J. Math. Kyoto Univ. (JMKYAZ)
23-2 (1983) 269-279

Every Noetherian uniformly coherent ring has dimension at most 2

By

Shiro GOTO*)

(Communicated by Prof. Nagata, Jan. 19, 1982. Revised, Feb. 12, 1982)

1. Introduction

The purpose of this paper is to give a theorem stated in the title.

Let A be a commutative ring and let $v_A(M)$ denote, for an A-module M, the least number of elements in systems of generators for M. Let $\boldsymbol{N}$ denote the set of positive integers. We put

$$f_A(n) = \sup_{h \in \mathrm{Hom}_A(A^n, A)} v_A(\mathrm{Ker}\, h)$$

for each $n \in \boldsymbol{N}$. Then $f_A(n)$ is possibly infinite, and the ring A is called uniformly coherent if the supremum $f_A(n)$ is finite for every $n \in \boldsymbol{N}$.

The concept of uniform coherence was introduced by Soublin [14], and he showed that a given ring A is uniformly coherent if and only if the direct product $A^{\boldsymbol{N}}$ is a coherent ring (Proposition 7). Quentel [12] succeeded Soublin and mentioned that if A is a Noetherian uniformly coherent ring, then

$$\sup_{p \in \mathrm{Spec} A} v_A(\mathfrak{p}) \leqq f_A(1) + f_A(4)$$

and, consequently, A must have finite dimension. This is a pretty application of Soublin's remark above and Gulliksen's theorem ([6]) that every prime ideal in a Noetherian ring belongs to some ideal generated by three elements. As was given in [12], it is not difficult to see that Noetherian semi-local rings of dimension at most one are uniformly coherent. Sally [13] has extended this result and proved that any two-dimensional Noetherian local ring is uniformly coherent (Ch. 3, 2.2 Theorem). In a subsequent joint paper [5] of the author and Suzuki, one may find a crucial use of her theorem, from which the motivation of the present research has started.

Nevertheless, no one knows any examples of higher dimension and, because polynomial rings $k[X_1, X_2,\ldots, X_d]$ and formal power series rings $k[|X_1, X_2,\ldots, X_d|]$ over a field k are not uniformly coherent for any $d \geqq 3$ (Recall the famous examples of Macaulay [11], cf. [12], Proposition 3.1.), it seems to be rather reasonalbe to

*) Partially supported by Grant-in-Aid for Co-operative Research.

doubt about the existence of Noetherian uniformly coherent rings with dimension at least three.

Now let us state explicitly our conclusion:

Theorem 1.1. *Let A be a Noetherian ring. Then the following conditions are equivalent.*

(1) *A is a uniformly coherent ring.*

(2) $\dim A \leqq 2$, *and the supremum*

$$\bar{f}_A(n) = \sup_{\mathfrak{m} \in \operatorname{Max} A} f_{A_{\mathfrak{m}}}(n)$$

is finite for every $n \in \boldsymbol{N}$.

Unfortunately the latter part of the condition (2) in Theorem 1.1 is, in general, not superfluous, and such an example shall be given in Section 4. However with some suitable additional assumption on A, one may easily omit this part.

Corollary 1.2. *Suppose that A is a Noetherian semi-local ring, or that A is a finitely generated algebra over a field. Then A is a uniformly coherent ring if and only if* $\dim A \leqq 2$.

We will prove Theorem 1.1 and its corollary in Section 3. The next section is deveoted to some preliminaries which we need to prove Theorem 1.1.

Throughout this paper, let A denote a commutative ring and $\boldsymbol{N}$ the set of positive integers.

2. Preliminaries

In this section let A denote a Noetherian local ring with maximal ideal $\mathfrak{m}$. Let $H^i_{\mathfrak{m}}(\cdot)$ be local cohomology functors.

We begin with the following

Lemma 2.1. *Suppose that* $\dim A = 2$ *and* $\operatorname{depth} A \geqq 1$. *Let a, b be a system of parameters for A and assume that* $(a, b) \cdot H^1_{\mathfrak{m}}(A) = (0)$. *Then the following equality*

$$(a^n, b^n) \colon (ab)^{n-1} = [(a) \colon b] + [(b) \colon a]$$

holds for every integer $n \geqq 2$.

