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An extension of Morishima's nonlinear Perron-Frobenius theorem

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1. Introduction

Many discussions have been done on the equations of the form:

$$\frac{du_j}{dt}=A_j(u_1,\dots, u_n),\ j=1,\dots, n$$

where each $A_j(u)$ is a quadratic form. Two important examples are Volterra's ecology equation (see Volterra [1], Jenks [2]), and Boltzmann's gas equation with a finite number of velocities. (See Carleman [3], Conner [4].) Note that in both cases, every solution starting from a non-negative initial data remains non-negative.

To study this type of equations, finding all the non-negative equilibrium points is very important. The author studied the special type of equation:

$$\frac{du_j}{dt}=A_j(u_1,\dots, u_n)-\mu u_j^2,\quad j=1,\dots, n$$

where each $A_j(u)$ is a quadratic form with non-negative coefficients, and $\mu\geqq 0$ is a parameter. In this case, each equilibrium point satisfies

$$A_j(u_1,\dots, u_n)=\mu u_j^2,\quad j=1,\dots, n$$

that is

$$H(u)=\mu^{1/2}u$$

where $H_j(u)=[A_j(u)]^{1/2}$. In this way, to find equilibrium points can be converted to the nonlinear eigen-value problem.

This type of nonlinear eigen-value problem has been studied in detail by economists. (See especially Morishima [5], Nikaido [6].) But the notion of indecomposability defined by them is a little too stringent in order to guarantee the existence of positive eigen-vectors. The author could loosen the sufficient condition for this, introducing the notion of non-sectionality. He could also represent the cannonical expansion of sectional homogeneous transformations of degree one with which we can establish the characterization of their maximal eigen-value.

We shall explain these facts in this paper.

2. Fundamental Notions

We shall use vector inequalities. $x \leqq y$ means $x_j \leqq y_j$ for all $j=1,\dots, n$. Similarly, $x<y$ implies $x_j<y_j$ for all $j=1,\dots, n$. Lastly, $x \leq y$ implies that $x \leqq y$ and $x \neq y$. We call a vector $x \geqq 0$ non-negative, a vector $x>0$ positive.

Definition 1. $H(x)$ is called a non-negative non-decreasing homogeneous transformation of degree one or simply a homogeneous transformation if it satisfies the following three conditions.

1) $H(x)$ is a continuous map from $[0, \infty) \times \cdots \times [0, \infty)$ into itself.
2) $H(x) \leqq H(y)$ for all $x \leqq y$.
3) $H(\mu x)=\mu H(x)$ for all $\mu \geqq 0$. (Therefore $H(0)=0$.)

Our problem throughout this paper concerns $\lambda \geqq 0$ and $x \geqq 0$ satisfying

$$H(x)=\lambda x$$

which we call an eigen-value and an eigen-vector of $H(x)$.

We introduce here two subsets of homogeneous transformations, the indecomposable ones defined by economists and the non-sectoinal ones which contain the indecomposable ones.

Definition 2. (See Morishima [5].) A homogeneous transformation $H(x)$ is called indecomposable if it satisfies the following:

For any given proper partition $\Theta \cup \Omega=\{1,\dots, n\}$ (i.e. $\Theta \cap \Omega=\phi$, $\Theta \neq \phi$, $\Omega \neq \phi$), there exists always $\bar{\omega} \in \Omega$ for which following 1) holds.

1) $H_{\bar{\omega}}(x)<H_{\bar{\omega}}(y)$ when $x_\theta<y_\theta$ for all $\theta \in \Theta$, $0 \leqq x_\omega=y_\omega$ for all $\omega \in \Omega$.

Example of indecomposable transformation:

$$H_1(x_1, x_2)=(2x_1^2+x_2^2)^{1/2}$$

$$H_2(x_1, x_2)=(x_1^2+2x_2^2)^{1/2}$$

whose unique eigen-vector is $H(1, 1)=3^{1/2}(1, 1)$.

Definition 3. A homogeneous transformation is called non-sectional if it satisfies the following:

For any given proper partition $\Theta \cup \Omega=\{1,\dots, n\}$, there exists always $\bar{\omega} \in \Omega$ for which following 1) and 2) hold at the same time.

