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Some remarks on high order derivations III

By

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By the theorem of Jacobson-Bourbaki correspondence, the following proposition is obvious.

If K is an extension field of finite degree over a field k, then all subrings of $\mathrm{Hom}_k(K, K)$ *which contain* K $(=K\cdot id_K)$ *are simple.*

In this short paper, we show that this proposition remains true even if we remove the finiteness condition for extension K/k and replace $\mathrm{Hom}_k(K, K)$ with $\mathscr{D}(K/k)$, where $\mathscr{D}(K/k)$ denotes the derivation algebra of K over k. That is, we prove the following proposition.

Let K be an arbitrary field extension of a field k, and $\mathscr{D}(K/k)$ be its derivation algebra. Then each subring of $\mathscr{D}(K/k)$ containing K must be simple.

If K is purely inseparable over k, this is a generalization of the above, for in the case of finite purely inseparable extension K/k, we have $\mathscr{D}(K/k)=\mathrm{Hom}_k(K, K)$. (Nakai-Kosaki-Ishibashi [2]).

As an application of this property, we shall give an another rapid proof of a part of our theorem of preveous paper [3]. That is, we shall show that the above proposition yields at once the following result.

If K is a field extension of a field k such that $\mathscr{D}(K/k)=\mathrm{Hom}_k(K, K)$, *then* $[K: k]$ *is finite.*

Notation and terminology. We adopt the notation and terminology in [1] and [2]. All rings are assumed to be commutative and have identities. When k is a ring and K is a commutative k-algebra, a q-th order derivation of K/k (or k-derivation of K) is, by definition, a k-homomorphism $D: K\to K$ satisfying the following identity:

$$D(x_0x_1\cdots x_q)=\sum_{s=1}^{q}(-1)^{s-1}\sum_{i_1<\cdots<i_s}x_{i_1}\cdots x_{i_s}D(x_0\cdots\hat{x}_{i_1}\cdots\hat{x}_{i_s}\cdots x_q)$$

for any set $\{x_0, x_1,\dots, x_q\}$ of $(q+1)$-elements in K. $\mathscr{D}_0^{(q)}(K/k)$ denotes the totality of q-th order k-derivations of K and $\mathscr{D}_0(K/k)$ denotes the union $\bigcup_{q=1}^{\infty}\mathscr{D}_0^{(q)}(K/k)$, which is a K-submodule of $\mathrm{Hom}_k(K, K)$. $\mathscr{D}(K/k)$ denotes the sum (necessarily a direct sum) of K-submodules K and $\mathscr{D}_0(K/k)$ in $\mathrm{Hom}_k(K, K)$, which has a natural structure of k-subalgebra of $\mathrm{Hom}_k(K, K)$. $\mathscr{D}(K/k)$ is called the derivation algebra of K over k. For any $D \in \mathscr{D}_0(K/k)$ and $a \in K$, we set $[D, a] = Da - aD - D(a)$ i.e. $[D, a](x) = D(ax) - aD(x) - D(a)x$. D belongs to $\mathscr{D}_0^{(q)}(K/k)$ if and only if $[D, a]$ belongs to $\mathscr{D}_0^{(q-1)}(K/k)$ for all $a \in K$. (Nakai [1], Ch. 1, Prop. 3)

Simplicity of derivation subalgebras of $\mathscr{D}$(K/k).

Proposition. *Let K be an arbitrary field extension of a field k, and $\mathscr{D}(K/k)$ be its derivation algebra. Then each subring of $\mathscr{D}(K/k)$ containing K is simple. (i.e. (0) and ring itself are the only two-sided ideals.)*

Proof. Let A be a subring of $\mathscr{D}(K/k)$ such that $A \supset K$, and let $\mathfrak{a}$ be any non-zero two-sided ideal of A, and let f be a non-zero element of $\mathfrak{a}$. Then f is uniquely written as follows: $f = a + D$, where $a \in K$ and $D \in \mathscr{D}_0^{(q)}(K/k)$ for some q. If $D = 0$, we are through. If $D \neq 0$, we shall show that $\mathfrak{a}$ contains an element of the form $a' + D'$, where $a' \in K \backslash \{0\}$ and $D' \in \mathscr{D}_0^{(q-1)}(K/k)$. Indeed there exists an element $x \in K$ such that $D(x) \neq 0$, and since we have $D(x) + [D, x] = Dx - xD = (a + D)x - x(a + D) = fx - xf$, it follows that $D(x) + [D, x]$ is in $\mathfrak{a}$. Thus $a' = D(x)$ and $D' = [D, x]$ satisfy desired condition.

Hence, by induction, we conclude that $\mathfrak{a}$ contains a non-zero element of K, because 0-th order derivation is 0 (the null map). Thus $\mathfrak{a}$ must coincide with A itself. q.e.d.

Corollary 1. *Let K be any field extension of a field k, then the derivation algebra $\mathscr{D}(K/k)$ is a simple ring.*

An application.

Corollary 2. *If K is a field extension of a field k such that $\mathscr{D}(K/k) = \mathrm{Hom}_k \cdot (K, K)$, then $[K: k]$ must be finite.*

This is obvious by the above Corollary 1 and the next lemma.

Lemma. (*Jacobson* [4], *Th.* 5, p. 258) *Let V be an infinite dimensional vector space over a field k. For each infinite cardinal α such that $\alpha \leqq \dim. V$, set*

$$\mathfrak{a}_\alpha = \{f \in \mathrm{Hom}_k(V, V) \mid \dim.\ \mathrm{Im}(f) < \alpha\}.$$

Then $\mathfrak{a}_\alpha$ is a proper two-sided ideal of $\mathrm{Hom}_k(V, V)$, and conversely any proper two-sided ideal coincides with one of the $\mathfrak{a}_\alpha$.

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