Proof. Let $f \in A$ and assume that $(ab)^{n-1} f = a^n g + b^n h$ where $g, h \in A$. Then $a^{n-1} f - bh \in (a^n) \colon b^{n-1}$ clearly. Hence, as $(a^n) \colon b^{n-1} = (a^n) \colon b$ (cf. [4], (2.7)), we get that $a^{n-1} bf = ax^n + b^2 h$ for some $x \in A$. Similarly we may express $abf = a^2 x + b^2 y$ with $y \in A$. Now notice that $x \in (b) \colon a$ $(=(b) \colon a^2)$, and express $ax = bz$ with $z \in A$. Then $abf = b(az) + b^2 y$, whence $af = az + by$ since b is a non-zerodivisor of A (cf. [4], (2.6) (1)). Consequently $f - z$ is in $(b) \colon a$, whence $f \in [(a) \colon b] + [(b) \colon a]$ as $z \in (a) \colon b$ by our choice. Thus we have proved the inclusion

$$(a^n, b^n) \colon (ab)^{n-1} \subset [(a) \colon b] + [(b) \colon a].$$

The opposite one is obvious.

Proposition 2.2. *Suppose that* $\dim A=3$ *and* $\operatorname{depth} A\geqq 2$. *Assume that the A-module* $H_{\mathfrak{m}}^2(A)$ *is finitely generated. Let a, b, c be a system of parameters for A and assume that* $(a, b)\cdot H_{\mathfrak{m}}^2(A)=(0)$. *Then the following equality*

$$(a^n, b^n, c^n): (abc)^{n-1}=[(a, c): b]+[(b, c): a]+(a, b): c^{n-1}]$$

holds for every integer $n\geqq 2$.

Proof. First of all, we prove the following

Claim. $(a^{n_1}, b^{n_2}): c^{n_3}=(a^{n_1}, b^{n_2})+a^{n_1-1}b^{n_2-1}\cdot[(a, b): c^{n_3}]$ for all n_1, n_2, $n_3\in \boldsymbol{N}$.

Let $f\in A$ and assume that $c^{n_3}f=a^{n_1}g+b^{n_2}h$ for some $g, h\in A$. Then $h\in (a^{n_1}, c^{n_3}): b^{n_2}$ clearly. Let $\bar{A}=A/(a^{n_1}, c^{n_3})$, and let $\bar{\cdot}$ denote the reduction mod(a^{n_1}, c^{n_3}). Then as $\bar{b}$ is a parameter for the one-dimensional local ring $\bar{A}$, it must be a non-zerodivisor mod the ideal $H_{\mathfrak{m}}^0(A/(a^{n_1}, c^{n_3}))$. Hence we get that $\bar{h}\in H_{\mathfrak{m}}^0(A/(a^{n_1}, c^{n_3}))$, as $h\in(a^{n_1}, c^{n_3}): b^{n_2}$ by our choice. Now recall that a^{n_1}, c^{n_3} is an A-regular sequence (cf. [4], (2.6)) and apply the functors $H_{\mathfrak{m}}^i(\cdot)$ to the following exact sequences

$$0\longrightarrow A\xrightarrow{c^{n_3}} A\longrightarrow A/(c^{n_3})\longrightarrow 0,$$

$$0\longrightarrow A/(c^{n_3})\xrightarrow{a^{n_1}} A/(c^{n_3})\longrightarrow A/(a^{n_1}, c^{n_3})\longrightarrow 0.$$

Then we get two inclusions

$$H_{\mathfrak{m}}^1(A/(c^{n_3}))\subset H_{\mathfrak{m}}^2(A) \text{ and } H_{\mathfrak{m}}^0(A/(a^{n_1},c^{n_3}))\subset H_{\mathfrak{m}}^1(A/(c^{n_3})),$$

which yield that $bh\in(a^{n_1}, c^{n_3})$, as $b\cdot H_{\mathfrak{m}}^2(A)=(0)$ by our standard assumption.

Let us express $bh=a^{n_1}x+c^{n_3}y$ with $x, y\in A$. Then $c^{n_3}f=a^{n_1}g+b^{n_2-1}(a^{n_1}x+c^{n_3}y)$, whence $f-b^{n_2-1}y\in(a^{n_1}): c^{n_3}$. As a^{n_1}, c^{n_3} is an A-regular sequence, this guarantees that $f-b^{n_2-1}y\in(a^{n_1})$ and so we have that $f\in(a^{n_1})+b^{n_2-1}\cdot[(a^{n_1}, b): c^{n_3}]$. (Notice that $y\in(a^{n_1}, b): c^{n_3}$.) Thus

$$(a^{n_1}, b^{n_2}): c^{n_3}\subset(a^{n_1})+b^{n_2-1}.\ [(a^{n_1}, b): c^{n_3}].$$

As the opposite inclusion is obvious, we have proved that

$$(a^{n_1}, b^{n_2}): c^{n_3}=(a^{n_1})+b^{n_2-1}.\ [(a^{n_1}b): c^{n_3}]$$

for all n_1, n_2, $n_3\in \boldsymbol{N}$.