1) $H_{\bar{\omega}}(x)<H_{\bar{\omega}}(y)$ when $x_\theta<y_\theta$ for all $\theta \in \Theta$, $0<x_\omega=y_\omega$ for all $\omega \in \Omega$.
2) Let $x_\omega>0$ be fixed for all $\omega \in \Omega$, then

$$H_{\bar{\omega}}(x) \longrightarrow \infty \quad \text{when all} \quad x_\theta(\theta \in \Theta) \longrightarrow \infty$$

Example of non-sectional transformation:

$$H_1(x_1, x_2)=x_1^{1/2}(2x_1+x_2)^{1/2}$$

$$H_2(x_1, x_2) = x_2^{1/2}(x_1 + 2x_2)^{1/2}$$

whose eigen-vectors are $H(1, 1) = 3^{1/2}(1, 1)$, $H(1, 0) = 2^{1/2}(1, 0)$, and $H(0, 1) = 2^{1/2}(0, 1)$.

Remark. An indecomposable homogeneous transformation $H(x)$ is always non-sectional. In fact, for any given proper partition $\Theta \cup \Omega$, there exists $\bar{\omega} \in \Omega$ satisfying 1) of the definition 2. This $\bar{\omega}$ clearly satisfies 1) of the definition 3. To consider 2) of the definition 3, we set $x = 0$ and $y \geq 0$ such that

$$y_\theta = 1 \quad \text{for all} \quad \theta \in \Theta$$

$$y_\omega = 0 \quad \text{for all} \quad \omega \in \Omega.$$

Then

$$0 = H_{\bar{\omega}}(0) < H_{\bar{\omega}}(y). \tag{1}$$

Let all $x_\omega > 0$ $(\omega \in \Omega)$ be fixed and all $x_\theta (\theta \in \Theta)$ go to infinity.

$$\begin{aligned} \lim_{x_\theta \to \infty} H_{\bar{\omega}}(x) &\geqq \lim_{t \to \infty} H_{\bar{\omega}}(ty) \\ &= \lim_{t \to \infty} t H_{\bar{\omega}}(y) \\ &= \infty \end{aligned}$$

because (1) holds. Q. E. D.

3. Preliminary Results

Theorem 1. (See Morishima [5].) *A homogeneous transformation has at least one non-negative eigen-value and a non-negative eigen-vector associated with it.*

Proof. We consider the following map from the set $S = \{(x_1, \ldots, x_n);\ x_j \geqq 0$ for all j and $\sum_{j=1}^{n} x_j = 1\}$ into itself.

$$F(x) = \frac{x + H(x)}{1 + \sum_{j=1}^{n} H_j(x)}. \tag{2}$$

Then, by virtue of Brouwer's fixed point theorem, there exists at least one fixed point $\gamma \in S$:

$$\gamma = F(\gamma) = \frac{\gamma + H(\gamma)}{1 + \sum_{j=1}^{n} H_j(\gamma)}.$$

Rewriting this, we get

$$H(\gamma) = \{\sum_{j=1}^{n} H_j(\gamma)\}\gamma. \tag{3}$$

Therefore, $\sum_{j=1}^{n} H_j(\gamma)$ and γ are the eigen-value and eigen-vector we have been looking for. Q. E. D.

We shall write down some fundamental results concerning the maximal eigen-value and the maximal eigen-vector of homogeneous transformations.

Lemma 1. (See Morishima [5].) *If there exist two non-negative eigen-vectors γ, $\bar{\gamma}$ for the same $H(x)$ such that*

$$\{j;\, \gamma_j > 0\} \subseteqq \{j;\, \bar{\gamma}_j > 0\},$$

then follows the next inequality concerning their eigen-values λ, $\bar{\lambda}$ respectively:

$$\lambda \leqq \bar{\lambda}.$$

And consequently, the eigen-vectors with the same position of zero-elements have the same eigen-value.

