\varphi(X(0)) = \varphi(X(0),\, d_0)$ and $\varphi(f_t;\, X(0)) = \varphi((f_t)_h^\#(X(0)),\, d_t)$ respectively. Then*

$$\lim_{t\to 0} \|\varphi(f_t;\, X(0))\|_t = \|\varphi(X(0))\|_0. \tag{2.6}$$

To prove this theorem we need some lemmas. At first, we consider a set E_0 on R_0 such that E_0 is the union of at most countable number of analytic curves on R_0, and $R_0' = R_0 - E_0$ is simply connected. And put $E_t = f_t^{-1}(E_0)$, $R_t' = R_t - E_t$.

We may consider $\Gamma_h(R_0)$ as a closed subspace of $\Gamma_{he}(R_0')$ and assume that d_0 is in R_0'. Then we have

Lemma 2.8. *Let* $\tilde{\varphi}(f_t; \Gamma_h(R_0))$ *be the reproducing differential for* $L_t|_{(f'_t)^{\#}_h(\Gamma_h(R_0))}$ *on* $(f'_t)^{\#}_h(\Gamma_h(R_0))$, *where* $f'_t = f_t|_{R_t}$. *Then*

$$\text{(2.7)} \qquad \lim_{t\to 0} \|\tilde{\varphi}(f'_t; \Gamma_h(R_0))\|_t = \|\varphi(\Gamma_h(R_0))\|_0.$$

Proof. By [17] Theorem 1, for each $\omega \in \Gamma_{he}(R'_0)$

$$\lim_{t\to 0} L_t((f'_t)^{\#}_h(\omega)) = L_0(\omega).$$

Hence from Proposition 2.3 and Theorem 1.3, (2.7) is easily obtained.

Lemma 2.9.

$$\text{(2.8)} \qquad \varliminf_{t\to 0} \|\varphi(f_t; \Gamma_h(R_0)\|_t \geqq \|\varphi(\Gamma_h(R_0))\|_0.$$

Proof. Suppose that each f_t is real analytic. For each $\omega \in \Gamma_h(R_0)$ we set $(f_t)^{\#}(\omega) = (f_t)^{\#}_h(\omega) + \omega^t_{e0}$, $\omega^t_{e0} \in \Gamma_{e0}(R_t)$. Then $\int_{d_0} \omega = \int_{d_t} (f_t)^{\#}(\omega) = \int_{d_t} (f_t)^{\#}_h(\omega) + \int_{d_t} \omega^t_{e0}$. Since $\|\omega^t_{e0}\|_t \to 0$ as $t \to 0$, we can show that

$$\lim_{t\to 0} \int_{d_t} (f_t)^{\#}_h(\omega) = \int_{d_0} \omega$$

by the same method as in [17] Theorem 1. Hence from Lemma 1.2 (2.8) is valid.

For arbitrary quasiconformal mappings $\{f_t\}$ it is known (cf. [8] Lemma 2) that for sufficiently small $t\,(>0)$ there are Riemann surfaces $\underline{R}_t$ and $\underline{K}_t$-quasiconformal mappings $\underline{f}_t$: $\underline{R}_t \to R_0$ such that each $\underline{f}_t$ is real analytic and $(\underline{R}_t, \underline{f}_t^{-1})$ is equivalent to (R_t, f_t^{-1}) in $T^{\#}(R_0)$, the *reduced Teichmüller space* of R_0 (cf. [6]). And we can show (2.8) by the similar method to that of [17] Lemma 4. For convenience, we shall sketch the proof.

