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The essential spectrum of two-dimensional Schrödinger operators with perturbed constant magnetic fields

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1. Results.

The properties of the Schrödinger operators with constant magnetic fields are well kown (see, e.g., Avron-Herbst-Simon [1]). If one chooses a suitable gauge taking the z axis parallel to the magnetic field, the operator has the form $H_0=-\left(\frac{\partial}{\partial x}-i\frac{B_0}{2}y\right)^2-\left(\frac{\partial}{\partial y}+i\frac{B_0}{2}x\right)^2-\frac{\partial^2}{\partial z^2}$ where B_0 is the intensity of the magnetic field. The most characteristic feature of H_0 is that the two-dimensional operator $-\left(\frac{\partial}{\partial x}-i\frac{B_0}{2}y\right)^2-\left(\frac{\partial}{\partial y}+i\frac{B_0}{2}x\right)^2$ has a complete set of eigenfunctions (the eigenvalues are B_0 times positive odd numbers and each of them is infinitely degenerate), which corresponds to the fact that classical orbits of charged particles in constant magnetic fields are bounded in the x and y directions. Our purpose of the present paper is to show that the same is true for the perturbed operator, namely, we shall prove the following theorem:

Theorem A. *Let L be the differential operator defined on $C_0^\infty(\boldsymbol{R}^2)$ by*

$$L=-\left(\frac{\partial}{\partial x}-ia\right)^2-\left(\frac{\partial}{\partial y}-ib\right)^2$$

where a and b are the multiplications by real-valued C^∞ functions $a(x, y)$ and $b(x, y)$, respectively. Suppose that $B(x, y)\equiv\frac{\partial a}{\partial y}(x, y)-\frac{\partial b}{\partial x}(x,y)$ tends to a positive constant B_0 as $\sqrt{x^2+y^2}$ tends to infinity. Let H be a self-adjoint extension of L in $\mathcal{H}=L^2(\boldsymbol{R}^2)$. Then $\sigma_{ess}(H)=\{(2k+1)B_0 \mid k \text{ is an integer } \geqq 0\}$.

Remark 1. $\sigma_{ess}(A)$ (=the essential spectrum of a self-adjoint operator A) is the set of all $\lambda\in\boldsymbol{R}$ such that the dimension of $\mathcal{R}(E((\lambda-\varepsilon, \lambda+\varepsilon)))$ ($\mathcal{R}(\cdot)$ denotes the range of an operator) is infinite for all $\varepsilon>0$ where E is the spectral measure associated with A. Thus, by the discreteness of $\sigma_{ess}(H)$ according to Theorem A, H has a complete set of eigenfunctions with eigenvalues which, with their multiplicities taken account of, have the accumulation points equal to $\sigma_{ess}(H)$.

Remark 2. L is essentially self-adjoint (see, e.g., Leinfelder-Simader [4]), and thus H is the unique self-adjoint extension of L.

The magnetic Schrödinger operators perturbed by scalar potentials V in $\boldsymbol{R}^3$, i.e., H_0+V, have been studied fairly well, e.g., by [1], but there seems to be only few researches on perturbations of magnetic fields. One difficulty in manipulating such perturbations lies in the fact that the Hamiltonian H depends explicitly not on the field $\vec{B}=\mathrm{rot}\ \vec{a}$, but on the vector potential $\vec{a}=(a(x,y),\ b(x,y))$ which allows the so-called gauge transformations which do not change the field. We avoid this ambiguity of the choice of $\vec{a}$ by noting that $B/B_0=i[\Pi_1,\ \Pi_2]\equiv i(\Pi_1\Pi_2-\Pi_2\Pi_1)$ where $\Pi_1=\frac{1}{\sqrt{B_0}}\left(i\frac{\partial}{\partial x}+a\right)$ and $\Pi_2=\frac{1}{\sqrt{B_0}}\left(i\frac{\partial}{\partial y}+b\right)$. This commutation relation can be regarded as a perturbation of C. C. R. (the canonical commutation relation, $i(PQ-QP)=1$) because $B/B_0=1+\text{small}$. C.C.R. is an old topic in quantum mechanics and has been studied by many authers (see e.g. Dixmier [2], Putnam [5]). Our first step is to prove a theorem (Theorem B below) concerned with a perturbation of C.C.R. which we think interesting in itself. Theorem A is obtained by applying the following theorem:

