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Note to Goto's paper "Every noetherian uniformly coherent ring has dimension at most 2"

By

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1. Introduction.

Let A be a commutative ring with unity. A is said to be coherent if every finitely generated ideal of A is finitely presented. A is said to be uniformly coherent if there is a function $\varphi: \boldsymbol{N} \rightarrow \boldsymbol{N}$ such that for each $n \in \boldsymbol{N}$ and any A-homomorphism $f: A^n \rightarrow A$, $\operatorname{Ker} f$ is generated by at most $\varphi(n)$ elements. The notion of uniform coherence was introduced by Soublin [9] and subsequently studied by many authors including Goto [4], Quentel [7] and Sally [8]. Sally [8] proved that a local ring is uniformly coherent if its (Krull) dimension is at most two. Goto [4] sharpened this result and proved that the converse also holds, i.e.,

(*) A local ring is uniformly coherent if and only if its dimension is at most two.

In this note we show that the ideal-adic completion of a noetherian uniformly coherent ring is again uniformly coherent. Using this and some other results of [4] we given an alternative and a short proof of (*).

The notations and terminology are in general that of Nagata [6] and Sally [8] unless stated otherwise.

2. Proof.

Let A be a commutative ring with unity and M an A-module. If $\nu_A(M)$ denotes the least number of elements in a system of generators of M, then for a fixed $n \in \boldsymbol{N}$ we denote by

$$\beta_A(n) = \sup_f \{\nu_A(\operatorname{Ker} f) \mid f \in \operatorname{Hom}_A(A^n, A)\}.$$

It is clear that A is uniformly coherent if and only if $\beta_A(n)$ is finite for all $n \in \boldsymbol{N}$.

We start with the following proposition.

Proposition 1. *Let A be a noetherian ring, $x \in \operatorname{rad}(A)$ and A^* the xA-adic completion of A. Then A is uniformly coherent if and only if A^* is uniformly coherent.*

Proof. If A^* is uniformly coherent then clearly so is A. Conversely, let n be a fixed natural number and $f^*: (A^*)^n \to A^*$ be an A^*-homomorphism. Let $K^* = \mathrm{Ker} f^*$. Let $f^*_\nu = f^* \otimes A^*/x^\nu A^*$: $(A^*/x^\nu A^*)^n \to A^*/x^\nu A^*$ and $K^*_\nu = \mathrm{Ker} f^*_\nu$ for all $\nu \in N$. Then by Artin-Rees Lemma, we have $\nu_{A^*}(K^*) \leqq \nu_{A^*}(K^*_\nu)$ for sufficiently large ν (cf. [2]).

Now we approximate f^* by f: $A^n \to A$, $f^* \equiv f \pmod{x^\nu A^*}$. Let $f_\nu = f \otimes A/x^\nu A$: $(A/x^\nu A)^n \to A/x^\nu A$ and $K_\nu = \mathrm{Ker} f_\nu$. Then $\nu_A(K_\nu) = \nu_{A^*}(K^*_\nu)$.

Define g: $A^n \oplus A \to A$ by $g|_{A^n} = f$ and $g|_A$: $A \xrightarrow{\times x^\nu} A$. Let $M = \mathrm{Ker}\, g$. Then we have a canonical surjection $M \to K_\nu$. Therefore we get $\nu_{A^*}(K^*) \leqq \nu_{A^*}(K^*_\nu) = \nu_A(K_\nu) \leq \nu_A(M) \leq \beta_A(n+1)$. This proves the proposition.

Corollary 2. *Let $(A, \mathfrak{m})$ be a local ring and $\hat{A}$ its completion. Then the following are equivalent*:

a) A is uniformly coherent.

b) $\hat{A}$ is uniformly coherent.

Proof. b) $\Rightarrow$ a) is obvious.
a) $\Rightarrow$ b): Let $\mathfrak{m} = (x_1, x_2, \dots, x_r)$. Let A_1 be the $x_1 A$-adic completion of A. Define inductively A_i as $x_i A_{i-1}$-adic completion of A_{i-1}. Then $\hat{A} = A_r$. Now the result follows.