From the definition of the equivalence class in $T^{\#}(R_0)$, there exist conformal mappings ϕ_t: $R_t \to \underline{R}_t$ such that ϕ_t is homotopic to $\underline{f}_t^{-1} \circ f_t$. Hence as $t \to 0$, $F_t = \underline{f}_t \circ \phi_t \circ f_t^{-1}$ which are homotopic to the identity on R_0, converges to the identity uniformly on every compact subset in R_0. Hence $\|\varphi(\Gamma_h(R_0))\|_0 = \|\varphi(\Gamma_h(R_0), d_0)\|_0 = \|\varphi(\Gamma_h(R_0), F_t(d_0))\|_0 + o(1)$ as $t \to 0$. From this result we can show that the difference between the Dirichlet norm of $\Gamma_h(\underline{R}_t)$-reproducing differential for $\underline{f}_t^{-1}(d_0)$, say $e(\underline{d}^t_0)$, and the Dirichlet norm of $\Gamma_h(\underline{R}_t)$-reproducing differential for $\phi_t \circ f_t^{-1}(d_0)$, say $e(\phi_t d_0)$, is $o(1)$ as $t \to 0$. Further $\|\varphi(f_t; \Gamma_h(R_0))\|_t = e(\phi_t d_0)$ because ϕ_t is conformal. Hence we have

$$\varliminf_{t\to 0} \|\varphi(f_t; \Gamma_h(R_0))\|_t = \varliminf_{t\to 0} e(\phi_t d_0) = \varliminf_{t\to 0} e(\underline{d}^t_0) \geqq \|\varphi(\Gamma_h(R_0))\|_0. \qquad \text{q.e.d.}$$

Lemma 2.10. *For* $\omega \in \Gamma_h(R_0)$, $(f'_t)^{\#}_h(\omega) \in \Gamma_c(R_t)$.

Proof. For each $dg_t \in \Gamma_{e0}(R_t)$, by Proposition 2.1, there is $dg_0 \in \Gamma_{e0}(R_0)$ such that $(f_t)^{\#}(dg_0) = dg_t$. So, by 2.3 Remark 1) we have

$$((f_t')_h^\sharp(\omega),\ dg_t^*)_t = -((f_t')_h^\sharp(\omega)^*,\ dg_t)_{R_t'}$$
$$= -((f_t')^\sharp(\omega)^*,\ dg_t)_{R_t'} \qquad ((\Gamma_{e0}^*)^\perp = \Gamma_c \supset \Gamma_e)$$
$$= -((f_t)^\sharp(\omega)^*,\ (f_t)^\sharp(dg_0))_t$$
$$= (\omega,\ dg_0^*)_0 = 0. \qquad (\Gamma_h \perp \Gamma_{e0}^*)$$

Therefore $(f_t')_h^\sharp(\omega) \in (\Gamma_{e0}^*(R_t))^\perp = \Gamma_c(R_t)$. q.e.d.

Lemma 2.11. *Let R_t $(t \geqq 0)$ be compact or compact bordered Riemann surfaces and let f_t: $R_t \to R_0$ be K_t-quasiconformal mappings with $\lim_{t\to 0} K_t = 1$. Then*

$$\lim_{t\to 0} \|\varphi(f_t;\ \Gamma_h(R_0))\|_t = \|\varphi(\Gamma_h(R_0))\|_0.$$

Proof. First we assume that f_t^{-1} are differentiable on R_0 and conformal on $R_0 - V_0$ for each $t > 0$, where V_0 is a fixed local disk on R_0'.

Denote by P_t $(t \geqq 0)$ the orthogonal projections of $\Gamma_c(R_t)$ onto $\Gamma_h(R_t)$, then we can show that $P_t \circ (f_t')_h^\sharp(\Gamma_h(R_0)) = \Gamma_h(R_0)$ as follows.

