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Embedding of Noetherian rings into faithful modules

By

Kikumichi YAMAGISHI*)

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1. Introduction.

Let M be a finitely generated faithful module over a commutative ring A and let M^r denote, for an integer $r>0$, the direct sum of r copies of M. Let $\{x_i\}_{1\leq i\leq n}$ be a system of generators for M and consider the A-linear map $f: A\rightarrow M^n$ defined by $f(a)=(ax_1, \ldots, ax_n)$ for each $a\in A$. Then f is a monomorphism and consequently we see that for every integer $r\geqq n$, the ring A can be embedded into M^r as one of its submodules. One may further take $r=1$ in certain cases (e.g., if A is an intergral domain). However in general, A is not contained in M itself (Counterexamples are easily given, cf. (3.1).).

The purpose of this research is to clarify when a given ring A can be embedded into *every* finitely generated faithful module over itself and our conclusion is stated as follows:

Theorem (1.1). *Suppose that A is a Noetherian ring. Then the following two conditions are equivalent.*
(1) *Every finitely generated faithful A-module contains A as one of its submodules.*
(2) $\dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}}[0 \underset{A_{\mathfrak{p}}}{:} \mathfrak{p}A_{\mathfrak{p}}]=1$ *for all* $\mathfrak{p}\in \operatorname{Ass} A$.

It follows from Theorem (1.1) that for Noetherian rings A, condition (1) is a local property. As an immediate consequence of it one also has the following

Corollary (1.2) ([7], Theorem). *Let A denote a Noetherian ring and assume that A contains no embedded associated prime ideals of* (0). *Then the following conditions are equivalent.*
(1) *Every finitely generated faithful A-module contains A as one of its submodules.*
(2) *The total quotient ring of A is Gorenstein.*

We shall prove Theorem (1.1) in Section 2. The behaviour of property (1) in Theorem (1.1) under flat base chang is discussed also in Section 2. Typical examples of rings satisfying condition (2) of Theorem (1.1) may be found in Section 3.

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Throughout this paper let A denote a commutative Noetherian ring.

2. Proof of Theorem (1.1).

We begin with the following

Lemma (2.1). *Assume that every finitely generated faithful A-module contains A and let S be a multiplicative system in A. Then every finitely generated faithful $S^{-1}A$-module also contains $S^{-1}A$ as one of its submodules.*

Proof. Let L be a finitely generated faithful $S^{-1}A$-module and choose a finitely generated A-submodule N of L so that $L=S^{-1}N$. Let

$$(0)=\bigcap_{\mathfrak{p}\in \operatorname{Ass} A} I(\mathfrak{p})$$

denote a primary decomposition of (0) in A. We put

$$M=N\oplus(\bigoplus_{\mathfrak{p}\cap S\neq\emptyset} A/I(\mathfrak{p})).$$

Then M is a faithful A-module, as

$$\begin{aligned}\operatorname{Ann}_A N&=\operatorname{Ker}(A\longrightarrow S^{-1}A)\\&=\bigcap_{\mathfrak{p}\cap S=\emptyset} I(\mathfrak{p}).\end{aligned}$$

Hence there is an embedding $A\hookrightarrow M$ of A-modules, which induces a required embedding $S^{-1}A\hookrightarrow L$ of $S^{-1}A$-modules. (Notice that $S^{-1}M=S^{-1}N=L$.)

Proposition (2.2). *Let $(A, \mathfrak{m})$ be a local ring and assume that* $\dim_{A/\mathfrak{m}}[0 \underset{A}{:} \mathfrak{m}]\geqq 2$. *Then there exists a finitely generated faithful A-module M which cannot contain A as one of its submodules.*

Proof. Let F be a free A-module of rank 2 and let $\{X, Y\}$ denote a basis of F. Choose $a, b\in[0 \underset{A}{:} \mathfrak{m}]$ so that a, b are linearly independent over $A/\mathfrak{m}$. Let N denote the A-submodule of F generated by the elements $a(X-Y)$, $(a-b)X$ and bY. We put $M=F/N$. Then

(i) M is a faithful A-module.

(ii) $[0 \underset{A}{:} x]\cap(a, b)A\neq(0)$ for all $x\in M$.