By virtue of this equality and the symmetry between a and b, we further get that

$$\begin{aligned}(a^{n_1}, b^{n_2}): c^{n_3}&=(a^{n_1})+b^{n_2-1}.\ [(a^{n_1}, b): c^{n_3}]\\&=(a^{n_1})+b^{n_2-1}.[(b)+a^{n_1-1}.\ [(a, b): c^{n_3}]]\\&=(a^{n_1}, b^{n_2})+a^{n_1-1}b^{n_2-1}.\ [(a, b): c^{n_3}].\end{aligned}$$

Thus

$$(a^{n_1}, b^{n_2}):c^{n_3}=(a^{n_1}, b^{n_2})+a^{n_1-1}b^{n_2-1}.\ [(a, b): c^{n_3}]$$

as required.

Now return to the main proof. Let $f \in A$ and assume that

$$(abc)^{n-1}f = a^n x + b^n y + c^n z$$

for some $x, y, z \in A$. Then, as $(ab)^{n-1}f - cz \in (a^n, b^n): c^{n-1}$, we see by the above claim that

$$(ab)^{n-1}f - cz \in (a^n, b^n) + (ab)^{n-1} \cdot [(a, b): c^{n-1}].$$

Accordingly $(ab)^{n-1}(f-g) \in (a^n, b^n, c)$ for some $g \in (a, b): c^{n-1}$. Let $\bar{A} = A/(c)$, and let $\bar{\cdot}$ denote the reduction mod (c). Then, as $(\bar{a}, \bar{b}) \cdot H^1_{\mathfrak{m}}(\bar{A}) = (0)$ by our choice of a and b (recall that $H^1_{\mathfrak{m}}(\bar{A}) \subset H^2_{\mathfrak{m}}(A)$), we see by (2.1) that $\bar{f} - \bar{g}$ is contained in $[(\bar{a}): \bar{b}] + [(\bar{b}): \bar{a}]$, i.e.,

$$f - g \in [(a, c): b] + [(b, c): a].$$

Hence

$$f \in [(a, c): b] + [(b, c): a] + [(a, b): c^{n-1}],$$

and we have proved that

$$(a^n, b^n, c^n): (abc)^{n-1} \subset [(a, c): b] + [(b, c): a] + [(a, b): c^{n-1}].$$

As the opposite inclusion is clear, this completes the proof of (2.2).

Corollary 2.3. *In the same situation as* (2.2),

$$(abc)^{n-1} \notin (a^n, b^n, c^n)$$

for every $n \in \boldsymbol{N}$.

Let us assume that $\dim A = 3$ and let $\underline{a} = a_1, a_2, a_3, a_4$ be a system of elements in $\mathfrak{m}$ such that $\dim A/(\underline{a}) = 0$. Let $H(\underline{a}; A)$ denote the homology of the Koszul complex $K(\underline{a}; A)$ generated by $\underline{a}$ over A. Let $h: A^4 \to A$ be the A-linear map defined by $h(e_i) = a_i$ for all i, $1 \leqq i \leqq 4$, where e_i are the canonical basis of the A-module A^4. Then

$$\mathrm{v}_A(H_1(\underline{a}; A)) \leqq \mathrm{v}_A(\mathrm{Ker}\, h) \leqq \mathrm{v}_A(H_1(\underline{a}; A)) + 6$$

clearly.

We put

$$\mathrm{s}(A) = \sup_{\underline{a}} \mathrm{v}_A(H_1(\underline{a}; A)),$$

where $\underline{a} = a_1, a_2, a_3, a_4$ runs over systems of elements in $\mathfrak{m}$ such that $\dim A/(a) = 0$. Notice that if this supremum $\mathrm{s}(A)$ is infinite, then A is not a uniformly coherent ring.

Proposition 2.4. (1) *Let* A^* *denote the completion of* A. *Then* $\mathrm{s}(A^*) = \mathrm{s}(A)$.
(2) *Suppose that* A *contains a regular local ring* R *over which* A *is module-finite. Then* $\mathrm{s}(A)$ *is infinite, if* R *is a direct summand of* A.