It suffices to show that $P_0 \circ (f_t'^{-1})_h^\sharp = \{P_t \circ (f_t')_h^\sharp\}^{-1}$. For each $\omega \in \Gamma_h(R_0)$ put $u_{0\omega}(p) = \int_{p_0}^{p} \omega$ $(p \in R_0')$ on R_0' and denote by $u_{t\omega}$ the solution of Dirichlet problem on R_t' with the boundary value $u_{0\omega} \circ f_t$ on $\partial R_t' = \partial R_t \cup E_t^+ \cup E_t^-$. Then $u_{t\omega}$ has the boundary value $u_{0\omega} \circ f_t$, where the boundary value means the non-tangential limit on the boundary almost everywhere if the boundary represents the unit circle by a conformal mapping. Then $(f_t')_h^\sharp(\omega) = du_{t\omega}$. In fact, $(f_t')^\sharp(\omega) - (f_t')_h^\sharp(\omega)$ is the differential of a *Dirichlet potential* v_t on R_t'. Since a Dirichlet potential is a *Wiener potential* (cf. [5] p. 81), there exists a potential U on R_t' such that $|v_t| \leqq U$ from [5] Hilfssatz 6.4. By Littlewood's theorem (cf. [19] Theorem IV. 33) U has the radial limit zero almost everywhere on $\partial R_t'$, and v_t also does. Hence $(f_t')_h^\sharp(\omega)$ is the differential of HD-function whose boundary value is $u_{0\omega} \circ f_t$, that is, $(f_t')_h^\sharp(\omega) = du_{t\omega}$.

Therefore for any closed curve γ_t on R_t, $\int_{\gamma_t} (f_t')_h^\sharp(\omega) = \int_{\gamma_0} \omega$ where $\gamma_0 = f_t(\gamma_t)$.

By Lemma 2.10 and the same argument as above, $P_t \circ (f_t')_h^\sharp(\omega)$ has the same γ_t-period as $(f_t')_h^\sharp(\omega)$ and $\int^p P_t \circ (f_t')_h^\sharp(\omega)$ on R_t' has the boundary value $u_{0\omega} \circ f_t$ on ∂R_t.

Considering the same argument for $P_0 \circ (f_t'^{-1})_h^\sharp$, we can verify that $\omega' = P_0 \circ (f_t'^{-1})_h^\sharp \circ P_t \circ (f_t')_h^\sharp(\omega)$ has the same period as ω for any closed curve on R_0 and has same boundary behavior near ∂R_0 as ω. So, we conclude $\omega' = \omega$.

Thus $\varphi(f_t;\ \Gamma_h(R_0))$, the reproducing differential on $\Gamma_h(R_t)$, is $P_t(\tilde{\varphi}(f_t';\ \Gamma_h(R_0)))$, the reproducing differential on $P_t \circ (f_t')_h^\sharp(\Gamma_h(R_0))$, and $\|\tilde{\varphi}(f_t';\ \Gamma_h(R_0))\|_{R_t'} \geqq \|\varphi(f_t;\ \Gamma_h(R_0))\|_t$. Hence we have from Lemma 2.8 and 2.9

$$\lim_{t\to 0} \|\varphi(f_t;\ \Gamma_h(R_0))\|_t = \|\varphi(\Gamma_h(R_0))\|_0.$$

For an arbitrary $\{f_t\}$ it is known (cf. [13] Proposition 6) that for sufficiently small t (>0) there are Riemann surfaces $\underline{R}_t$ and $\underline{K}_t$-quasiconformal mappings $\underline{f}_t$: $\underline{R}_t \to R_0$ such that $\{\underline{f}_t\}$ satisfies the same condition as above and $(\underline{R}_t, \underline{f}_t^{-1})$ is equivalent to (R_t, f_t^{-1}) in $T^{\#}(R_0)$. By the same proof as in Lemma 2.9 we can prove our conclusion for an arbitrary $\{f_t\}$.

2.5. *Proof of Theorem* 2.7. We may assume that d_0 is in $R_0'=R_0-E_0$. Further, as in the proof of Lemma 2.8 we may assume that each f_t is real analytic on R_t. Then by Theorem 1.3 it suffices to show that $\lim_{t\to 0}\|\varphi(f_t;\,\Gamma_h(R_0))\|_t=\lim_{t\to 0}\|\varphi(\Gamma_h(R_t), d_t)\|_t=\|\varphi(\Gamma_h(R_0))\|_0$.

