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On Galois correspondence between intermediate fields and closed derivation subalgebras

By

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Nakai-Kosaki-Ishibashi [2] has proved that if K is a purely inseparable field extension of finite exponent of a field k, then there exists a bijective correspondence between intermediate fields of K/k and closed subrings of $\mathscr{D}(K/k)$ containing K such that the corresponding field E and closed subring $\mathfrak{a}$ are related by the formulae $E=Z(\mathfrak{a})$ and $\mathfrak{a}=\mathscr{D}(K/E)$, where $\mathscr{D}(K/k)$ denotes the derivation algebra of K over k, $Z(\mathfrak{a})$ denotes the center of $\mathfrak{a}$ and $\mathscr{D}(K/k)$ is regarded as a topological space by the topology induced by the Krull topology of $\mathrm{Hom}_k(K, K)$.

This is a generalization of the theorem of Jacobson-Bourbaki correspondence in the case of purely inseparable finite extension.

In this paper we shall prove that a similar theorem of Galois correspondence still holds if the Krull topology is replaced by the finite topology and K is replaced by any field extension satisfying one of the following equivalent conditions (0), (1), (2) and (3).

(0) $\mathscr{D}(K/k)$ is dense in $\mathrm{Hom}_k(K, K)$.

(1) $Z(\mathscr{D}(K/k))=k$.

(2) If x is an element of $K\backslash k$, then there exists a high order derivation D of some order such that $D(x)\neq 0$.

(3) $\bigcap_{n=1}^{\infty} I^n_{K/k}=(0)$, where $I_{K/k}=\mathrm{Ker}\,(\text{multiplication } K\otimes_k K\to K)$.

Not only purely inseparable extension K/k of finite exponent but also purely transcendental extension K/k satisfy the conditions above. And there exists an example of a purely inseparable extension, not of finite exponent, satisfying the above conditions.*)

Notation and terminology. We adopt the notation and terminology in [1] and [2]. All rings are assumed to be commutative and have identities. When k is a ring and K is a commutative k-algebra, a q-th order derivation of K/k (or k-derivation of K) is, by definition, a k-homomorphism D: $K\to K$ satisfying the following identity:

*) Moreover, K/k is purely inseparable of finite exponent if and only if, for every intermediate field E of K/k, K/E satisfies the above conditions. (cf. Mordeson-Vinograde [5])

$$D(x_0x_1\cdots x_q)=\sum_{s=1}^{q}(-1)^{s-1}\sum_{i_1<\cdots<i_s}x_{i_1}\cdots x_{i_s}D(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_q)$$

for any set $\{x_0, x_1,\dots, x_q\}$ of $(q+1)$-elements in K. $\mathscr{D}_0^{(q)}(K/k)$ denotes the totality of q-th order k-derivations of K and $\mathscr{D}_0(K/k)$ denotes the union $\bigcup_{q=1}^{\infty}\mathscr{D}_0^{(q)}(K/k)$, which is a K-submodule of $\mathrm{Hom}_k(K, K)$. $\mathscr{D}(K/k)$ denotes the sum (necessarily a direct sum) of K-submodules K and $\mathscr{D}_0(K/k)$ in $\mathrm{Hom}_k(K, K)$, which has a natural structure of k-subalgebra of $\mathrm{Hom}_k(K, K)$. $\mathscr{D}(K/k)$ is called the derivation algebra of K over k.

For any $D\in\mathscr{D}_0(K/k)$ and $a\in K$, we set $[D, a]=Da-aD-D(a)$ i.e. $[D, a](x)=D(ax)-aD(x)-D(a)x$. D belongs to $\mathscr{D}_0^{(q)}(K/k)$ if and only if $[D, a]$ belongs to $\mathscr{D}_0^{(q-1)}(K/k)$ for all $a\in K$.

Always $\otimes$ means $\otimes_k$ (tensoring over k). $I_{K/k}$ denotes the kernel of the multiplication map $\mu\colon K\otimes K\to K$. We regard $K\otimes K$ as a left K-module by the K-operation $a(x\otimes y)=ax\otimes y$.