Theorem B. *Let P and Q be symmetric operators in a Hilbert space $\mathcal{H}$ defined on Ω, dense in $\mathcal{H}$, such that $P\Omega\subset\Omega$, $Q\Omega\subset\Omega$. Suppose that*

(*a*) *P^2+Q^2 is essentially self-adjoint (let H denote the self-adjoint extension of P^2+Q^2),*

(*b*) *$i(PQ-QP)u=(1+K)u$ for $u\in\Omega$ where K is a relatively compact operator with respect to H (i.e., $D(K)$ (=the domain of K) contains $D(H)$ and $K(H+i)^{-1}$ is compact).*

Then either of the following (*i*) *or* (*ii*) *holds*:

(*i*) $\sigma_{ess}(H)=\phi$,

(*ii*) $\sigma_{ess}(H)=\{2k+1\mid k$ *is an integer* $\geqq 0\}$.

2. A Lemma and Proofs.

Lemma. *Let A be a densely defined closed operator on a Hilbert space $\mathcal{H}$. Suppose that*

(1) *$AA^*=A^*A+1+K$, where K is a relatively compact operator with respect to A^*A,*

that is, $D(AA^*)=D(A^*A)\equiv D$, $D\subset D(K)$, $K(A^*A+1)^{-1}$ is compact and $AA^*=A^*A+1+K$ holds on D. Then either of the following (*i*) *or* (*ii*) *holds*:

(*i*) $\sigma_{ess}(A^*A)=\phi$,

(*ii*) $\sigma_{ess}(A^*A)=\{k\mid k$ *is an integer* $\geqq 0\}$.

Moreover, in the case (*i*), *$N(A)$=the null space of $A\equiv\{u\in D(A)\mid Au=0\}$ is finite dimensional, and, in the case* (*ii*), *$N(A)$ is infinite dimensional.*

Proof. Let $S=\sigma_{ess}(A^*A)$. Then since $A^*A\geqq 0$,

(2) $\quad S\subset[0,\ \infty)$.

On the other hand, we have

(3) $\quad S\backslash\{0\}(=\{a\mid a\in S,\ a\neq 0\})=\sigma_{ess}(AA^*)\backslash\{0\}$

because A^*A restricted to $N(A^*A)^{\perp}$ ($\perp$ denotes the orthogonal complement) is unitarily

equivalent to AA^* restricted to $N(AA^*)^{\perp}$ by using the polar decomposition of A (see Kato [3], p.334).
Moreover,
(4) $\sigma_{ess}(AA^*)=\sigma_{ess}(A^*A+1)=S+1$,
by (1) (Reed-Simon [6], p. 113) where $S+1=\{a+1|a\in S\}$. Hence we have, from (3) and (4),
(5) $S\backslash\{0\}=S+1$
since $(S+1)\backslash\{0\}=S+1$ by (2). If $S\neq\phi$, it is not difficult to verify $S=\{k|k$ is an integer $\geqq 0\}$ using (2) and (5). Thus we have proved that either (i) or (ii) holds.

Next, let $\sigma_{ess}(A^*A)=\phi$ (the case (i) hold). Since $N(A)=N(A^*A)$, $N(A)$ is finite dimensional by the definition of essential spectrum (see Remark 1 after Theorem A).

Finally, let $\sigma_{ess}(A^*A)=\{k|k$ is an integer $\geqq 0\}$ (the case (ii) hold). Suppose that $N(A)$ is finite dimensional. Let E_1 and E_2 be the spectral measure associated with A^*A and AA^*, respectively. Then, as stated above, the unitary equivalence between $E_1(B)$ and $E_2(B)$ holds if B is a Borel set contained in $(0, \infty)$. Since $\sigma_{ess}(AA^*)=\{k|k$ is an integer$>0\}$ by (4), $\mathfrak{R}(E_2((0, c)))$ is finite dimensional if $0<c<1$. Thus $\mathfrak{R}(E_1((0, c)))$ is also finite dimensional. Since $E_1([0, c))=E_1(\{0\})+E_1((0, c))$ where $E_1(\{0\})$ is the orthogonal projection onto $N(A)=N(A^*A)$, $\mathfrak{R}(E_1([0, c)))$ is finite dimensional if $N(A)$ is so. Hence, noting that $E_1((-\infty, c))=E_1([0, c))$ since $A^*A\geqq 0$, we have $\sigma_{ess}(A^*A)\not\ni 0$. This contradicts the supposition that $\sigma_{ess}(A^*A)=\{k|k$ is an integer $\geqq 0\}$. Hence $N(A)$ must be infinite dimensional in the case (ii). We have thus concluded the proof of the lemma.