First we shall show that for each R_t

$$\|\varphi(\Gamma_h(R_t), d_t)\|_t=\inf\{\|\varphi_h(W_t)\|_{W_t}:\ W_t \text{ is a regular subregion of } R_t \text{ and contains } d_t.\}, \tag{2.9}$$

where $\varphi_h(W_t)$ is the $\Gamma_h(W_t)$-reproducing differential for $L_t|_{\Gamma_h(w_t)}$.

Since $\Gamma_h(R_t)\subset\Gamma_h(W_t)$, $\|\varphi(\Gamma_h(R_t), d_t)\|_t\leqq\|\varphi_h(W_t)\|_{W_t}$. On the other hand, it is known (cf. [15] p. 100) that $\varphi(\Gamma_h(R_t), d_t)=dp_{I\sigma}^t+dp_{I\tau}^{t*}$ where $p_{I\sigma}^t$ is the (I)-L_1 principal function for the singularity $\sigma=\log|(z-q_t)/(z-p_t)|$ $(\partial d_t=q_t-p_t)$ on R_t and $p_{I\tau}^t$ is the (I)-L_1 principal function for the singularity $\tau=\arg(z-q_t)/(z-p_t)$ on R_t-d_t. The similar result is true for $\varphi_h(W_t)$. Hence by [14] II 1. H Theorem, we conclude $\lim_{W_t\nearrow R_t}\|\varphi_h(W_t)\|_{W_t}=\|\varphi(\Gamma_h(R_t), d_t)\|_t$. Consequently (2.9) follows.

If $\lim_{t\to 0}\|\varphi(f_t;\,\Gamma_h(R_0))\|_t=\|\varphi(\Gamma_h(R_0))\|_0$ is not true, then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n=0$ and $\lim_{n\to\infty}\|\varphi(f_{t_n};\,\Gamma_h(R_0))\|_{t_n}>\|\varphi(\Gamma_h(R_0))\|_0$ by (2.8). Then we take a regular subregion W_0 of R_0 such that

$$\lim_{n\to\infty}\|\varphi(f_{t_n};\,\Gamma_h(R_0))\|_{t_n}>\|\varphi_h(W_0)\|_{W_0}\geqq\|\varphi(\Gamma_h(R_0))\|_0.$$

On the other hand, from Lemma 2.11 for sufficiently large n we have

$$\|\varphi_h(W_{t_n})\|_{W_{t_n}}\leqq\|\varphi_h(W_0)\|_{W_0}+y/2,$$

where $W_{t_n}=f_{t_n}^{-1}(W_0)$ and $y=\lim_{n\to\infty}\|\varphi(f_{t_n};\,\Gamma_h(R_0))\|_{t_n}-\|\varphi_h(W_0)\|_{W_0}>0$.

Hence from (2.9) we have for sufficiently large n

$$\begin{aligned}\|\varphi(f_{t_n};\,\Gamma_h(R_0))\|_{t_n}&=\|\varphi(\Gamma_h(R_{t_n}), d_{t_n})\|_{t_n}\\ &\leqq\|\varphi_h(W_{t_n})\|_{W_{t_n}}<\lim_{n\to\infty}\|\varphi(f_{t_n};\,\Gamma_h(R_0))\|_{t_n}-y/2.\end{aligned}$$

This is a contradiction. q.e.d.

By the similar proof to that of Theorem 2.3 and Corollary 2.5, we can prove the followings.

