Note to Goto's paper
"Every noetherian uniformly coherent ring has dimension at most 2"

By

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1. Introduction.

Let $A$ be a commutative ring with unity. $A$ is said to be coherent if every finitely generated ideal of $A$ is finitely presented. $A$ is said to be uniformly coherent if there is a function $\phi : N \to N$ such that for each $n \in N$ and any $A$-homomorphism $f : A^n \to A$, $\ker f$ is generated by at most $\phi(n)$ elements. The notion of uniform coherence was introduced by Soublin [9] and subsequently studied by many authors including Goto [4], Quentel [7] and Sally [8]. Sally [8] proved that a local ring is uniformly coherent if its (Krull) dimension is at most two. Goto [4] sharpened this result and proved that the converse also holds, i.e.,

(*) A local ring is uniformly coherent if and only if its dimension is at most two.

In this note we show that the ideal-adic completion of a noetherian uniformly coherent ring is again uniformly coherent. Using this and some other results of [4] we give an alternative and a short proof of (*).

The notations and terminology are in general that of Nagata [6] and Sally [8] unless stated otherwise.

2. Proof.

Let $A$ be a commutative ring with unity and $M$ an $A$-module. If $\nu_{A}(M)$ denotes the least number of elements in a system of generators of $M$, then for a fixed $n \in N$ we denote by

$$\beta_{A}(n) = \sup_{f} \{ \nu_{A}(\ker f) | f \in \text{Hom}_{A}(A^{n}, A) \}.$$ 

It is clear that $A$ is uniformly coherent if and only if $\beta_{A}(n)$ is finite for all $n \in N$.

We start with the following proposition.

**Proposition 1.** Let $A$ be a noetherian ring, $x \in \text{rad}(A)$ and $A^{*}$ the $xA$-adic completion of $A$. Then $A$ is uniformly coherent if and only if $A^{*}$ is uniformly coherent.
Proof. If \( A^* \) is uniformly coherent then clearly so is \( A \). Conversely, let \( n \) be a fixed natural number and \( f^*: (A^*)^n \to A^* \) be an \( A^* \)-homomorphism. Let \( K^* := \ker f^* \). Let \( f^* = f_0 \otimes A^* | x^* A^* : (A^* | x^* A^*)^n \to A^* | x^* A^* \) and \( K^* := \ker f^* \) for all \( \nu \in \mathbb{N} \). Then by Artin-Rees Lemma, we have \( \nu_{A^*}(K^*) \leq \nu_{A^*}(K^*) \) for sufficiently large \( \nu \) (cf. [2]).

Now we approximate \( f^* \) by \( f: A^n \to A, f^* \equiv f \mod x^* A^* \). Let \( f^*: f \otimes A^* | x^* A^* : (A^* | x^* A^*)^n \to A^* | x^* A^* \) and \( K = \ker f. \) Then we have a canonical surjection \( M \to K. \) Therefore we get \( \nu_{A^*}(K^*) \leq \nu_{A^*}(K^*) = \nu_{A^*}(K^*) \leq \nu_{A^*}(M) \leq \beta_A(n+1). \) This proves the proposition.

Corollary 2. Let \((A, \mathfrak{m})\) be a local ring and \( \mathring{A} \) its completion. Then the following are equivalent:

a) \( A \) is uniformly coherent.

b) \( \mathring{A} \) is uniformly coherent.

Proof. b) \( \Rightarrow \) a) is obvious.

a) \( \Rightarrow \) b): Let \( \mathfrak{m} = (x_1, x_2, \ldots, x_r). \) Let \( A_i \) be the \( x_i A \)-adic completion of \( A. \) Define inductively \( A_i \) as \( x_i A_{i-1} \)-adic completion of \( A_{i-1}. \) Then \( A = A_r. \) Now the result follows.

Before proving the next proposition, we remark that if \( A \) is uniformly coherent then for any prime ideal \( \mathfrak{p} \) of \( A, A_\mathfrak{p} \) is also uniformly coherent.