Proof of Theorem B. Let X and Y be operators defined on Ω by $X=\frac{1}{\sqrt{2}}(P-iQ)$, $Y=\frac{1}{\sqrt{2}}(P+iQ)$. Since P and Q are symmetric, we have $(Xu, v)=(u, Yv)$ $(u, v\in\Omega)$. Hence, $X^*\supset Y$ and $X^{**}\subset Y^*$. Therefore, X has the closure A $(=X^{**})$, whose adjoint extends Y, and
(6) $YX\subset A^*A,\ XY\subset AA^*$.
On the other hand, by the assumption (b),

$$(7)\quad \begin{cases} YX=\frac{1}{2}(P+iQ)(P-iQ)=\frac{1}{2}(P^2+Q^2)-i\frac{1}{2}(PQ-QP) \\ \quad =\frac{1}{2}(P^2+Q^2)-\frac{1}{2}(1+K), \\ XY=\frac{1}{2}(P^2+Q^2)+\frac{1}{2}(1+K). \end{cases}$$

Moreover, note that, when $D(S)\supset D(T)$ and $\rho(T)$ (the resolvent set of T) $\neq\phi$ for operators S, T in some Banach space, $S(T+z)^{-1}$ is compact for some $z\in\rho(T)$ if and only if $\{Su_n\}$ contains a convergent subsequence for any sequence $u_n\in D(T)$ with both $\{u_n\}$ and $\{Tu_n\}$ bounded. That is, our definition of relative compactness is equivalent to that in [3], p.194, except that the latter can also be applied to non-closed T. Therefore, K is H-compact and hence P^2+Q^2-compact in the sence of [3]. Hence we obtain
(8) $\overline{P^2+Q^2\pm K}=H\pm K$
(bar denotes the closure of an operator) because $H\pm K$ is closed as well as H, $P^2+Q^2\pm K$

is closable as well as P^2+Q^2, the closures of P^2+Q^2 and $P^2+Q^2\pm K$ have the same domain ([3], p. 194, Theorem 1.11), and $P^2+Q^2\pm K\subset H\pm K$. Moreover, we have by the same theorem in [3]

(9) K is $(H\pm K)$-compact,

and it is not difficult to see $\overline{K|_\Omega}\supset K|_{D(H)}$ ($K|_\Omega$ denotes K restricted to Ω, etc.), which implies that $K|_{D(H)}$ is symmetric since $K|_\Omega= i(PQ-QP)-1$ is symmetric as P, Q are so. Hence we have ([6], p.113)

(10) $H\pm K$ is self-adjoint.

Therefore, we have from (6), (7) and (8)

$$(11)\quad \begin{cases} \frac{1}{2}H-\frac{1}{2}(1+K)=\overline{YX}\subset A^*A, \\ \frac{1}{2}H+\frac{1}{2}(1+K)=\overline{XY}\subset AA^*, \end{cases}$$

since A^*A and AA^* are self-adjoint ([3], p. 275) and thus closed. Moreover we have from (10), (11) and the self-adjointness of A^*A and AA^*

$$(12)\quad \begin{cases} \frac{1}{2}H-\frac{1}{2}(1+K)=A^*A, \\ \frac{1}{2}H+\frac{1}{2}(1+K)=AA^*. \end{cases}$$

Hence we have shown that $D(A^*A)=D(AA^*)=D(H)$ and $AA^*=A^*A+1+K$ where K is relatively compact with respect to $A^*A=\frac{1}{2}(H+1+K)$ by (9). Thus the closed operator A suffices the assumption (1) of Lemma. Therefore, we have by Lemma either (i) $\sigma_{ess}(A^*A)=\phi$ or (ii) $\sigma_{ess}(A^*A)=\{k|k \text{ is an integer}\geqq 0\}$. Finally, by noting that $H=2A^*A+1+K$ with K relatively compact with respect to A^*A, we obtain $\sigma_{ess}(H)=\sigma_{ess}(2A^*A+1)$ ([6]. p. 113) and thus the conclusion of the theorem.