After proving these claims, it is clear that M is a required example.

Now let $\lambda\in\operatorname{Ann}_A M$ and express

$$\begin{aligned}\lambda X&=\alpha a(X-Y)+\beta(a-b)X+\gamma bY,\\ \lambda Y&=\rho a(X-Y)+\sigma(a-b)X+\tau bY\end{aligned}$$

with α, β, γ, ρ, σ and $\tau\in A$. Then comparing coefficients of X and Y in both the above expressions, we get

$$(\#)\qquad \lambda=(\alpha+\beta)a-\beta b=(-\rho)a+\tau b,$$

$$(\#\#) \qquad 0=(-\alpha)a+\gamma b=(\rho+\sigma)a-\sigma b.$$

It follows from equalities (##) that σ, ρ, $\alpha\in\mathfrak{m}$ since a, b are, by our choice, linearly independent over $A/\mathfrak{m}$. Therefore by equalities (#), we see that

$$\lambda=\beta(a-b)=\tau b$$

as $\alpha\in[0\underset{A}{:}\mathfrak{m}]$. Thus by linear independence of a and b again, we have $\beta\in\mathfrak{m}$ whence $\lambda=\beta(a-b)=0$. This proves claim (i).

We put $I=(a, b)A$. Then

$$\begin{aligned} IM&=(AaX+AaY+AbX+AbY)/N \\ &=(AaX+Aa(X-Y)+A(a-b)X+AbY)/N \\ &=A\overline{aX}, \end{aligned}$$

where $\overline{aX}$ denotes the reduction of aX mod N. As M is faithful, $IM\neq(0)$ and hence we get that $IM\cong A/\mathfrak{m}$, i.e., $\dim_{A/\mathfrak{m}}IM=1$. Let $x\in M$ and consider the A-linear map $g: I\rightarrow IM$ defined by $g(c)=cx$ for each $c\in I$. Then since

$$\dim_{A/\mathfrak{m}} I=2>\dim_{A/\mathfrak{m}} IM=1,$$

we get that Ker $g\neq(0)$, i.e., $[0\underset{A}{:}x]\cap I\neq(0)$. This finishes the proof of claim (ii).

Lemma (2.3). *Let $\{N_i\}_{1\leqq i\leqq s}$ denote a family of proper submodules of an A-module M and assume that there exist distinct prime ideals $\mathfrak{p}_i$ $(1\leqq i\leqq s)$ of A such that*

$$\text{(i)}\ \ \mathfrak{p}_iM\subset N_i \ \ \textit{and}\ \ \text{(ii)}\ \ (N_i)_{\mathfrak{p}_i}\cap M=N_i$$

for all $1\leqq i\leqq s$. Then

$$M\neq\bigcup_{i=1}^{s}N_i.$$

Proof ([2], Hilfssatz 1). Assume the contrary and choose s as small as possible among such counterexamples. After relabelling $\mathfrak{p}_i$'s we may assume that $\mathfrak{p}_s$ is minimal among the ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, ..., $\mathfrak{p}_s$. Take an element x of M so that $x\notin\bigcup_{i=1}^{s-1}N_i$. Then $x\in N_s$ clearly. Let y (resp. a) be an element of M (resp. $\bigcap_{i=1}^{s-1}\mathfrak{p}_i$) such that $y\notin N_s$ (resp. $a\notin\mathfrak{p}_s$). Then we get by conditions (i) and (ii) that $ay\in(\bigcap_{i=1}^{s-1}N_i)\setminus N_s$, whence the element $z=x+ay$ cannot be contained in $\bigcup_{i=1}^{s}N_i$. This is a contradiction.

Let M be a finitely generated faithful A-module. We put

$$M(\mathfrak{p})=[0\underset{M_{\mathfrak{p}}}{:}[0\underset{A_{\mathfrak{p}}}{:}\mathfrak{p}A_{\mathfrak{p}}]]\cap M$$

for each $\mathfrak{p}\in\operatorname{Ass}A$. Then it is routine to check that the family $\{M(\mathfrak{p})\}_{\mathfrak{p}\in\operatorname{Ass}A}$ of submodules of M fulfills all the requirements in Lemma (2.3), and hence we have the following

Proposition (2.4). $M\neq\bigcup_{\mathfrak{p}\in\operatorname{Ass}A}M(\mathfrak{p})$.