Corollary 2.12. *For each closed subspace $X(0)$ of $\Gamma_h(R_0)$,*

$$\lim_{t\to\infty}\|(f_t)_h^{\#}(\varphi(X(0)))-\varphi(f_t;\,X(0))\|_t=0. \tag{2.10}$$

Corollary 2.13. *Let d_0' be another Jordan arc on R_0 and $d_t'=f_t^{-1}(d_0')$. Then for each closed subspace $X(0)$ of $\Gamma_h(R_0)$*

$$\lim_{t\to 0}\int_{d_t'}\varphi(f_t;\ X(0))=\int_{d_0'}\varphi(X(0)).$$

2.6. We note that for $\varphi(X(0))$ and $\varphi(f_t;\ X(0))$ the same inequalities as in Theorem 2.3 do *not* hold generally.

In fact, when $R_0=R_t=\{z;\ |z|<1\}$ and $f_t(z)=z|z|^t$ $(t>0)$, it is known (cf. [11]) that

$$\|\varphi(\Gamma_h(R_0))\|_0^2=-\frac{1}{\pi}\log(1-r^2),\ \text{and}$$

$$\|\varphi(f_t;\ \Gamma_h(R_0))\|_t^2=-\frac{1}{\pi}\log(1-r^{2(t+1)}),$$

where $p_0=0$ and $q_0=r$ $(0<r<1)$. Hence the simple calculation gives the fact $\sup\{\|\varphi(\Gamma_h(R_0))\|_0/\|\varphi(f_t;\ \Gamma_h(R_0))\|_t:\ 0<r<1\}=+\infty$, and we can obtain the desired results.

2.7. As 2.3 Remark 2), we know that Theorem 2.7 implies the continuity of Γ_x-reproducing differentials under quasiconformal deformations, where $\Gamma_x=\Gamma_{hm}$, Γ_{h0}, Γ_{hse}, or Γ_{he}. Furthermore, we can show the continuity of Γ_x^*-reproducing differentials by considering the orthogonal decompositions $\Gamma_h=\Gamma_{hse}\dotplus\Gamma_{hm}^*=\Gamma_{he}\dotplus\Gamma_{h0}^*=\Gamma_{hm}\dotplus\Gamma_{hse}^*=\Gamma_{h0}\dotplus\Gamma_{he}^*$ and Theorem 2.7. More generally,

Theorem 2.7′. *Let R_t, f_t, d_t be the same as in Theorem 2.7. For each closed subspace $X(0)$ of $\Gamma_h(R_0)$ we put $\varphi^{\perp}(X(0))=\varphi(X(0)^{\perp},\ d_0)$ and $\varphi^{\perp}(f_t;\ X(0))=\varphi((f_t)_h^{\#}(X(0))^{\perp},\ d_t)$ respectively, where $X^{\perp}$ is considered in Γ_h. Then*

$$\lim_{t\to 0}\|\varphi^{\perp}(f_t;\ X(0))\|_t=\|\varphi^{\perp}(X(0))\|_0. \tag{2.6}$$

In [11], Minda studied the pseudo distance

$$d_K^R(a,\ b)=\sup\{|u(a)-u(b)|/\sqrt{D_R(u)};\ u\in KD(R),\ D_R(u)\neq 0.\}$$

where $a,\ b\in R$ and $KD(R)$ is the space of $u\in HD(R)$ such that $du^*\in\Gamma_{hse}(R)$. It is easily seen that $d_H^R(a,\ b)$ is the Dirichlet norm of $\Gamma_{he}(R)\cap\Gamma_{hse}(R)^*$-reproducing differential for a Jordan arc from a to b.

Hence from Theorem 2.7′ and the orthogonal decomposition $\Gamma_{he}=\Gamma_{hm}\dotplus\Gamma_{he}\cap\Gamma_{hse}^*$, we have:

Corollary 2.14. *Let R_t, f_t, d_t be the same as in Theorem 2.7 and put $p_t=f_t^{-1}(p_0)$, $q_t=f_t^{-1}(q_0)$. Then*

$$\lim_{t\to 0}d_K^{R_t}(p_t,\ q_t)=d_K^{R_0}(p_0,\ q_0).$$

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