Proof. (1) Let $\underline{b}=b_1, b_2, b_3, b_4$ be elements of $\mathfrak{m}A^*$ with $\dim A^*/(\underline{b})A^*=0$. Choose elements $\underline{a}=a_1, a_2, a_3, a_4$ of $\mathfrak{m}$ so that $(\underline{a})A^*=(\underline{b})A^*$. Then, because $K(\underline{b}; A^*)\cong A^*\otimes_A K(\underline{a}; A)$ as complexes of A^*-modules, we get an isomorphism $H_1(\underline{b}; A^*)\cong A^*\otimes_A H_1(\underline{a}; A)$ of A^*-modules. Consequently $\mathrm{v}_{A^*}(H_1(\underline{b}; A^*))=\mathrm{v}_A(H_1(\underline{a}; A))$, whence $\mathrm{s}(A^*)\leqq \mathrm{s}(A)$. The opposite inequality is similarly proved.

(2) Let $\underline{a}=a_1, a_2, a_3, a_4$ be elements of R. Then, because $K(\underline{a}; R)$ is a direct summand of $K(\underline{a}; A)$ as a complex of R-modules, the R-module $H_1(\underline{a}; R)$ is also contained in $H_1(a; A)$ as a direct summand. Consequently, we get that

$$\mathrm{v}_R(H_1(\underline{a}; R))\leqq \mathrm{v}_R(H_1(\underline{a}; A))\leqq \mathrm{v}_R(A)\cdot \mathrm{v}_A(H_1(\underline{a}; A)).$$

Therefore $\mathrm{s}(R)\leqq \mathrm{v}_R(A)\cdot \mathrm{s}(A)$, which tells us that it suffices to show that $\mathrm{s}(R)$ is infinite.

Now let x, y, z be a system of generators for the maximal ideal $\mathfrak{n}$ of R, and let $n\geqq 5$ be an odd integer. Let H_n denote the n by n alternating matrix defined by

$$\begin{aligned}[H_n]_{ij} &= x && (i \text{ odd and } j=i+1)\\ &= y && (i \text{ even and } j=i+1)\\ &= z && (i+j=n+1)\\ &= 0 && \text{otherwise}\end{aligned}$$

$(1\leqq i<j\leqq n)$, and let I_n be the ideal of R generated by $n-1$ by $n-1$ Pfaffians of the matrix H_n (cf. [1]. See also [13], Ch. 5.). Then as $x^e, y^e, z^e\in I_n$ $(e=(n-1)/2)$, the ideal I_n is certainly $\mathfrak{n}$-primary and therefore, by virtue of Theorem 2.1 in [1], we see that R/I_n is an Artinian Gorenstein local ring with $\mathrm{v}_R(I_n)=n$. Let $J_n=(x^e, y^e, z^e): I_n$. Then, according to Proposition 3.1 in [10], we get that $\mathrm{v}_{R/(x^e,y^e,z^e)}(J_n/(x^e, y^e, z^e))=1$ and that the dimension $\mathrm{r}(R/J_n)$ of the socle of the ring R/J_n is equal to $n-3$. Thus $\mathrm{v}_R(J_n)=4$. Let $J_n=(x^e, y^e, z^e, v_n)$ for some element v_n of $\mathfrak{n}$. Then, as $\mathrm{r}(R/J_n)=\mathrm{v}_R(\mathrm{Ext}_R^3(R/J_n, R))$ (cf. [7], 6.10), we find that the R-module R/J_n has a minimal free resolution of the following form:

$$0\longrightarrow R^{n-3}\longrightarrow R^n\longrightarrow R^4\xrightarrow[{[x^e,y^e,z^e,v_n]}]{} R\longrightarrow R/J_n\longrightarrow 0.$$

Therefore $\mathrm{v}_R(H_1(x^e, y^e, z^e, v_n; R))\geqq n-6$, whence we know that $\mathrm{s}(R)$ is infinite.

3. Proof of Theorem 1.1

First of all, we note the following two results due to Quentel [12]. (A detailed proof of Lemma 3.2 may be found also in [13].)

Lemma 3.1. *Let S be a multiplicative system in A. Then*

$$f_{S^{-1}A}(n)\leqq f_A(n)$$

for every $n\in \mathbf{N}$. Hence $S^{-1}A$ is a uniformly coherent ring if so is A.

Lemma 3.2. *Let $f: A \to B$ be a homomorphism of commutative rings making B a finitely presented A-module. Then B is a uniformly coherent ring if so is A. In case* $\mathrm{Ker} f$ *is a finitely presented and nilpotent ideal of A, the converse is also true.*

Proof of Theorem 1.1.