§1. Preliminaries on the finite topology on $\mathrm{Hom}_k(K, K)$

Let V and W be vector spaces over a field k. For any pair of finite ordered sets $\{v_1,\dots, v_m\}\subset V$ and $\{w_1,\dots, w_m\}\subset W$, we set $U_k(v_1,\dots, v_m; w_1,\dots, w_m)=\{f\in\mathrm{Hom}_k\cdot(V, W)\,|\,f(v_i)=w_i \text{ for all } i\}$. The whole of the subsets of $\mathrm{Hom}_k(V, W)$ of $U_k(v_i; w_i)$ type forms a basis for a topology on $\mathrm{Hom}_k(V, W)$ which is called the finite topology on $\mathrm{Hom}_k(V, W)$. (cf. Jacobson [3], Ch. IX, §6) This is nothing but a topology with the fundamental system of neighborhood of zero consisting of all the subsets of the form $U_k(E)=\{f\in\mathrm{Hom}_k(V, W)\,|\,f|_E=0\}$, where E is a finite dimensional subspace of V. This topology is discrete if and only if $\dim_k V<\infty$. If $V=W$ and V is an algebraic extension field of k, the finite topology is identical with the Krull topology. (cf. Nakai-Kosaki-Ishibashi [2])

By definition, next lemma is obvious, which means that the basic open sets $U_k(v_i; w_i)$'s are also closed.

Lemma 1. *Let V and W be vector spaces over a field k, $\mathfrak{a}$ be a subset of $\mathrm{Hom}_k(V, W)$ and $\bar{\mathfrak{a}}$ be its closure in $\mathrm{Hom}_k(V, W)$ with respect to the finite topology. Let v and w be elements of V and W respectively such that $f(v)=w$ for every $f\in\mathfrak{a}$. Then we have $g(v)=w$ for every $g\in\bar{\mathfrak{a}}$.*

Now let K be any field extension of a field k. When M is a K-submodule of the left K-module $K\otimes K$, we identify $\mathrm{Hom}_K((K\otimes K)/M, K)$ with a subset of $\mathrm{Hom}_K\cdot(K\otimes K, K)$ consisting of the elements f such that $f|_M=0$. Then, from the above lemma, we obtain immediately the following.

Corollary 2. *Let K be any field extension of a field k and M be any K-submodule of $K\otimes K$. Then $\mathrm{Hom}_K((K\otimes K)/M, K)$ is closed in $\mathrm{Hom}_K(K\otimes K, K)$ with respect to the finite topology.*

Next we show the following

Lemma 3. *Let K be any field extension of a field k. Then the mapping φ: $\mathrm{Hom}_K(K\otimes K, K)\to\mathrm{Hom}_k(K, K)$ defined by $\varphi(f)(x)=f(1\otimes x)$ is a homeomorphism with respect to the finite topology.*

Proof. If we define ψ: $\mathrm{Hom}_k(K, K)\to\mathrm{Hom}_K(K\otimes K, K)$ by the formula $\psi(g)(\sum_i x_i\otimes y_i)=\sum_i x_i g(y_i)$, it is clear that $\psi\circ\varphi=id$ and $\varphi\circ\psi=id$, hence ψ is the inverse of φ. Consequently it is obvious that $\varphi^{-1}(U_k(x_i;\ y_i))=\psi(U_K(x_i;\ y_i))=U_k(1\otimes x_i;\ y_i)$ for any (x_i), $(y_i)\subset K$, which shows that φ is continuous. On the other hand, if $U_K(\xi; 0)$ is any basic open neighborhood of zero in $\mathrm{Hom}_K(K\otimes K, K)$ where $\xi=\sum_i x_i\otimes y_i=\sum_i x_i(1\otimes y_i)$ $(x_i, y_i\in K)$ and $y_1,\dots, y_n$ are linearly independent over k, then we see easily that $\psi^{-1}(U_K(\xi; 0))\subset U_k(y_1,\dots, y_n; 0,\dots, 0)$. Hence ψ is also continuous. q.e.d.