Proposition 3. If \((A, \mathfrak{m})\) is a regular local ring, then \( A \) is uniformly coherent if and only if \( \dim A \) is at most two.

Proof. If \( \dim A \leq 2 \) then the result follows from Sally [8]. Conversely, assume that \( \dim A \geq 3. \) In the light of the above remark, we may assume that \( A \) is of dimension 3.

We show that \( \beta_A(4) \) is not finite. We reproduce Goto's proof [4].

Let \( x, y, z \) be a regular system of parameters of \( A \) and let \( n \geq 5 \) be an odd integer. Consider the \( n \) by \( n \) alternating matrix \( H_n \) defined as follows:

\[
(H_n)_{ij} = \begin{cases} x & \text{if } i \text{ is odd and } j = i+1 \\ y & \text{if } i \text{ is even and } j = i+1 \\ z & \text{if } i+1=n+1 \\ 0 & \text{otherwise} \end{cases}
\]

where \( 1 \leq i < j \leq n. \) Let \( I_n \) be the ideal of \( A \) generated by all \( n-1 \) by \( n-1 \) Pfaffians of the matrix \( H_n \). Let \( e = \frac{n-1}{2}. \) Then \( x^e, y^e, z^e \in I_n \) so that \( I_n \) is \( \mathfrak{m} \)-primary. \( I_n \) being an ideal generated by the Pfaffians, we get that \( A/I_n \) is a Gorenstein ring and \( \nu_{A}(I_n) = n \) (cf. [1]). Let \( J_n = (x^e, y^e, z^e) : I_n. \) Then \( \nu_{A}(x^e, y^e, z^e)(J_n/x^e, y^e, z^e)) = 1 \) and the dimension of the socle of \( A/J_n \) is \( n-3 \) by [5, Proposition 3.1]. Thus \( \nu_{A}(J_n) = 4. \) Let \( J_n = (x^e, y^e, z^e, t_n) \) for some \( t_n \in \mathfrak{m}. \) Then \( \nu_{A}(\text{Ext}_A^3(A/J_n, A)) \) equals \( n-3, \) the dimension of the socle of \( A/J_n. \) Thus \( A/J_n \) has a minimal free resolution of the following form:
where $\rho(e_1) = x^t$, $\rho(e_2) = y^t$, $\rho(e_3) = z^t$ and $\rho(e_4) = l_n$; $e_1$, $e_2$, $e_3$, $e_4$ being a canonical base of $A^4$. Since $\nu_A(\text{Ker } \rho) = n$, it is easily seen that $\beta_A(4)$ is not finite. Hence $A$ is not uniformly coherent.

Following proposition can be found in Quentel [7].

**Proposition 4.** Let $f: R \to S$ be a ring-homomorphism making $S$ an $R$-module of finite presentation. If $R$ is uniformly coherent so is $S$. Conversely, if $\text{Ker } f$ is a nil ideal then $S$ uniformly coherent implies $R$ is uniformly coherent.

**Theorem.** Let $(A, \mathfrak{m})$ be a local ring. Then $A$ is uniformly coherent if and only if dimension of $A$ is at most two.

**Proof.** If dimension of $A$ is at most two, the result follows from Sally [8]. Assume that $A$ is uniformly coherent. Let $\mathfrak{p}$ be a prime ideal of $A$ such that $\dim A/\mathfrak{p} = 3$. By Proposition 4 and Corollary 2, we may assume that $A$ is a complete uniformly coherent domain of dimension 3. Again using Proposition 4, it amounts to saying that a complete regular local ring of dimension 3 is uniformly coherent, which is a contradiction.

Using a result of Forster [3], now we deduce the main result of Goto [4].

**Corollary.** Let $A$ be a noetherian ring, then the following are equivalent:
1. $A$ is uniformly coherent.
2. $\dim A \leq 2$ and $\beta_A(n) = \sup_{\mathfrak{m} \in \text{Max } A} \beta_A(n)$ is finite for all $n \in \mathbb{N}$.

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**References**