(2)⇒(1) (cf. [12], Corollaire) Let $h: A^n \to A$ be an A-linear map. Then $v_{A_{\mathfrak{m}}}(A_{\mathfrak{m}} \otimes_A \mathrm{Ker}\, h) \leqq \bar{f}_A(n)$ for every maximal ideal $\mathfrak{m}$ of A. Consequently, by Satz 2 in [3], we see that $v_A(\mathrm{Ker}\, h) \leqq \bar{f}_A(n)+2$, whence $f_A(n) \leqq \bar{f}_A(n)+2$ for every $n \in \boldsymbol{N}$. Thus A is a uniformly coherent ring by definition.

(1)⇒(2) By virtue of (3.1), it is enough to show that $\dim A \leqq 2$.

Assume the contrary and choose a prime ideal $\mathfrak{p}$ of A so that $\dim A_{\mathfrak{p}} = 3$. Then, as the ring $A_{\mathfrak{p}}$ is, by (3.1), again uniformly coherent, to produce a contradiction we may assume that A is a local ring of dimension 3.

Suppose that A contains a field and let k be a coefficient field of the completion A^* of A. Choose a system x, y, z of parameters for A^* and put $R = k[|x, y, z|]$. Then R is a regular local ring, over which A^* is module-finite. Moreover R is, by Theorem 2 in [8], a direct summand of A^*, whence from (2.4) it follows that $s(A)$ is infinite — this is a contradiction, and we find that A does not contain any field.

Let $\mathfrak{m}$ denote the maximal ideal of A and put $p = ch\, A/\mathfrak{m}$, the characteristic of the field $A/\mathfrak{m}$. Then $0 \neq p \in \mathfrak{m}$. On the other hand, as A/pA certainly contains a field and is, by (3.2), uniformly coherent, we get, by the above argument, that $\dim A/pA \leqq 2$. Consequently, the element p of $\mathfrak{m}$ may be extended to a system of parameters for A, say p, x, y. Let W denote a coefficient ring of A^* and put $R = W[|x, y|]$. Then R is a regular local ring and A^* is module-finite over R. Take a prime ideal P of A^* such that $\dim A^*/P = 3$, and let B denote the normalization of the ring A^*/P. Then B is a module-finite extension of A^*/P and is again a local ring. Moreover depth $B \geqq 2$, and the B-module $H^2_{\mathfrak{m}B}(B)$ is finitely generated. Now choose the elements x, y so that $(x, y) \cdot H^2_{\mathfrak{m}B}(B) = (0)$. Then, according to (2.3), we get that $(pxy)^{n-1} \notin (p^n, x^n, y^n)B$ for every $n \in \boldsymbol{N}$. Consequently

$$(pxy)^{n-1} \notin (p^n, x^n, y^n)A^*$$

for all $n \in \boldsymbol{N}$, which guarantees, by Theorem 1 in [8], that R is a direct summand of A^*. Thus $s(A)$ must be infinite by (2.4) — this is the final contradiction, and we conclude that $\dim A \leqq 2$.

Corollary 3.3. *Let R be a regular ring. Then R is a uniformly coherent ring if and only if* $\dim R \leqq 2$.

Proof. The only if part is a direct consequence of Theorem 1.1, and the if part is due to [14] (cf. Théorème 3).

Proof of Corollary 1.2.

In case A is a Noetherian semi-local ring, our assertion immediately follows from (1.1). Assume that A is a finitely generated algebra over a field k. Let $d = \dim A$, and choose elements $X_1, X_2, \ldots, X_d$ of A so that A is module-finite over the polynomial ring $R=k[X_1, X_2,\dots, X_d]$. Then by (3.2), we see that A is a uniformly coherent ring if and only if so is R. The latter statement is clearly equivalent to the condition that $d\leqq 2$ (see (3.3)).

4. Example

In this section, we shall construct a Noetherian integral domain A containg an algebraically closed field and satisfying the following conditions:

(1) *A contains countably many maximal ideals, say* $\{\mathfrak{m}_n\}_{n\in N}$.
(2) $\dim A_{\mathfrak{m}_n}=2$ *for every* $n\in N$.
(3) *For each* $n\in N$, *there is an A-linear map* h_n: $A^2\to A$ *such that*

$$v_A(\operatorname{Ker} h_n)=v_{A_{\mathfrak{m}_n}}(A_{\mathfrak{m}_n}\underset{A}{\otimes}\operatorname{Ker} h_n)=n+1.$$

Of course, this ring A is not uniformly coherent, and one knows by this example that the latter part of the condition (2) in Theorem 1.1 is, in general, not superfluous.