Remark 4. (Nakai-Kosaki-Ishibashi [2]) Let k, K and φ be as in Lemma 3, and let $I=I_{K/k}$. Then we have $\varphi^{-1}(\mathscr{D}(K/k))\subset\mathrm{Hom}_K((K\otimes K)/\bigcap_{n=1}^{\infty} I^n, K)$, where $\mathscr{D}(K/k)$ denotes the derivation algebra of K over k. (However $\varphi^{-1}(\mathscr{D}(K/k))\neq\mathrm{Hom}_K((K\ \ K)/\bigcap_{n=1}^{\infty} I^n, K)$ in general.)

Proposition 5. *Let K be any field extension of a field k, and E be an intermediate field between k and K. Then $\mathrm{Hom}_E(K, K)$ is closed in $\mathrm{Hom}_k(K, K)$ with respect to the finite topology.*

Proof. The proof is similar to that of [2], Prop. 8. Let f be any element of the closure of $\mathrm{Hom}_E(K, K)$ and let $x\in K$ and $a\in E$. Since the neighborhood $U_k(x, ax; f(x), f(ax))$ of f contains an element of $\mathrm{Hom}_E(K, K)$, there exists an element $g\in\mathrm{Hom}_E(K, K)$ such that $g(x)=f(x)$ and $g(ax)=f(ax)$. Then we have $f(ax)=g(ax)=ag(x)=af(x)$, which shows that $f\in\mathrm{Hom}_E(K, K)$. q.e.d.

Next we characterize the dense subrings of $\mathrm{Hom}_k(K, K)$ containing $K(=K\cdot id)$ as follows.

Proposition 6. *Let K be a field extension of a field k, and $\mathfrak{a}$ be a subring of $\mathrm{Hom}_k(K, K)$ containing K. Then $\mathfrak{a}$ is dense in $\mathrm{Hom}_k(K, K)$ with respect to the finite topology if and only if $Z(\mathfrak{a})=k$, where $Z(\mathfrak{a})$ denotes the center of $\mathfrak{a}$.*

Proof. The proof of the if part is the same as that of [2], Th. 7. That is, regarding K as a left $\mathfrak{a}$-module, K is a simple $\mathfrak{a}$-module. And the commutant of $\mathfrak{a}$-module K is nothing but $\mathrm{Hom}_{\mathfrak{a}}(K, K)$. However, since $Z(\mathfrak{a})=k$, we have $\mathrm{Hom}_{\mathfrak{a}}(K, K)=k$. Hence the bicommutant of $\mathfrak{a}$-module K is $\mathrm{Hom}_k(K, K)$. Therefore by the density theorem (Bourbaki [4], ch. 8, §4, $n°2$.), $\mathfrak{a}$ is dense in $\mathrm{Hom}_k(K, K)$. Conversely suppose $\mathfrak{a}$ be dense in $\mathrm{Hom}_k(K, K)$. If $f\in Z(\mathfrak{a})$, $\varphi\in\mathrm{Hom}_k(K, K)$ and $x\in K$, there exists an element $\alpha\in\mathfrak{a}$ such that $\alpha(x)=\varphi(x)$ and $\alpha(f(x))=\varphi(f(x))$ i.e. $\alpha\in U_k(x, f(x); \varphi(x), \varphi(f(x)))$. Then we have $(\varphi f)(x)=\varphi(f(x))=\alpha(f(x))=f(\alpha(x))=f(\varphi(x))=(f\varphi)(x)$ i.e. $\varphi f=f\varphi$. This shows that $Z(\mathfrak{a})\subset Z(\mathrm{Hom}_k(K, K))$. On the other hand it is clear that $Z(\mathrm{Hom}_k(K, K))=k$. Hence we have $Z(\mathfrak{a})=k$. q.e.d.

