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On the homogeneous space E_8/E_7

By

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In [I-Y], Imai and Yokota showed that the group

$$E_8=\{\alpha\in \mathrm{Iso}_C(\mathfrak{e}_8^C,\ \mathfrak{e}_8^C)\,|\,\alpha[R_1, R_2]=[\alpha R_1, \alpha R_2],\ <\alpha R_1, \alpha R_2>=<R_1, R_2>\}$$

is a simply connected compact simple Lie group of type E_8 and it contains a subgroup

$$E_7=\{\beta\in E_8\,|\,\beta\underline{1}=\underline{1}\}$$

which is a simply connected compact simple Lie group of type E_7. In the present paper, we consider the homogeneous space E_8/E_7. The result is

$$E_8/E_7=\mathfrak{W}_1=\{R\in\mathfrak{e}_8^C\,|\,R\times R=0,\ <R, R>=4\}.$$

This paper is a continuation of [I-Y] and we use the same notations as [I-Y]. So the numbering of sections and Theorems of this paper starts from 7 and 29 respectively. The authors wish to thank Prof. Tetsuo Ishihara for his advices.

7. The manifold $\mathfrak{W}^C$.

For $R\in\mathfrak{e}_8^C$, Freudenthal defined in [F] a linear transformation $R\times R$ of $\mathfrak{e}_8^C$ by

$$(R\times R)R_1=(\mathrm{ad}R)^2R_1+\frac{1}{30}B(R, R_1)R, \qquad R_1\in\mathfrak{e}_8^C$$

(where B is the Killing form of the Lie algebra $\mathfrak{e}_8^C$) and considered a subspace $\mathfrak{W}^C$ of $\mathfrak{e}_8^C$:

$$\mathfrak{W}^C=\{R\in\mathfrak{e}_8^C\,|\,R\times R=0,\ R\neq 0\}.$$

By the use of [I-Y] Theorem 28 (and Proposition 27, E_7 : (2)), we have immediately the following

Proposition 29. *For $R=(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8^C$, $R\neq 0$, R belongs to $\mathfrak{W}^C$ if and only if R satisfies*

(1) $2s\Phi-P\times P=0$ (2) $2t\Phi+Q\times Q=0$ (3) $2r\Phi+P\times Q=0$

(4) $\Phi P-3rP-3sQ=0$ (5) $\Phi Q+3rQ-3tP=0$ (6) $\{P, Q\}-16(st+r^2)=0$

(7) $2(\Phi P\times Q_1+2P\times\Phi Q_1-rP\times Q_1-sQ\times Q_1)-\{P, Q_1\}\Phi=0$

(8) $2(\Phi Q\times P_1+2Q\times\Phi P_1+rQ\times P_1-tP\times P_1)-\{Q, P_1\}\Phi=0$

(9) $8((P\times Q_1)Q-stQ_1-r^2Q_1-\Phi^2Q_1+2r\Phi Q_1)+5\{P, Q_1\}Q-2\{Q, Q_1\}P=0$

(10) $8((Q\times P_1)P+stP_1+r^2P_1+\Phi^2P_1+2r\Phi P_1)+5\{Q, P_1\}P-2\{P, P_1\}Q=0$

(11) $18((\mathrm{ad}\Phi)^2\Phi_1+Q\times\Phi_1P-P\times\Phi_1Q)+B_7(\Phi,\ \Phi_1)\Phi=0$

(12) $18(\Phi_1\Phi P-2\Phi\Phi_1P-r\Phi_1P-s\Phi_1Q)+B_7(\Phi,\ \Phi_1)P=0$

(13) $18(\Phi_1\Phi Q-2\Phi\Phi_1Q+r\Phi_1Q-t\Phi_1P)+B_7(\Phi,\ \Phi_1)Q=0$

(*where* B_7 *is the Killing form of the Lie algebra* $\mathfrak{e}_7^C$) *for any* $\Phi_1\in\mathfrak{e}_7^C$, P_1, $Q_1\in\mathfrak{P}^C$.