Before proving the next proposition, we remark that if A is uniformly coherent then for any prime ideal $\mathfrak{p}$ of A, $A_\mathfrak{p}$ is also uniformly coherent.

Proposition 3. *If $(A, \mathfrak{m})$ is a regular local ring, then A is uniformly coherent if and only if* $\dim A$ *is at most two.*

Proof. If $\dim A \leq 2$ then the result follows from Sally [8]. Conversely, assume that $\dim A \geq 3$. In the light of the above remark, we may assume that A is of dimension 3. We show that $\beta_A(4)$ is not finite. We reproduce Goto's proof [4].

Let x, y, z be a regular system of parameters of A and let $n \geq 5$ be an odd integer. Consider the n by n alternating matrix H_n defined as follows:

$$(H_n)_{ij} = \begin{cases} x & \text{if } i \text{ is odd and } j = i+1 \\ y & \text{if } i \text{ is even and } j = i+1 \\ z & \text{if } i+j = n+1 \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leqslant i < j \leqslant n$. Let I_n be the ideal of A generated by all $n-1$ by $n-1$ Pfaffians of the matrix H_n. Let $e = \frac{n-1}{2}$. Then $x^e, y^e, z^e \in I_n$ so that I_n is $\mathfrak{m}$-primary. I_n being an ideal generated by the Pfaffians, we get that A/I_n is a Gorenstein ring and $\nu_A(I_n) = n$ (cf. [1]). Let $J_n = (x^e, y^e, z^e) : I_n$. Then $\nu_{A/(x^e, y^e, z^e)}(J_n/x^e, y^e, z^e)) = 1$ and the dimension of the socle of A/J_n is $n-3$ by [5, Proposition 3.1]. Thus $\nu_A(J_n) = 4$. Let $J_n = (x^e, y^e, z^e, t_n)$ for some $t_n \in \mathfrak{m}$. Then $\nu_A(\mathrm{Ext}^3_A(A/J_n, A))$ equals $n-3$, the dimension of the socle of A/J_n. Thus A/J_n has a minimal free resolution of the following form:

$$0 \longrightarrow A^{n-3} \longrightarrow A^n \longrightarrow A^4 \xrightarrow{\rho} A \longrightarrow A/J_n \longrightarrow 0,$$

where $\rho(e_1)=x^e$, $\rho(e_2)=y^e$, $\rho(e_3)=z^e$ and $\rho(e_4)=t_n$; e_1, e_2, e_3, e_4 being a canonical base of A^4. Since $\nu_A(\mathrm{Ker}\ \rho)=n$, it is easily seen that $\beta_A(4)$ is not finite. Hence A is not uniformly coherent.

Following proposition can be found in Quentel [7].

Proposition 4. *Let f: $R \rightarrow S$ be a ring-homomorhpism making S an R-module of finite presentation. If R is uniformly coherent so is S. Conversely, if* Ker f *is a nil ideal then S uniformly coherent implies R is uniformly coherent.*

Theorem. *Let $(A, \mathfrak{m})$ be a local ring. Then A is uniformly coherent if and only if dimension of A is at most two.*

Proof. If dimension of A is at most two, the result follows from Sally [8]. Assume that A is uniformly coherent. Let $\mathfrak{p}$ be a prime ideal of A such that dim $A/\mathfrak{p}=3$. By Proposition 4 and Corollary 2, we may assume that A is a complete uniformly coherent domain of dimension 3. Again using Proposition 4, it amounts to saying that a complete regular local ring of dimension 3 is uniformly coherent, which is a contradiction.

Using a result of Forster [3], now we deduce the main result of Goto [4].

Corollary. *Let A be a noetherian ring, then the following are equivalent*:
a) *A is uniformly coherent.*
b) dim $A \leq 2$ *and* $\beta_A(n)=\sup_{\mathfrak{m}\in \mathrm{Max}\,A} \beta_{A_\mathfrak{m}}(n)$ *is finite for all* $n \in \mathbf{N}$.

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