Corollary 7. *Let K be a field extension of a field k. Then $\mathscr{D}(K/k)$ is dense in* $\mathrm{Hom}_k(K, K)$ *if and only if* $Z(\mathscr{D}(K/k))=k$.

§ 2. Galois correspondence

First we shall investigate the condition for $Z(\mathscr{D}(K/k))=k$.

Lemma 8. *Let K be a field extension of a field k, and $x\in K$. Then the following conditions are equivalent.*

(1) $x\in Z(\mathscr{D}(K/k))$.

(2) *For any* $D\in\mathscr{D}_0(K/k)$, *we have* $Dx=xD$.

(3) *For any* $D\in\mathscr{D}_0(K/k)$, *we have* $D(x)=0$.

(4) $\delta(x)=1\otimes x-x\otimes 1$ *belongs to* $\bigcap_{n=1}^{\infty} I_{K/k}^{n}$.

Proof. Since (1)⇔(2)⇒(3)⇔(4) is obvious, we have only to prove (3)⇒(2). Suppose $D(x)=0$ for all $D\in\mathscr{D}_0(K/k)$, and let y be any element in K. Then we have

$$0=[D, y](x)=D(yx)-yD(x)-D(y)x$$
$$=D(xy)-xD(y)=(Dx-xD)y,$$

which shows that $Dx=xD$. q.e.d.

From this lemma and results in the last section we obtain the following

Theorem 9. *Let K be any field extension of a field k. Then the following conditions are equivalent.*

(1°) $Z(\mathscr{D}(K/k))=k$.

(2°) *If x is an element of $K\backslash k$, then there exists a derivation $D\in\mathscr{D}_0(K/k)$ of some order such that* $D(x)\neq 0$.

(3°) $\bigcap_{m=1}^{\infty} I_{K/k}^{n}=(0)$

Proof. Since it is obvious that $Z(\mathscr{D}(K/k))\subset K$, the implications (3°)⇒(1°)⇔(2°) are clear from the above lemma. Hence we have only to prove (1°)⇒(3°). (The following proof is essentially the same as that of [2], Prop. 11.) Suppose that $Z(\mathscr{D}(K/k))=k$ and set $M=\bigcap_{n=1}^{\infty} I_{K/k}^{n}$. Let φ: $\mathrm{Hom}_K(K\otimes K, K)\rightarrow\mathrm{Hom}_k(K, K)$ be the homeomorphism in Lemma 3. Then, by Remark 4, we have $\varphi^{-1}(\mathscr{D}(K/k))\subset\mathrm{Hom}_K((K\otimes K)/M, K)\subset\mathrm{Hom}_K(K\otimes K, K)$. Thus, since $\mathscr{D}(K/k)$ is dense in $\mathrm{Hom}_k(K, K)$ by Cor. 7, $\mathrm{Hom}_K((K\otimes K)/M, K)$ must be dense in $\mathrm{Hom}_K(K\otimes K, K)$. On the other hand, $\mathrm{Hom}_K((K\otimes K)/M, K)$ is closed in $\mathrm{Hom}_K(K\otimes K, K)$ by Cor. 2. Hence we have $\mathrm{Hom}_K((K\otimes K)/M, K)=\mathrm{Hom}_K(K\otimes K, K)$, which shows that $M=(0)$. q.e.d.

Corollary 10. *Let K be any field extension of a field k such that K/k satisfies (one of) the equivalent conditions of Theorem 9, and let E be an intermediate field between k and K. Then the extension E/k still satisfies the same conditions of Theorem 9. (K/E does not, in general.)*

Proof. Since we have $I_{E/k} \subset I_{K/k}$, the condition $\bigcap_{n=1}^{\infty} I_{K/k}^n = (0)$ implies $\bigcap_{n=1}^{\infty} I_{E/k}^n = (0)$.

Now we can establish a Galois correspondence between special intermediate fields of K/k and closed derivation subalgebras of $\mathscr{D}(K/k)$ provided that K/k satisfies the conditions of Theorem 9, in a similar way as [2].