Proof of Theorem A. Let P and Q be the operators defined on $C_0^\infty(\boldsymbol{R}^2)$ by

$$Pu=\frac{1}{\sqrt{B_0}}\left(i\frac{\partial u}{\partial x}+au\right),\quad Qu=\frac{1}{\sqrt{B_0}}\left(i\frac{\partial u}{\partial y}+bu\right)$$

and let K be the operator of multiplication by the function $K(x,y)\equiv B(x,y)/B_0-1$. Then $C_0^\infty(\boldsymbol{R}^2)$ is invariant under P and Q, P and Q are symmetric, and $B_0(P^2+Q^2)=L$. We have by direct computation

$$\begin{aligned}(PQ-QP)u&=\frac{-1}{B_0}\left(\frac{\partial}{\partial x}-ia\right)\left(\frac{\partial}{\partial y}-ib\right)u+\frac{1}{B_0}\left(\frac{\partial}{\partial y}-ib\right)\left(\frac{\partial}{\partial x}-ia\right)u\\ &=\frac{-i}{B_0}\left(\frac{\partial a}{\partial y}-\frac{\partial b}{\partial x}\right)u\\ &=-i(1+K)u.\end{aligned}$$

Moreover, it is known that $P^2+Q^2=\frac{1}{B_0}L$ is essentially self-adjoint as remarked after Theorem A, and that K is a relatively compact operator with respect to H since $K(x,y)\to 0$ as $\sqrt{x^2+y^2}\to\infty$ (see [1], Theorem 2.6 and [6], p. 117). Thus the assumption of Theorem B is satisfied for P and Q if we put $\Omega=C_0^\infty(\boldsymbol{R}^2)$. According to Theorem

B, it suffices to show $\sigma_{ess}(H)\neq\phi$ for obtaining the assertion of Theorem A.

Let $\{(c_j, d_j)\}_{j=1,2,\dots}$ be a sequence of points in $\boldsymbol{R}^2$ and M, r be positive constants such that Q_j, the disks in $\boldsymbol{R}^2$ with radius r about (c_j, d_j), do not intersect each other and

(13) $$|B(x, y)|=\left|\frac{\partial a}{\partial y}(x, y)-\frac{\partial b}{\partial x}(x, y)\right|\leqq M$$

for $(x, y)\in Q_j$ $(j=1, 2, \dots)$. (There exist such $\{(c_j, d_j)\}$ and M, r since $B(x, y)$ is bounded in the whole plane by the assumption of the theorem.) Then we can construct functions $u_j\in C_0^\infty(\boldsymbol{R}^2)$ such that supp $u_j\subset Q_j$,

(14) $$\|u_j\|=\left(\int|u_j(x, y)|^2\,dxdy\right)^{1/2}=1,$$

(15) $$(Hu_j, u_j)=\left\|\left(\frac{\partial}{\partial x}-ia\right)u_j\right\|^2+\left\|\left(\frac{\partial}{\partial y}-ib\right)u_j\right\|^2\leqq C$$

where C is a constant independent of j. This can be done as follows:

Let Φ_j be defined by

$$\Phi_j(x, y)=\int_{(c_j,d_j)}^{(x,y)}(a(x, y)\,dx+b(x, y)\,dy)$$

where the integral is taken along the straight line from (c_j, d_j) to (x, y).
Then Φ_j is a real-valued C^∞ function and we have

$$\Phi_j(x, y)=\int_0^1(x'\tilde{a}(tx', ty')+y'\tilde{b}(tx', ty'))\,dt$$

where $(x', y')\equiv(x-c_j, y-d_j)$ and $(\tilde{a}(x', y'), \tilde{b}(x', y'))\equiv(a(x, y), b(x, y))$. Hence we obtain, by using $\left(\frac{\partial\tilde{a}}{\partial y}-\frac{\partial\tilde{b}}{\partial x}\right)(x', y')=B(c_j+x', d_j+y')\equiv\tilde{B}(x', y')$ and integrating by parts,