Proof. We redefine as γ the vector $\rho\gamma$ with the scalar $\rho > 0$ such that

$$\rho = \min_{j \in J} \frac{\bar{\gamma}_j}{\gamma_j} \qquad \text{where} \quad J = \{j;\, \gamma_j > 0\}.$$

So $\gamma \leqq \bar{\gamma}$ and there exists i such that

$$\gamma_i = \bar{\gamma}_i > 0.$$

From the monotonicity of $H(x)$,

$$\lambda = \frac{H_i(\gamma)}{\gamma_i} = \frac{H_i(\gamma)}{\bar{\gamma}_i} \leqq \frac{H_i(\bar{\gamma})}{\bar{\gamma}_i} = \bar{\lambda}. \tag{4}$$

We have proved the first statement.

If γ and $\bar{\gamma}$ have the same zero-position, then

$$\{j;\, \gamma_j > 0\} \subseteqq \{j;\, \bar{\gamma}_j > 0\}$$

and

$$\{j;\, \gamma_j > 0\} \supseteqq \{j;\, \bar{\gamma}_j > 0\}.$$

Therefore, from the result just obtained,

$$\lambda \leqq \bar{\lambda} \quad \text{and} \quad \lambda \geqq \bar{\lambda}, \quad \text{i.e.} \quad \lambda = \bar{\lambda}. \qquad \text{Q.E.D.}$$

From the latter half of the lemma 1, we know that there exists at most one eigen-value for any given zero-position. Thus we get the following theorem.

Theorem 2. (See Morishima [5].) *A homogeneous transformation has only a finite number (at most $2^n - 1$) of non-negative eigen-values associated with non-negative eigen-vectors.*

Definition 4. The maximal eigen-value of $H(x)$ is denoted by λ_0 or $\lambda_0(H)$. And an eigen-vector associated with it is called a maximal eigen-vector.

Theorem 3. *If there exists a positive eigen-vector $\gamma > 0$ for $H(x)$, its eigen-value is $\lambda_0(H)$.*

Proof. Let λ be its eigen-value. Suppose there exists any other eigen-vector $\bar{\gamma}$ with the eigen-value $\bar{\lambda}$. Then, from the lemma 1,

$$\lambda \geqq \bar{\lambda}.$$

This shows that λ is maximal. Q. E. D.

We shall cite without proof a few theorems about indecomposable homogeneous transformations from Morishima [5], which will be the basis of our discussion.

Theorem 4. (See Morishima [5].) *The eigen-vector of an indecomposable homogeneous transformation $H(x)$ is unique up to the scalar multiplication, and is positive. Its eigen-value is also positive.*

Theorem 5. (See Morishima [5].) *If $H(x) \leqq \bar{H}(x)$ for all $x \geqq 0$, then*

$$\lambda_0(H) \leqq \lambda_0(\bar{H}).$$

If, moreover, $\bar{H}(x)$ is indecomposable and $H(x) \leq \bar{H}(x)$ for all $x > 0$, then

$$\lambda_0(H) < \lambda_0(\bar{H}).$$

Using the theorem 4 and the theorem 5, we get the important lemma 2, its proof will also be omitted.

Lemma 2. (See Morishima [5].) *Let $H(x)$ be a homogeneous transformation. If we define a new homogeneous transformation $H(\varepsilon)(x)$ in the following manner:*

$$H(\varepsilon)(x) = H(x) + \varepsilon U(x) \tag{5}$$

where $U_j(x) = x_1 + \cdots + x_n$ for all $j = 1, \ldots, n$, then this $H(\varepsilon)(x)$ is indecomposable for $\varepsilon > 0$, and $\lambda_0(H(\varepsilon))$ is a strictly increasing function in $\varepsilon \geqq 0$, satisfying

$$\lim_{\varepsilon \downarrow 0} \lambda_0(H(\varepsilon)) = \lambda_0(H).$$

Using the preceding theorems, we get the following propositions which will be essential to our argument below.