Definition. An intermediate field E of K/k is called allowable if K/E satisfies the conditions of Theorem 9.

Theorem 11. *Let K be any field extension of a field k such that K/k satisfies (one of) the conditions of Theorem 9, and endow $\mathscr{D}(K/k)$ with the induced topology from the finite topology of* $\mathrm{Hom}_k(K, K)$.

(1) *Let E be any allowable intermediate field of K/k. Then $\mathscr{D}(K/E)$ is a closed subring of $\mathscr{D}(K/k)$ containing K, and we have $Z(\mathscr{D}(K/E)) = E$.*

(2) *Let $\mathfrak{a}$ be a closed subring of $\mathscr{D}(K/k)$ containing K. Then the center $Z(\mathfrak{a})$ of $\mathfrak{a}$ is an anllowable intermediate field of K/k, and we have $\mathscr{D}(K/Z(\mathfrak{a})) = \mathfrak{a}$.*

Thus there exists a bijective correspondence between allowable intermediate fields of K/k and closed subrings of $\mathscr{D}(K/k)$ containing K such that the corresponding field E and closed subring $\mathfrak{a}$ are related by the formulae $E = Z(\mathfrak{a})$ and $\mathfrak{a} = \mathscr{D}(K/E)$.

Proof. (1) is obvious from the fact $\mathscr{D}(K/E) = \mathscr{D}(K/k) \cap \mathrm{Hom}_E(K, K)$ and Prop. 5.

(2) Since $\mathfrak{a} \supset K$, we must have $Z(\mathfrak{a}) \subset K$, from which we can easily show that $Z(\mathfrak{a})$ is an intermediate field between k and K. On the other hand, since $\mathfrak{a} \subset \mathscr{D}(K/Z(\mathfrak{a}))$ and $\mathfrak{a}$ is dense in $\mathrm{Hom}_{Z(\mathfrak{a})}(K, K)$ by Prop. 6, $\mathfrak{a}$ is dense and closed in $\mathscr{D}(K/Z(\mathfrak{a})) = \mathscr{D}(K/k) \cap \mathrm{Hom}_{Z(\mathfrak{a})}(K, K)$. Hence we must have $\mathfrak{a} = \mathscr{D}(K/Z(\mathfrak{a}))$.

q.e.d.

Corollary 12. *Let K be a field extension of a field k such that K/k satisfies the conditions of Theorem 9. Let $\{E_\lambda \mid \lambda \in \Lambda\}$ be any collection of allowable intermediate fields of K/k and $\{\mathfrak{a}_\lambda \mid \lambda \in \Lambda\}$ be any collection of closed subrings of $\mathscr{D}(K/k)$ containing K. Then we have the following formulae, where $\bigvee_\lambda$ denotes the generated object.*

(1) $\mathscr{D}(K/\bigvee_\lambda E_\lambda) = \bigcap_\lambda \mathscr{D}(K/E_\lambda)$.

(2) $\mathscr{D}(K/\bigcap_\lambda E_\lambda) =$ *the closure of* $\bigvee_\lambda \mathscr{D}(K/E_\lambda)$ *in* $\mathscr{D}(K/k)$.

(3) $Z(\bigcap_\lambda \mathfrak{a}_\lambda) =$ *the smallest allowable subfield containing* $\bigvee_\lambda Z(\mathfrak{a}_\lambda)$.

(4) $Z(\bigvee_\lambda \mathfrak{a}_\lambda) = \bigcap_\lambda Z(\mathfrak{a}_\lambda)$.

Proposition 13. *If K is a field extension of a field k satisfying the conditions of Theorem 9, then we have the following.*

(1) *k is separably closed in K. That is, no element of $K \backslash k$ is separably algebraic over k.*

(2) *If K/k is algebraic, then K/k is purely inseparable.*

Proof. If an element $x \in K \backslash k$ is separably algebraic over k, we have $I_{k[x]/k}^2 = I_{k[x]/k}$. Hence we have

$$(0) \neq I_{k[x]/k} = \bigcap_{n=1}^{\infty} I_{k[x]/k}^n \subset \bigcap_{n=1}^{\infty} I_{K/k}^n,$$

which proves (1). (2) follows from (1). q.e.d.