Theorem 30. *The group* E_8^C *acts transitively on* $\mathfrak{W}^C$ (*which is connected*) *and the isotropy subgroup* $(E_8^C)_{\underline{1}}$ *of* E_8^C *at* $\underline{1}\in\mathfrak{W}^C$ *is* $\exp(\mathfrak{P}^C)\exp(\boldsymbol{C})E_7^C$ (*where* $\exp(\mathfrak{P}^C)\exp(\boldsymbol{C})=\{\exp(\Theta(0,\ 0,\ Q,\ 0,\ 0,\ t))\,|\,P\in\mathfrak{P}^C,\ t\in\boldsymbol{C}\}$, $E_7^C=\{\beta\in E_8^C\,|\,\beta\bar{1}=\bar{1},\ \beta 1=1,\ \beta\underline{1}=\underline{1}\}$). *Therefore we have the following homeomorphism*:

$$E_8^C/(\exp(\mathfrak{P}^C)\exp(\boldsymbol{C})E_7^C)\simeq\mathfrak{W}^C.$$

In particular, $\mathfrak{W}^C$ *is a* 56 *dimensional connected complex manifold.*

Proof. Obviously the group E_8^C acts on $\mathfrak{W}^C$. Since $\underline{1}=(0,\ 0,\ 0,\ 0,\ 0,\ 1)\in\mathfrak{W}^C$, in order to prove the transitivity of E_8^C, it suffices to show that any element $R\in\mathfrak{W}^C$ can be transformed to $\underline{1}$ by a certain element $\alpha\in E_8^C$.

Case (1) $R=(\Phi,\ P,\ Q,\ r,\ s,\ t)$, $t\neq 0$. In this case, from (2), (5), (6) of Proposition 29, we have

$$\Phi=-\frac{1}{2t}Q\times Q,\quad P=\frac{r}{t}Q-\frac{1}{6t^2}(Q\times Q)Q,\quad s=-\frac{r^2}{t}+\frac{1}{96t^3}\{Q,\ (Q\times Q)Q\}.$$

Now, for $\Theta=\Theta(0,\ P_1,\ 0,\ r_1,\ s_1,\ 0)\in\mathrm{ad}\,\mathfrak{e}_8^C$, we shall calculate $(\exp\Theta)\underline{1}$.

$$\Theta\underline{1}=\begin{pmatrix} 0 & 0 & P_1 & 0 & 0 & 0\\ -P_1 & r_1 & s_1 & -P_1 & 0 & 0\\ 0 & 0 & -r_1 & 0 & 0 & -P_1\\ 0 & 0 & -\frac{1}{8}P_1 & 0 & 0 & s_1\\ 0 & \frac{1}{4}P_1 & 0 & -2s_1 & 2r_1 & 0\\ 0 & 0 & 0 & 0 & 0 & -2r_1 \end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\-P_1,\\s_1\\0\\-2r_1\end{pmatrix}$$

$$\Theta^2\underline{1}=\begin{pmatrix}-P_1\times P_1\\-2s_1P_1\\3r_1P_1\\-2r_1s_1\\-2s_1^2\\4r_1^2\end{pmatrix},\qquad \Theta^3\underline{1}=\begin{pmatrix}3r_1P_1\times P_1\\3r_1s_1P_1+(P_1\times P_1)P_1\\-7r_1^2P_1\\4r_1^2s_1\\0\\-8r_1^3\end{pmatrix},\ \cdots,$$

in general $(n\geqq 4)$

$$\Theta^n\underline{1}=\begin{pmatrix}((-2)^{n-1}+(-1)^n)r_1^{n-2}P_1\times P_1\\ \left((-2)^{n-1}-\frac{1+(-1)^{n-1}}{2}\right)r_1^{n-2}s_1P_1+\left(\frac{1-(-2)^n}{6}+\frac{(-1)^n}{2}\right)r_1^{n-3}(P_1\times P_1)P_1\\ ((-2)^n+(-1)^{n+1})r_1^{n-1}P_1\\ (-2)^{n-1}r_1^{n-1}s_1\end{pmatrix}.$$

$$\left.\begin{matrix} -((-2)^{n-2}+2^{n-2})\, r_1^{n-2}s_1^2+\dfrac{2^{n-2}+(-2)^{n-2}-(-1)^n-1}{24}r_1^{n-4}\{P_1,\ (P_1\times P_1)P_1\} \\ (-2)^n r_1^n \end{matrix}\right)$$