(16) $$\begin{aligned}\frac{\partial\Phi_j}{\partial x}(x, y)&=\int_0^1\tilde{a}(tx', ty')\,dt+x'\int_0^1 t\frac{\partial\tilde{a}}{\partial x}(tx', ty')\,dt\\&\qquad+y'\int_0^1 t\frac{\partial\tilde{b}}{\partial x}(tx', ty')\,dt\\&=\int_0^1\tilde{a}(tx', ty')dt+\int_0^1 t\frac{d}{dt}\{\tilde{a}(tx', ty')\}\,dt\\&\qquad-y'\int_0^1 t\tilde{B}(tx', ty')\,dt\\&=\tilde{a}(x', y')-y\int_0^1 t\tilde{B}(tx', ty')\,dt\\&=a(x, y)-(y-d_j)\int_0^1 tB(c_j+t(x-c_j), d_j+t(y-d_j))\,dt\end{aligned}$$

Therefore, by (13) and (16),

(17) $$\left|\frac{\partial\Phi_j}{\partial x}(x, y)-a(x, y)\right|\leqq M|y-d_j|\leqq Mr\quad\text{for all }(x, y)\in Q_j,$$

and similarly we have

(18) $$\left|\frac{\partial\Phi_j}{\partial y}(x, y)-b(x, y)\right|\leqq Mr\quad\text{for all }(x, y)\in Q_j.$$

Let $u(x, y)\in C_0^\infty(\boldsymbol{R}^2)$ such that $u(x, y)=0$ if $\sqrt{x^2+y^2}\geqq\frac{r}{2}$ and $\|u\|=1$. Define $u_j(x, y)=\exp\{i\Phi_j(x, y)\}\,u(x-c_j, y-d_j)$. Then we have (14),

$$\left\|\left(\frac{\partial}{\partial x}-ia\right)u_j\right\|=\left\|\left[\left(\frac{\partial}{\partial x}-ia\right)\exp\{i\Phi_j\}\right]u(x-c_j,y-d_j)+\right.$$
$$\left.+\exp\{i\Phi_j\}\frac{\partial u}{\partial x}(x-c_j,\ y-d_j)\right\|$$
$$\leqq\left\|i\left(\frac{\partial\Phi_j}{\partial x}-a\right)u(x-c_j,\ y-d_j)\right\|+\left\|\frac{\partial u}{\partial x}\right\|,$$

and a similar estimate for $\left\|\left(\frac{\partial}{\partial y}-ib\right)u_j\right\|$, which together with (17) and (18) show that (15) holds.

Thus we have shown that there exists an orthonormal sequence of functions $\{u_j\}_{j=1,2,\ldots}$ in $D(H)$ such that (Hu_j, u_j) is bounded. Suppose that $\sigma_{ess}(H)=\phi$. Then, by the definition of essential spectrum (see Remark 1 after Theorem A), the range of $E([-\lambda, \lambda])$ is finite dimensional and hence $E([-\lambda, \lambda])$ is compact for all $\lambda>0$ where E is the spectral measure associated with H. Since $H\geqq 0$, we have also

(19) $E_\lambda\equiv E((-\infty, \lambda])=E([-\lambda, \lambda])$ is compact.

Moreover, for any positive number A, we have

$$(20)\qquad \begin{aligned}(Hu_j, u_j)&=\int_0^\infty\lambda d(E_\lambda u_j, u_j)\\ &\geqq\int_A^\infty\lambda d(E_\lambda u_j, u_j)\\ &\geqq A(E((A, \infty))u_j, u_j)\\ &=A(\|u_j\|^2-(E_A u_j, u_j)).\end{aligned}$$

Since $\{u_j\}$ is orthonormal, $E_A u_j\to 0$ strongly as $j\to\infty$ by (19). Hence we obtain, with the use of (20) and $\|u_j\|=1$,

$$\liminf_{j\to\infty}(Hu_j, u_j)\geqq A,$$

which contradicts the boundedness of $\{(Hu_j, u_j)\}$ since A is arbitrary. Thus $\sigma_{ess}(H)\neq\phi$. This completes the proof of the theorem.

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