Now we are ready to prove Theorem (1.1).

(1) $\Rightarrow$ (2) Let $\mathfrak{p}\in \operatorname{Ass} A$. Then by (2.1), the local ring $A_{\mathfrak{p}}$ also satisfies condition (1) of Theorem (1.1). Hence the assertion that $\dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}}[0 \underset{A_{\mathfrak{p}}}{:} \mathfrak{p}A_{\mathfrak{p}}]=1$ follows from (2.2).

(2) $\Rightarrow$ (1) Let M be a finitely generated faithful A-module and let $\mathrm{t}M=\{x\in M \mid [0 \underset{A}{:} x]\neq(0)\}$.

Claim. $\mathrm{t}M=\bigcup_{\mathfrak{p}\in \operatorname{Ass} A} M(\mathfrak{p})$.

Let $x\in \mathrm{t}M$. Then $[0 \underset{A}{:} x]A_{\mathfrak{p}} \neq(0)$ for some $\mathfrak{p}\in \operatorname{Ass} A$. Choose such a prime ideal $\mathfrak{p}$ of A so that the height of $\mathfrak{p}$ is as small as possible. Then $[0 \underset{A}{:} x]A_{\mathfrak{q}}=(0)$ for all $\mathfrak{q}\in \operatorname{Ass} A$ which are properly contained in $\mathfrak{p}$, whence we find that the ideal $[0 \underset{A}{:} x]A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ has finite length. Therefore

$$[0 \underset{A}{:} x]A_{\mathfrak{p}}\cap[0 \underset{A_{\mathfrak{p}}}{:} \mathfrak{p}A_{\mathfrak{p}}]\neq(0),$$

and so we get $[0 \underset{A}{:} x]A_{\mathfrak{p}}\supset[0 \underset{A_{\mathfrak{p}}}{:} \mathfrak{p}A_{\mathfrak{p}}]$ since $\dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}}[0 \underset{A_{\mathfrak{p}}}{:} \mathfrak{p}A_{\mathfrak{p}}]=1$ by our assumption. Thus $x \in M(\mathfrak{p})$ and the inclusion

$$\mathrm{t}M\subset\bigcup_{\mathfrak{p}\in \operatorname{Ass} A} M(\mathfrak{p})$$

is established. The opposite inclusion is clear.

By this claim and (2.4) we get that $M\neq \mathrm{t}M$. Let y be an element of M such that $y\notin \mathrm{t}M$. Then $A\cong Ay\subset M$ and hence we have a required embedding $A\hookrightarrow M$ of A-modules. This completes the proof of Theorem (1.1).

Proposition (2.5). *Let B denote a Noetherian flat A-algebra. Then B satisfies condition* (1) *in* Theorem (1.1) *if and only if all the rings A and $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}\otimes_A B$* ($\mathfrak{p}\in \operatorname{Ass} A$) *satisfy the same condition.*

Proof. Let $\mathfrak{P}\in \operatorname{Ass} B$. We put $\mathfrak{p}=\mathfrak{P}\cap A$. Then we see by [3, Theorem] that

$$\dim_{B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}}[0 \underset{B_{\mathfrak{P}}}{:} \mathfrak{P}B_{\mathfrak{P}}]=\dim_{C_{\mathfrak{P}}/\mathfrak{P}C_{\mathfrak{P}}}[0 \underset{C_{\mathfrak{P}}}{:} \mathfrak{P}C_{\mathfrak{P}}]\ \dim_{A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}}[0 \underset{A_{\mathfrak{p}}}{:} \mathfrak{p}A_{\mathfrak{p}}],$$

where C denotes the ring $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}\otimes_A B$. Thus the assertion follows from (1.1). (Recall that $\operatorname{Ass} B=\bigcup_{\mathfrak{p}\in \operatorname{Ass} A} \operatorname{Ass}_B B/\mathfrak{p}B$.)

We close this section with the next two results, both of which immediately follow from (2.5).