Let $S=k[X_1, X_2, X_3, X_4]$ be a polynomial ring over an algebraically closed field k, and let $M=S_+$, the irrelevant maximal ideal of S. Let E be a graded S-module. We denote by E_q $(q\in Z)$ the graduation of E. Let $H^i_M(E)$ stand for local cohomology modules of E relative to M, which we regard as graded S-modules. For an integer p, we denote by $E(p)$ the graded S-module whose underlying S-module coincides with that of E and whose graduation is given by $[E(p)]_q=E_{p+q}$ for all $q\in Z$.

We put $\underline{k}=S/M$.

Lemma 4.1. *Let* $n\in N$. *Then there exists a graded prime ideal* P_n *of* S *such that* $\dim S/P_n=2$ *and* $H^1_M(S/P_n)\cong\underline{k}^n(3-5n)$ *as graded S-modules.*

Proof. Let

$$0\longrightarrow F_4\longrightarrow F_3\longrightarrow F_2\longrightarrow F_1\longrightarrow F_0=S\longrightarrow \underline{k}\longrightarrow 0$$

be a graded minimal free resolution of $\underline{k}$. We put

$$Z=\operatorname{Im}(F_2\longrightarrow F_1) \quad\text{and}\quad E=Z^n(2).$$

Then, since $\operatorname{depth}_{S_M}E_M=2$ and $E_\mathfrak{p}$ is a free $S_\mathfrak{p}$-module for every prime ideal $\mathfrak{p}$ of S $(\mathfrak{p}\neq M)$, one may take, by virtue of Theorem B in [2], an exact sequence

$$0\longrightarrow S^{3n-1}(-1)\xrightarrow{\ h\ } E\longrightarrow P_n(r)\longrightarrow 0 \ (*)$$

of graded S-modules, where r is an integer and P_n is a graded prime ideal of S with $\dim S/P_n=2$. Now consider the following exact sequence

$$0\longrightarrow P_n\longrightarrow S\longrightarrow S/P_n\longrightarrow 0\ (**).$$

Then we get, by both the exact sequences $(*)$ and $(**)$, that

$$H^1_M(S/P_n)\cong[H^2_M(E)]\,(-r),$$

which yields an isomorphism

$$H^1_M(S/P_n) \cong \underline{k}^n(2-r)$$

of graded S-modules since $E = Z^n(2)$ and $H^2_M(Z) \cong \underline{k}$.

Let us check that $r = 5n-1$.

Claim. The S-module $P_n(r)$ has a graded minimal free resolution of the following form:

$$0 \longrightarrow F^n_4(2) \longrightarrow F^n_3(2) \oplus S^{3n-1}(-1) \longrightarrow F^n_2(2) \longrightarrow P_n(r) \longrightarrow 0.$$

Proof. Recall that E has a resolution

$$0 \longrightarrow F^n_4(2) \longrightarrow F^n_3(2) \longrightarrow F^n_2(2) \xrightarrow{\ f\ } E \longrightarrow 0.$$

Choose a homomorphism $g\colon S^{3n-1}(-1) \to F^n_2(2)$ of graded S-modules making the following triangle

$$\begin{array}{ccc} & S^{3n-1}(-1) & \\ {}^{g}\swarrow & & \searrow^{h} \\ F^n_2(2) & \xrightarrow{\quad f \quad} & E \end{array}$$

commutative. Then it is a routine work to get that the mapping cone of the following homomorphism

$$\begin{array}{ccccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & S^{3n-1}(-1) & \longrightarrow & 0 & \longrightarrow \cdots & & & & \\ & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & & & & \\ \cdots \longrightarrow 0 \longrightarrow & F^n_4(2) & \longrightarrow & F^n_3(2) & \longrightarrow & F^n_2(2) & \longrightarrow & 0 & \longrightarrow \cdots \end{array}$$

of complexes of graded S-modules provides a graded minimal free resolution of $P_n(r)$.

Let $H(t)$ denote the Hilbert-Samuel series of the graded ring S/P_n, i.e., $H(t) = \sum_{q=0}^{\infty} \dim_k [S/P_n]_q) \cdot t^q$. Then, since $F_i = S^{\binom{4}{i}}(-i)$ and the Hilbert-Samuel series of S is given by $\frac{1}{(1-t)^4}$, we get, by the resolution of $P_n(r)$ obtained by the above claim, that

$$H(t) = \frac{1 - 6nt^r + (7n-1)t^{r+1} - nt^{r+2}}{(1-t)^4}.$$

On the other hand, as $\dim S/P_n = 2$, we know that $H(t) = \frac{f(t)}{(1-t)^2}$ for some polynomial $f(t)$ in t. Consequently, in the following equation

$$1 - 6nt^r + (7n-1)t^{r+1} - nt^{r+2} = 0,$$

$t=1$ must be a multiple root, whence we get that $r = 5n-1$ as required.