Hence, by simple calculations, we have

$$\exp(\Theta(0,\ P_1,\ 0,\ r_1,\ s_1,\ 0))\underline{1}=(\exp\Theta)\underline{1}=\sum_{n=0}^{\infty}\frac{1}{n!}\ \Theta^n\underline{1}$$

$$=\begin{pmatrix} -\dfrac{1}{2r_1^2}(e^{-2r_1}-2e^{-r_1}+1)P_1\times P_1 \\ \dfrac{s_1}{2r_1^2}(-e^{-2r_1}-e^{r_1}+e^{-r_1}+1)P_1+\dfrac{1}{6r_1^3}(-e^{-2r_1}+e^{r_1}+3e^{-r_1}-3)(P_1\times P_1)P_1 \\ \dfrac{1}{r_1}(e^{-2r_1}-e^{-r_1})P_1 \\ \dfrac{s_1}{2r_1}(1-e^{-2r_1}) \\ -\dfrac{s_1^2}{4r_1^2}(e^{-2r_1}+e^{2r_1}-2)+\dfrac{1}{96r_1^4}(e^{2r_1}+e^{-2r_1}-4e^{r_1}-4e^{-r_1}+6)\{P_1,(P_1\times P_1)P_1\} \\ e^{-2r_1} \end{pmatrix}$$

(if $r_1=0$, $\dfrac{f(r_1)}{r_1^k}$ means $\lim\limits_{r_1\to 0}\dfrac{f(r_1)}{r_1^k}$). Find out $P_1\in\mathfrak{P}^C$, r_1, $s_1\in\boldsymbol{C}$ satisfying

$$\frac{1}{r_1}(e^{-2r_1}-e^{-r_1})P_1=Q,\qquad \frac{s_1}{2r_1}(1-e^{-2r_1})=r,\qquad e^{-2r_1}=t.$$

Then we have

$$(\exp\Theta)\underline{1}=\begin{pmatrix} -\dfrac{1}{2t}Q\times Q \\ \dfrac{r}{t}Q-\dfrac{1}{6t^2}(Q\times Q)\,Q \\ Q \\ r \\ -\dfrac{r^2}{t}+\dfrac{1}{96t^3}\{Q,\ (Q\times Q)\ Q\} \\ t \end{pmatrix}=\begin{pmatrix}\Phi\\ P\\ Q\\ r\\ s\\ t\end{pmatrix}=R.$$

Thus R is transformed to $\underline{1}$ by $\exp(-\Theta)\in E_8^C$.

Case (2) $R=(\Phi,\ P,\ Q,\ r,\ s,\ t)$, $s\neq 0$. Similarly as (1), we see that $R=(\exp\Theta)\bar{1}$ for some $\Theta=\Theta(0,\ 0,\ Q_1,\ r_1,\ 0,\ t_1)\in\mathrm{ad}\mathfrak{e}_8^C$, where $\bar{1}=(0,\ 0,\ 0,\ 0,\ 1,\ 0)$. On the other hand, $\bar{1}$ can be transformed to $-\underline{1}$ by

$$\omega=\begin{pmatrix} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&0&-1\\ 0&0&0&0&-1&0 \end{pmatrix}=\exp\left(\Theta\left(0,\ 0,\ 0,\ 0,\ \frac{\pi}{2},\ -\frac{\pi}{2}\right)\right)\in E_8^C.$$

Thus R is transformed to $-\underline{1}$ by $\omega\exp(-\Theta)\in E_8^C$. So this case can be reduced to the case (1).

Case (3) $R=(\Phi, P, Q, r, 0, 0)$, $r\neq 0$. In this case, from (2), (5), (6) of Proposition 29, we have

$$Q\times Q=0, \qquad \Phi Q=-3rQ, \qquad \{P, Q\}=16r^2.$$

Now, for $\Theta=\Theta(0, Q, 0, 0, 0, 0)\in \mathrm{ad}\mathfrak{e}_8^C$, $\Theta R=(Q\times Q, -\Phi Q-rQ, 0, 0, \frac{1}{4}\{Q, P\}, 0)=(0, 2rQ, 0, 0, -4r^2, 0)$, $\Theta^2 R=0$. Hence we have

$$(\exp\Theta)R=(\Phi, P+2rQ, Q, r, -4r^2, 0), \qquad -4r^2\neq 0.$$

So we can reduce to the case (2).

Case (4) $R=(\Phi, P, Q, 0, 0, 0)$, $Q\neq 0$. For $\Theta=\Theta(0, P_1, 0, 0, 0, 0)\in \mathrm{ad}\mathfrak{e}_8^C$, we have

$$(\exp\Theta)R=\Big(*, *, *, -\frac{1}{8}\{P_1, Q\}, *, 0\Big).$$

Choose $P_1\in\mathfrak{P}^C$ such that $\{P_1, Q\}\neq 0$. Then we can reduce to the case (3).