Proposition 14. (1) *In the following cases, K/k satisfies the conditions of Theorem* 9.

(i) *K/k is a purely inseparable extension of finite exponent.*

(ii) *K/k is a purely transcendental field extension.*

(2) *In the case of* (i), *every intermediate field of K/k is allowable.**)

Proof. (i) has been already proved in [2] (Th. 2, Th. 3 or Prop. 11) (ii) Let K be the quotient field of a polynomial ring $A = k[X_\lambda, \lambda \in \Lambda]$ where X_λ's are indeterminates. Then by the arguments of [1], Ch. 2, §2 we have $\bigcap_{n=1}^{\infty} I_{A/k}^n = (0)$. On the other hand, if we denote $\{s \otimes s \mid s \in A, s \neq 0\}$ by S, then we have $K \otimes K = (A \otimes A)_S$ and $I_{K/k} = I_{A/k}(K \otimes K)$, $I_{A/k}^n$ is $I_{A/k}$-primary. Hence $\bigcap_{n=1}^{\infty} I_{K/k}^n = \bigcap_{n=1}^{\infty} I_{A/k}^n(K \otimes K) = (0)$. q.e.d.

Examples

(1) An example of purely inseparable extension K/k, not of finite exponent, satisfying the conditions of Theorem 9.

Let k_0 be a perfect field of characteristic $p > 0$, set $k = k_0(x_1^p, x_2^{p^2}, \dots, x_n^{p^n}, \dots)$ where x_i's are independent variables over k_0 and set $K = k(x_1, x_2, \dots, x_n, \dots)$.

Then we have $I_{K/k} = \bigoplus_{n=1}^{\infty} I_n$ (direct sum of K-modules) where I_{n_i} denotes the K-submodule with a basis $\{\prod_{i=1}^{\infty} \delta(x_i)^{n_i} \mid 0 \leqq n_i \leqq p-1, \sum_i n_i = n\}$ $(\delta(x_i) = 1 \otimes x_i - x_i \otimes 1)$. Therefore we have $\bigcap_{n=1}^{\infty} I_{K/k}^n = (0)$ and exponent $(K/k) = \infty$.

(2) An example of imperfect purely inseparable extension K/k such that $\bigcap_{n=1}^{\infty} I_{K/k}^n = I_{K/k} \neq (0)$.

Let k_0 be a perfect field and x and y be independent variables over k_0. Set $k = k_0(x, y)$ and $K = k(y^{p^{-1}}, y^{p^{-2}}, \dots, y^{p^{-n}}, \dots) = k_0(x)\ k_0(y)^{p^{-\infty}}$. Then we have $K^{p^\infty} = k_0(y)^{p^{-\infty}}$ and $k(K^{p^\infty}) = K$, therefore $\bigcap_{n=1}^{\infty} I_{K/k}^n = I_{K/k}$.

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*) By the THEOREM of Mordeson-Vinograde [5], K/k is purely inseparable of finite exponent if and only if every intermediate field of K/k is allowable.

References

[1] Y. Nakai, High order derivations I, Osaka J. Math., **7** (1970), 1–21.
[2] Y. Nakai, K. Kosaki and Y. Ishibashi, High order derivations II, J. Sci. Hiroshima Univ., Ser A-1, **34** (1970), 17–27.
[3] N. Jacobson, Lectures in abstract algebra, Vol. II, Van Nostrand, Princeton, New Jersey.
[4] N. Bourbaki, Algèbre, Hermann, Paris.
[5] J. N. Mordeson and B. Vinograde, On high order derivations of fields, Proc. A. M. S., **32** (1972), 421–422.