We put $R_n = S/P_n$ and choose a linear system x_n, y_n of parameters for the ring

R_n. Let $g_n\colon R_n^2 \to R_n$ denote the R_n-linear map defined by $g_n\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax_n + by_n$ for each $\begin{bmatrix} a \\ b \end{bmatrix} \in R_n^2$.

Lemma 4.2. $\mathrm{v}_{R_n}(\operatorname{Ker} g_n) = n+1$.

Proof. Let $\overline{R_n} = R_n/x_nR_n$, and let $\overline{\cdot}$ denote the reduction mod x_nR_n. Then, since $x_n \cdot H_M^1(R_n) = (0)$, by the exact sequence

$$0 \longrightarrow R_n(-1) \xrightarrow{x_n} R_n \longrightarrow \overline{R_n} \longrightarrow 0$$

we see that $H_M^0(\overline{R_n}) \cong [H_M^1(R_n)](-1)$. Because $H_M^1(R_n) \cong \underline{k}^n(3-5n)$ by (4.1), this yields an isomorphism $H_M^0(\overline{R_n}) \cong \underline{k}^n(2-5n)$. Take homogeneous elements z_i $(1 \leqq i \leqq n)$ of R_n with degree $5n-2$ so that $\overline{z_i}$ form a k-basis of $H_M^0(\overline{R_n})$ and, subsequently, express $y_nz_i = -x_nv_i$ $(1 \leqq i \leqq n)$ with homogeneous elements v_i in R_n of degree $5n-2$.

Claim. The R_n-module $\operatorname{Ker} g_n$ is generated by the following elements

$$\left\{\begin{bmatrix} v_i \\ z_i \end{bmatrix}\right\}_{1 \leqq i \leqq n} \text{ and } \begin{bmatrix} -y_n \\ x_n \end{bmatrix}.$$

Proof. Let a, b be elements of R_n such that $ax_n + by_n = 0$. Then, since y_n is a non-zerodivisor on the ring $\overline{R_n}/H_M^0(\overline{R_n})$, we get that $\overline{b}$ is in $H_M^0(\overline{R_n})$, which allows us to express $b = \sum_{i=1}^{n} c_iz_i + ux_n$ with $c_i \in k$ and $u \in R_n$. Therefore $by_n = \sum_{i=1}^{n} c_i(y_nz_i) + (y_nu)x_n$, whence $by_n = \sum_{i=1}^{n} c_i(-x_nv_i) + (y_nu)x_n$ because $y_nz_i = -x_nv_i$ by our choice. Accordingly, we get that $a = \sum_{i=1}^{n} c_iv_i + u(-y_n)$, since $ax_n = -by_n$ and since x_n is a non-zerodivisor of R_n. Thus $\begin{bmatrix} a \\ b \end{bmatrix} = \sum_{i=1}^{n} c_i \begin{bmatrix} v_i \\ z_i \end{bmatrix} + u \begin{bmatrix} -y_n \\ x_n \end{bmatrix}$, and we conclude that $\operatorname{Ker} g_n$ is generated by $n+1$ elements $\begin{bmatrix} v_i \\ z_i \end{bmatrix}$ $(1 \leqq i \leqq n)$ and $\begin{bmatrix} -y_n \\ x_n \end{bmatrix}$.

Because $\begin{bmatrix} v_i \\ z_i \end{bmatrix}$ and $\begin{bmatrix} -y_n \\ x_n \end{bmatrix}$ are homogeneous elements of R_n^2 with degree $5n-2$ and 1 respectively, to prove that $\mathrm{v}_{R_n}(\operatorname{Ker} g_n) = n+1$ it is enough to check that they form a minimal basis of the graded R_n-module $\operatorname{Ker} g_n$, i.e., $\begin{bmatrix} v_i \\ z_i \end{bmatrix}$ $(1 \leqq i \leqq n)$ and $\begin{bmatrix} -y_n \\ x_n \end{bmatrix}$ are linearly independent mod $M \cdot (\operatorname{Ker} g_n)$ over the field k. This is routine and we omit it.

Let $Q_n = [R_n]_+$ $(= M/P_n)$ for each $n \in \boldsymbol{N}$ and put $B = \bigotimes_{i=1}^{\infty} R_n$, where the tensor product is taken over the ground field k. Then, as k is an algebraically closed field, the ring B must be an integral domain. Clearly Q_nB is a prime ideal of B for all $n \in \boldsymbol{N}$. We put $A = T^{-1}B$, where $T = B \setminus \bigcup_{n=1}^{\infty} Q_nB$.