Lemma 3. *For a homogeneous transformation and its maximal eigen-value λ_0, the following hold.*

1) *For any $\lambda > \lambda_0$, there exists a positive vector $\gamma > 0$ which satisfies*

$$H(\gamma) - \lambda\gamma < 0.$$

2) *For any $\lambda < \lambda_0$, there exists a non-negative vector $\gamma \geq 0$ which satisfies*

$$H_j(\gamma) - \lambda\gamma_j > 0 \qquad \text{for all} \quad j \in \{j;\, \gamma_j > 0\}.$$

Proof of 1). We use $H(\varepsilon)(x)$ of the lemma 2. Then

$$\lambda_0(H(\varepsilon)) \downarrow \lambda_0(H) \quad \text{when} \quad \varepsilon \downarrow 0. \tag{6}$$

Therefore there exists a certain ε such that

$$\lambda_0 < \lambda_0(H(\varepsilon)) < \lambda.$$

And

$$H(\varepsilon)[\gamma(\varepsilon)] = \lambda_0(H(\varepsilon))\gamma(\varepsilon).$$

Here $\gamma(\varepsilon) > 0$ since it is the eigen-vector of an indecomposable $H(\varepsilon)(x)$. (See the theorem 4.) Rewriting the last equality,

$$H[\gamma(\varepsilon)] + \varepsilon U[\gamma(\varepsilon)] = \lambda_0(H(\varepsilon))\gamma(\varepsilon)$$

$$H[\gamma(\varepsilon)] - \lambda\gamma(\varepsilon) = [\lambda_0(H(\varepsilon)) - \lambda]\gamma(\varepsilon) - \varepsilon U[\gamma(\varepsilon)].$$

Since $\gamma(\varepsilon) > 0$ and $\lambda_0(H(\varepsilon)) < \lambda$,

$$H[\gamma(\varepsilon)] - \lambda\gamma(\varepsilon) < 0. \tag{7}$$

Thus the vector $\gamma(\varepsilon) > 0$ is the vector we have been looking for.

Proof of 2). Let $\gamma \geq 0$ be the maximal eigen-vector, then

$$\begin{aligned} H(\gamma) - \lambda\gamma &= H(\gamma) - \lambda_0\gamma + (\lambda_0 - \lambda)\gamma \\ &= (\lambda_0 - \lambda)\gamma \geq 0. \end{aligned} \tag{8}$$

With this vector $\gamma \geq 0$, the statement of the lemma is clearly satisfied. The proof of the lemma 3 is completed.

We can prove the converse of this lemma as follows.

Lemma 4. *For a homogeneous transformation $H(x)$ and its maximal eigen-value λ_0, the following hold.*

1) *If there exists a positive vector $\gamma > 0$ for a given λ which satisfies*

$$H(\gamma) - \lambda\gamma \leqq 0,$$

then

$$\lambda \geqq \lambda_0.$$

2) *If there exists a non-negative vector $\gamma \geq 0$ for a given λ which satisfies*

$$H(\gamma) - \lambda\gamma \geqq 0,$$

then

$$\lambda \leqq \lambda_0.$$

Proof of 1). Assume $\lambda < \lambda_0$. Then, from the lemma, there would exist such a $\bar{\gamma} \geq 0$ that

$$H_j(\bar{\gamma}) > \lambda\bar{\gamma}_j \qquad \text{for all} \quad j \in \{j;\ \bar{\gamma}_j > 0\}.$$

The sign of inequality remains unchanged with a certain $\rho > 1$:

$$H_j(\bar{\gamma}) > \rho\lambda\bar{\gamma}_j \qquad \text{for all} \quad j \in \{j;\ \bar{\gamma}_j > 0\}. \tag{9}$$

After n times iteration H^n, we get the next inequality, using the homogeneity and the monotonicity,

(10) $$\frac{1}{\lambda^n} H_j^n(\bar{\gamma}) > \rho^n \bar{\gamma}_j \qquad \text{for all} \quad j \in \{j;\ \bar{\gamma}_j > 0\}.$$

On the other hand, because $\gamma > 0$, there exists an appropriate $C > 0$ such that $\bar{\gamma} \leqq C\gamma$. Then

$$\begin{aligned} \frac{1}{\lambda} H(\bar{\gamma}) &\leqq \frac{1}{\lambda} H(C\gamma) \\ &= \frac{C}{\lambda} H(\gamma) \\ &\leqq C\gamma \end{aligned}$$

Further

(11) $$\frac{1}{\lambda^n} H^n(\bar{\gamma}) \leqq C\gamma.$$

This means that after any number of iterations, $\frac{1}{\lambda^n} H^n(\bar{\gamma})$ remains bounded. This fact contradicts (10) where $\rho > 1$.