Case (5) $R=(\Phi, P, Q, 0, 0, 0)$, $P\neq 0$. This is similar to the case (4).

Case (6) $R=(\Phi, 0, 0, 0, 0, 0)$, $\Phi\neq 0$. In this case, from (10) of Proposition 29, we have $\Phi^2=0$. Now, for $\Theta=\Theta(\Phi, P_1, 0, 0, 0, 0)\in \mathrm{ad}\mathfrak{e}_8^C$, $\Theta R=(0, -\Phi P_1, 0, 0, 0, 0)$, $\Theta^3 R=(0, -\Phi^2 P_1, 0, 0, -\frac{1}{4}\{P_1, \Phi P_1\}, 0)=(0, 0, 0, 0, \frac{1}{4}\{\Phi P_1, P_1\}, 0)$, $\Theta^3 R=0$. Hence we have

$$(\exp\Theta)R=\Big(\Phi, -\Phi P_1, 0, 0, \frac{1}{8}\{\Phi P_1, P_1\}, 0\Big).$$

So, if we choose $P_1\in\mathfrak{P}^C$ such that $\Phi P_1\neq 0$, then we can reduce to the case (5). Thus the transitivity of E_8^C on $\mathfrak{W}^C$ is proved, so we see also the connectedness of $\mathfrak{W}^C$. Next we shall determine the isotropy subgroup

$$(E_8^C)_{\underline{1}}=\{\alpha\in E_8^C \mid \alpha\underline{1}=\underline{1}\}.$$

Since $\mathfrak{P}^C\oplus\underline{C}=\{\underline{Q}+\underline{t}=(0, 0, Q, 0, 0, t) \mid Q\in\mathfrak{P}^C, t\in C\}$ is a subalgebra of $\mathfrak{e}_8^C$ and $[\underline{Q}, \underline{t}]=0$, $\exp(\underline{Q})=\exp(\Theta(0, 0, Q, 0, 0, 0))$, $\exp(\underline{t})=\exp(\Theta(0, 0, 0, 0, 0, 0, t))$ commute with each other and $\exp(\underline{\mathfrak{p}}^C)\exp(\underline{C})=\exp(\mathrm{ad}(\underline{\mathfrak{P}}^C\oplus\underline{C}))$ is a connected subgroup of E_8^C. Now, let $\alpha\in(E_8^C)_{\underline{1}}$ and put

$$\alpha 1=(\Phi, P, Q, r, s, t). \qquad \alpha\bar{1}=(\Phi_1, P_1, Q_1, r_1, s_1, t_1)$$

where $1=(0, 0, 0, 1, 0, 0)$. Then, from the relations $[1, \underline{1}]=-2\underline{1}$, $[\bar{1}, \underline{1}]=1$, $[1, \bar{1}]=2\bar{1}$, that is, $[\alpha 1, \underline{1}]=-2\underline{1}$, $[\alpha\bar{1}, \underline{1}]=\alpha 1$, $[\alpha 1, \alpha\bar{1}]=2\alpha\bar{1}$, we have

$$P=0, \qquad s=0, \qquad r=1,$$

$$\Phi=0, \qquad P_1=-Q, \qquad s_1=1, \qquad r_1=-\frac{t}{2},$$

$$\Phi_1=\frac{1}{2}Q\times Q, \qquad Q_1=-\frac{t}{2}Q-\frac{1}{3}\Phi_1 Q, \qquad t_1=-\frac{t^2}{4}-\frac{1}{16}\{Q, Q_1\}$$

respectively. So α has the form

$$\alpha=\begin{pmatrix} * & * & * & 0 & \frac{1}{2}Q\times Q & 0 \\ * & * & * & 0 & -Q & 0 \\ * & * & * & Q & -\frac{t}{2}Q-\frac{1}{6}(Q\times Q)\,Q & 0 \\ * & * & * & 1 & -\frac{t}{2} & 0 \\ * & * & * & 0 & 1 & 0 \\ * & * & * & t & -\frac{t^2}{4}+\frac{1}{96}\{Q,\,(Q\times Q)\,Q\} & 1 \end{pmatrix}.$$