Corollary (2.6). *Let $B=A[X_1, X_2,\dots, X_n]$* ($n>0$) *denote a polynomial ring. Then B satisfies condition* (1) *in* Theorem (1.1) *if and only if so does A.*

Corollary (2.7). *Assume that A is a local ring. If the completion $\hat{A}$ of A satisfies condition* (1) *in* Theorem (1.1), *so does A. In case A is a homomorphic image of a Gorenstein ring, the converse is also true.*

Remark (2.8). Unless A is a homomorphic image of a Gorenstein ring, the second assertion in Corollary (2.7) is not necessarily true. In fact let A be an integral local domain of $\dim A=1$. Then $\hat{A}$ satisfies condition (2) of Theorem (1.1) if and only if the total quotient ring of $\hat{A}$ is Gorenstein (cf. (1.2)). The latter condition is equivalent to saying that A is a homomorphic image of a Gorenstein ring (cf. [4] and [6]) and, on the other hand, there is given by [1] an example of a one-dimensional integral local domain A which is not a homomorphic image of any Gorenstein ring. Therefore the completion $\hat{A}$ of such a local ring A does not satisfy condition (2) of Theorem (1.1), while A obviously fulfills it.

3. Examples.

(3.1). Let $(A, \mathfrak{m})$ be an Artinian local ring and let $E_A(A/\mathfrak{m})$ denote the injective envelope of $A/\mathfrak{m}$. Then $E_A(A/\mathfrak{m})$ is a finitely generated faithful A-module. It contains A if and only if $\dim_{A/\mathfrak{m}}[0 \underset{A}{:} \mathfrak{m}]=1$, i.e., A is a Gorenstein ring.

(3.2) ([5], Theorem 2). Let $s \geqq 4$ be an integer. Then there exists an Artinian local ring $(A, \mathfrak{m})$ and a finitely generated faithful A-module M such that $\dim_{A/\mathfrak{m}}[0 \underset{A}{:} \mathfrak{m}] = s$ and $l_A(M) < l_A(A)$, where $l_A(M)$ (resp. $l_A(A)$) denotes the length of the A-module M (resp. A). Of course M cannot contain A in this case.

(3.3). Let $k[[X_1, X_2, \ldots, X_n]]$ be a formal power series ring over a field k and let

$$A = k[[X_1, X_2, \ldots., X_n]] \ltimes k$$

denote the idealization. Then Goto showed by direct calculation that A is embedded into every finitely generated faithful module. This follows also from our theorem, since the ring A satisfies condition (2) of Theorem (1.1).

(3.4). Let $d > t \geqq 0$ be integers and let $R = k[[X_0, X_1, \ldots, X_d]]$ denote a formal power series ring over a field k. For each integer $0 \leqq s \leqq d-t$, we put

$$I_s = (X_0^2, \ldots, X_{s-1}^2) + (X_s),$$
$$\mathfrak{p}_s = (X_0, X_1, \ldots, X_s).$$

Let $I = \bigcap_{s=0}^{d-t} I_s$ and $A = R/I$. Then

(i) $\dim A = d$ and depth $A = t$.

(ii) $\operatorname{Ass} A = \{\mathfrak{p}_s/I \mid 0 \leqq s \leqq d-t\}$.

(iii) $\dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}[0 \underset{A_\mathfrak{p}}{:} \mathfrak{p}A_\mathfrak{p}] = 1$ for every $\mathfrak{p} \in \operatorname{Ass} A$.

This ring A can be embedded into every finitely generated faithful module.

Proof. Because $\prod_{i \neq s} X_i \notin I_s$ and $\prod_{i \neq s} X_i \in \bigcap_{i \neq s} I_i$, the intersection $I = \bigcap_{s=0}^{d-t} I_s$ is irredundant. As I_s is an irreducible ideal of R with $\mathfrak{p}_s = \sqrt{I_s}$, we get assertion (ii) together with assertion (iii). Hence $\dim A = d$ and depth $A \leqq \inf_{\mathfrak{p} \in \operatorname{Ass} A} \dim A/\mathfrak{p} = t$. On the other hand as $X_{d-t+1}, X_{d-t+2}, \ldots, X_d$ is an A-regular sequence, we see that depth $A \geqq t$ whence depth $A = t$.

The last assertion follows from our theorem (1.1).

DEPARTMENT OF MATHEMATICS
SCIENCE UNIVERSITY OF TOKYO

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