Proof of 2). Assume $\lambda > \lambda_0$. Then, from the lemma 3, there would exist a positive $\bar{\gamma} > 0$ such that

$$H(\bar{\gamma}) - \lambda\bar{\gamma} < 0.$$

With an appropriate $\rho > 1$,

$$\frac{1}{\lambda} H(\bar{\gamma}) \leqq \frac{1}{\rho} \bar{\gamma}.$$

After n times iteration, we get

(12) $$\frac{1}{\lambda^n} H^n(\bar{\gamma}) \leqq \frac{1}{\rho^n} \bar{\gamma}.$$

From the assumption of the lemma, after n times iteration,

$$\frac{1}{\lambda^n} H^n(\gamma) \geqq \gamma \geq 0.$$

Since there exists an appropriate $C > 0$ such that $\gamma \leqq C\bar{\gamma}$,

(13) $$\frac{1}{\lambda^n} H^n(\bar{\gamma}) \geqq \frac{1}{C\lambda^n} H^n(\gamma) \geqq \frac{1}{C} \gamma \geq 0.$$

(12) implies $\frac{1}{\lambda^n} H^n(\bar{\gamma})$ tends to zero, while (13) means it does not. This is a contradiction. Thus we finished the proof of the lemma 4.

Combining the lemma 3 and the lemma 4, we get a rough characterization of the maximal eigen-value as follows.

Theorem 6. *Following statements hold.*

1) *There exists a positive vector* $\gamma > 0$ *which satisfies*

$$H(\gamma)-\lambda\gamma<0$$

if and only if $\lambda>\lambda_0$.

2) *There exists a non-negative vector* $\gamma\geq 0$ *which satisfies*

$$H_j(\gamma)-\lambda\gamma_j>0 \qquad \textit{for all} \quad j\in\{j;\ \gamma_j>0\}$$

if and only if $\lambda<\lambda_0$.

Proof of the sufficiency. Evident from the lemma 3.

Proof of the necessity of 1). From the lemma 4, we get $\lambda\geqq\lambda_0$. Assume $\lambda=\lambda_0$. If $\lambda=\lambda_0=0$, then $H(\gamma)<0$, a contradiction. So we can assume $\lambda_0>0$. With a sufficiently small $\varepsilon>0$, we get

$$H(\gamma)-(\lambda_0-\varepsilon)\gamma<0.$$

This means from the lemma 4,

$$\lambda_0-\varepsilon\geqq\lambda_0.$$

A contradiction.

Proof of the necessity of 2). From the lemma 4, $\lambda\leqq\lambda_0$. Assume $\lambda=\lambda_0$. Then, with a sufficiently small $\varepsilon>0$, we also get

$$H_j(\gamma)-(\lambda_0+\varepsilon)\gamma_j>0 \qquad \text{for all} \quad j\in\{j;\ \gamma_j>0\}.$$

Again from the lemma 4,

$$\lambda_0+\varepsilon\leqq\lambda_0.$$

This is also a contradiction. Thus the proof of the theorem 6 is completed.

From the theorem 6, we can prove the next theorem.

Theorem 7. *Let* $x>0$ *be an arbitrary positive vector. Then*

$$\lim_{n\to\infty}(x,\ H^n(x))^{1/n}=\lambda_0(H).$$

Proof. Let $\gamma\geq 0$ be the maximal eigen-vector. With an appropriate $C>0$, $x\geqq C\gamma$. Then

$$\begin{aligned}(x,\ H^n(x))&\geqq(C\gamma,\ H^n(C\gamma))\\ &=(C\gamma,\ C(\lambda_0(H))^n\gamma)\\ &=C^2(\lambda_0(H))^n(\gamma,\ \gamma).\end{aligned}$$