On the other hand, we have

$$\exp\left(\underset{-}{\frac{t}{2}}\right)\exp(\underset{-}{Q})\bar{1}=\begin{pmatrix} \frac{1}{2}Q\times Q \\ -Q \\ -\frac{t}{2}Q-\frac{1}{6}(Q\times Q)\,Q \\ -\frac{t}{2} \\ 1 \\ -\frac{t^2}{4}+\frac{1}{96}\{Q,\,(Q\times Q)\,Q\} \end{pmatrix}=\alpha\bar{1}$$

and also we have

$$\exp\left(\underset{-}{\frac{t}{2}}\right)\exp(\underset{-}{Q})1=\alpha 1,\quad \exp\left(\underset{-}{\frac{t}{2}}\right)\exp(\underset{-}{Q})\underset{-}{1}=\alpha\underset{-}{1}.$$

Therefore $\exp(-\underset{-}{Q})\exp\left(-\underset{-}{\frac{t}{2}}\right)\alpha\in E_7^C=\{\beta\in E_8^C\,|\,\beta 1=1,\beta\bar{1}=\bar{1},\beta\underset{-}{1}=\underset{-}{1}\}\cong\{\beta\in \mathrm{Iso}_C(\mathfrak{P}^C,\mathfrak{P}^C)$ $|\,\beta(P\times Q)^{-1}=\beta P\times\beta Q\}$ (which is a simply connected complex Lie group of type E_7). Hence

$$(E_8^C)_{\underset{-}{1}}=\exp(\underset{-}{\mathfrak{P}^C})\exp(\underset{-}{C}))E_7^C.$$

Furthermore, for $\beta\in E_7^C$, it is easy to see that

$$\beta(\exp(\underset{-}{Q}))\beta^{-1}=\exp(\underset{-}{\beta Q}),\qquad \beta(\exp(\underset{-}{t}))\beta^{-1}=\exp(\underset{-}{t}).$$

This shows that $\exp(\underset{-}{\mathfrak{P}^C})\exp(\underset{-}{C})$ is a normal subgroup of $(E_8^C)_{\underset{-}{1}}$. Hence we have a split exact sequence

$$1\longrightarrow \exp(\underset{-}{\mathfrak{P}^C})\exp(\underset{-}{C})\longrightarrow (E_8^C)_{\underset{-}{1}}\longrightarrow E_7^C\longrightarrow 1.$$

Therefore $(E_8^C)_{\underset{-}{1}}$ is the semi-direct product of $\exp(\underset{-}{\mathfrak{P}^C})\exp(\underset{-}{C})$ and E_7^C. Thus we have the hemeomorphism

$$E_8^C/(\exp(\underset{-}{\mathfrak{P}^C})\exp(\underset{-}{C}))E_7^C\simeq\mathfrak{W}^C.$$

In particular, $\mathfrak{W}^C$ is a 248—(56+1+133)=56 dimensional complex manifold.

Remark. Theorem 30 gives another proof of the connectedness of the group E_8^C (see [I-Y] Theorem 18). In fact, the proof of Theorem 30 shows that the connected component $(E_8^C)_0$ of E_8^C containing the identity acts transitively on $\mathfrak{W}^C$, so $\mathfrak{W}^C$ is connected, and from the homeomorphism $E_8^C/(\exp(\mathfrak{P}^C)\exp(\underline{C}))E_7^C \simeq \mathfrak{W}^C$ we see that the group E_8^C is also connected.

Proposition 31. *$\mathfrak{W}^C$ is a complex submanifold of $\mathfrak{e}_8^C$.*

Proof. Consider a subset U of $\mathfrak{W}^C$:

$$U=\{(\Phi, P, Q, r, s, t)\in\mathfrak{W}^C \mid t\neq 0\}$$

$$=\left\{(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8^C \;\middle|\; \begin{array}{c} \Phi=-\dfrac{1}{2t}Q\times Q \\ P=\dfrac{r}{t}Q-\dfrac{1}{6t^2}(Q\times Q)\,Q \\ s=-\dfrac{t}{r^2}+\dfrac{1}{96t^3}\{Q,\ (Q\times Q)\,Q\} \end{array}\right\}$$

$$=\exp(\mathrm{ad}\,(\bar{\mathfrak{P}}^C\oplus C\oplus\bar{C}))\,\underline{1} \quad \text{(see Theorem 30, Proof of Case (1)).}$$