Proposition 4.3. (1) *A is a Noetherian integral domain.*

(2) $\operatorname{Max} A = \{Q_nA / n \in \boldsymbol{N}\}$.

(3) $\dim A_{Q_nA} = 2$ *for every* $n \in \boldsymbol{N}$.

Proof. See Proposition 1, [**9**].

Let h_n: $A^2 \to A$ denote, for each $n \in \boldsymbol{N}$, the A-linear map induced by the R_n-linear map g_n: $R_n^2 \to R_n$, i.e., $h_n = A \underset{R_n}{\otimes} g_n$. Recall the following commutative diagram

$$\begin{array}{ccc} R_n & \longrightarrow & A \\ \downarrow & & \downarrow \\ (R_n)_{Q_n} & \longrightarrow & A_{Q_n A} \end{array}$$

of rings, where all homomorphisms are flat. (The bottom one is, especially, faithfully flat.) Then we find that

$$\begin{aligned} \mathrm{v}_{R_n}(\mathrm{Ker}\, g_n) &\geqq \mathrm{v}_A(\mathrm{Ker}\, h_n) \\ &\geqq \mathrm{v}_{A_{Q_n A}}(\mathrm{Ker}\,(A_{Q_n A} \underset{R_n}{\otimes} g_n)) \\ &= \mathrm{v}_{(R_n)_{Q_n}}(\mathrm{Ker}\,((R_n)_{Q_n} \underset{R_n}{\otimes} g_n)). \end{aligned}$$

Because Q_n is the irrelevant maximal ideal of R_n and $\mathrm{Ker}\, g_n$ is a graded R_n-module, we further get

$$\begin{aligned} \mathrm{v}_{(R_n)_{Q_n}}(\mathrm{Ker}\,((R_n)_{Q_n} \underset{R_n}{\otimes} g_n)) &= \mathrm{v}_{(R_n)_{Q_n}}((R_n)_{Q_n} \underset{R_n}{\otimes} \mathrm{Ker}\, g_n) \\ &= \mathrm{v}_{R_n}(\mathrm{Ker}\, g_n). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{v}_A(\mathrm{Ker}\, h_n) &= \mathrm{v}_{A_{Q_n A}}(\mathrm{Ker}\,(A_{Q_n A} \underset{A}{\otimes} h_n)) \\ &= \mathrm{v}_{R_n}(\mathrm{Ker}\, g_n), \end{aligned}$$

whence $\mathrm{v}_A(\mathrm{Ker}\, h_n) = \mathrm{v}_{A_{Q_n A}}(A_{Q_n A} \underset{A}{\otimes} \mathrm{Ker}\, h_n) = n+1$ by (4.2). This completes our construction, and the ring A is a required example.

Department of Mathematics
Nihon University

References

[1] D. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math., **99** (1977), 447–485.

[2] E. G. Evans, Jr. and P. A. Griffith, Local cohomology modules for normal domains, J. London Math. Soc., **19** (1979), 277–284.

[3] O. Forster, Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring, Math. Z., **84** (1964), 80–87.

[4] S. Goto, Blowing-up of Buchsbaum rings, London Mathematical Society Lecture Note Series 72 (Commutative Algebra: Durham 1981), 140–162, Cambridge Univ. Press, 1983.

[5] S. Goto and N. Suzuki, Index of reducibility of parameter ideals in a local ring, to appear.

[6] T. H. Gulliksen, Tout idéal premier d'un anneau noethérien est associè á un idéal engendré par trois éléments, C. R. Acad. Sc. Paris, **271** (1970), 1206–1207.

[7] J. Herzog and E. Kunz et al., Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Mathematics 238, Springer Verlag, 1971.

[8] M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J., **51** (1973), 25–43.

[9] ———, Non-openness of loci in Noetherian rings, Duke Math. J., **40** (1973), 215–219.

[10] E. Kunz, Almost complete intersections are not Gorenstein rings, J. Alg., **28** (1974), 111–115.

[11] F. S. Macaulay, The Algebraic Theory of Modular Systems, Cambridge University, 1916.

[12] Y. Quentel, Sur l'uniforme cohérence des anneaux noethériens, C. R. Acad. Sc. Paris, **275** (1972), 753–755.

[13] J. Sally, Numbers of Generators of Ideals in Local Rings, Lecture Notes in Pure and Applied Mathematics 35, Marcel Dekker, Inc., 1978.

[14] J. -P. Soublin, Anneaux et modules cohérents, J. Alg., **15** (1970), 455–472.