Then U is an open set of $\mathfrak{W}^C$ and salo a submanifold of $\mathfrak{e}_8^C$. Now, for an open set $V=\{(\Phi, P, Q, r, s, t)\in\mathfrak{e}_8^C \mid t\neq 0\}$ of $\mathfrak{e}_8^C$, we have

$$U=V\cap\mathfrak{W}^C$$

(in general, $\alpha U=\alpha V\cap\mathfrak{W}_C$ and $\mathfrak{W}_C=\bigcup_{\alpha\in E_8^C}\alpha U$). This implies that $\mathfrak{W}^C$ is a (regular) submanifold of $\mathfrak{e}_8^C$.

8. The manifold $\mathfrak{W}_1$.

We define a space $\mathfrak{W}_1$ in $\mathfrak{e}_8^C$ by

$$\mathfrak{W}_1=\{R\in\mathfrak{W}^C \mid \langle R, R\rangle=4\}.$$

In order to prove that $\mathfrak{W}_1$ is a 115 dimensional manifold, we use the following well-known

Lemma 32. *Let M be a differentiable manifold, $f: M\to \boldsymbol{R}$ a differentiable mapping and $N=\{p\in M \mid f(p)=0\}$. Suppose rank $(df)_p\neq 0$ for all $p\in N$, then N is a submanifold of M with codimension* 1.

Proposition 33. *$\mathfrak{W}_1$ is a* 115 *dimensional connected compact manifold.*

Proof. Define a mapping $f: \mathfrak{W}^C\to\boldsymbol{R}$ by $f(R)=\langle R, R\rangle-4$. Then f is obviously differentiable because $\mathfrak{W}^C$ is a submanifold of $\mathfrak{e}_8^C=\boldsymbol{C}^{248}=\boldsymbol{R}^{496}$ (Proposition 31). We shall show that $(df)_R\neq 0$ for $R\in f^{-1}(0)=\mathfrak{W}_1$. Consider a curve λR, $0<\lambda<\infty$, in $\mathfrak{W}^C$ through $R\in\mathfrak{W}^C$. Then it is a differentiable curve with respect to λ from Proposition 31. Now, for $R\in\mathfrak{W}_1$,

$$(df)_R\left(\frac{\partial}{\partial\lambda}\right)_R=\frac{\partial f}{\partial\lambda}(\lambda R)\,|_{\lambda=1}=\frac{\partial}{\partial\lambda}(\langle\lambda R, \lambda R\rangle-4)|_{\lambda=1}=\frac{\partial}{\partial\lambda}4\lambda^2|_{\lambda=1}=8\neq 0.$$

Hence $\mathrm{rank}\,(df)_R=1$ for $R\in\mathfrak{W}_1$. Therefore $\mathfrak{W}_1$ is a $\dim\mathfrak{W}^C-1=116-1=115$ dimensional submanifold of $\mathfrak{W}^C$ from Lemma 32. Clearly $\mathfrak{W}_1$ is compact. $\mathfrak{W}_1$ is connected, since $\mathfrak{W}_1$ is the image of $\mathfrak{W}^C$ (which is connected) by a continuous mapping $h : \mathfrak{W}^C \to \mathfrak{W}_1$, $h(R)=\dfrac{2R}{<R, R>}$.

Theorem 34. *The homogeneous space* E_8/E_7 *is homeomorphic to the manifold* $\mathfrak{W}_1$:

$$E_8/E_7 \simeq \mathfrak{W}_1 = \{R \in \mathfrak{e}_8^C \mid R \times R = 0,\ <R, R> = 4\}.$$

Proof. Obviously the group E_8 acts on $\mathfrak{W}_1$ and the isotropy subgroup at $\underline{1}$ is E_7 ([I-Y] Theorem 26). Therefore the orbit $E_8\underline{1}$ (which is homeomorphic to E_8/E_7) through $\underline{1}$ is a $248-133=115$ dimensional submanifold of $\mathfrak{W}_1$, because E_8 is a compact Lie group. Since $E_8\underline{1}$ and $\mathfrak{W}_1$ are both connected manifolds, have the same dimension 115 and $E_8\underline{1}$ is a compact submanifold of $\mathfrak{W}_1$, they must coincide: $E_8\underline{1}=\mathfrak{W}_1$. Thus we have $E_8/E_7 \simeq E_8\underline{1} = \mathfrak{W}_1$.

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