Because $(\gamma,\ \gamma)>0$,

$$\varliminf_{n\to\infty}(x,\ H^n(x))^{1/n}\geqq\lambda_0(H). \tag{14}$$

From the theorem 6, for an arbitrary $\lambda>\lambda_0$, there exists a positive vector $\bar{\gamma}>0$ such that

$$H(\bar{\gamma})<\lambda\bar{\gamma}.$$

Choosing an appropriate $\bar{C}$, we have $x\leqq\bar{C}\bar{\gamma}$. Then

$$(x,\ H^n(x)) \leqq (\bar{C}\bar{\gamma},\ H^n(\bar{C}\bar{\gamma}))$$
$$\leqq \bar{C}^2 \lambda^n(\bar{\gamma},\ \bar{\gamma}).$$

Therefore

$$\overline{\lim_{n\to\infty}}\,(x,\ H^n(x))^{1/n} \leqq \lambda.$$

Since $\lambda > \lambda_0$ is arbitrary,

(15) $$\overline{\lim_{n\to\infty}}\,(x,\ H^n(x))^{1/n} \leqq \lambda_0(H).$$

Together with (14), this means

(16) $$\lim_{n\to\infty}\,(x,\ H^n(x))^{1/n} = \lambda_0(H).$$ Q. E. D.

4. Non-Sectional Homogeneous Transformations

In this section, we look into non-sectional homogeneous transformations. In this case, the necessary consitions of the theorem 6 can be replaced with somewhat looser ones which will be useful to prove the theorem 8.

Lemma 5. *Assume $H(x)$ is non-sectional. Then the following statements hold.*

1) *If there exists a positive vector $\gamma > 0$ for any given λ such that*

$$H(\gamma) - \lambda\gamma \leq 0,$$

then

$$\lambda > \lambda_0.$$

2) *If there exists a positive vector $\gamma > 0$ for any given λ such that*

$$H(\gamma) - \lambda\gamma \geq 0,$$

then

$$\lambda < \lambda_0.$$

Proof of 1). If $H(\gamma) - \lambda\gamma < 0$, there is nothing to prove. (See the theorem 6.) Assume the other case. Let us denote

$$J = \{j;\ H_j(\gamma) - \lambda\gamma_j < 0\}$$
$$K = \{k;\ H_k(\gamma) - \lambda\gamma_k = 0\}.$$

We shall construct a positive vector $\bar{\gamma} > 0$ such that

(17) $$H(\bar{\gamma}) - \lambda\bar{\gamma} \leq 0, \quad J \subsetneqq \bar{J} = \{j;\ H_j(\bar{\gamma}) - \lambda\bar{\gamma}_j < 0\}.$$

We consider the following vector with a sufficiently small ε:

$$\gamma_j(\varepsilon)\begin{cases} = \gamma_j - \varepsilon & \text{for all} \quad j \in J \\ = \gamma_j & \text{for all} \quad j \in K. \end{cases}$$

From the non-sectionality of $H(x)$, at least one $H_k[\gamma(\varepsilon)]$ $(k \in K)$ decreases as ε increases. So, for a sufficiently small $\varepsilon > 0$,

$$H_j[\gamma(\varepsilon)] - \lambda\gamma_j(\varepsilon) < 0 \qquad \text{for all} \quad j \in J$$

$$H_k[\gamma(\varepsilon)] - \lambda\gamma_k(\varepsilon) < 0 \qquad \text{for a certain} \quad k \in K.$$

Therefore this positive vector $\gamma(\varepsilon)$ has the property of (17). Repeating this procedure, we finally obtain $\tilde{\gamma} > 0$ such that

$$H(\tilde{\gamma}) - \lambda\tilde{\gamma} < 0.$$

The rest is clear from the theorem 6.

Proof of 2). This time, we have only to introduce

$$\gamma_j(\varepsilon) \begin{cases} = \gamma_j + \varepsilon & \text{for all} \quad j \in J \\ = \gamma_j & \text{for all} \quad j \in K \end{cases}$$

where $J = \{j;\ H_j(\gamma) - \gamma_j > 0\}$, $K = \{k;\ H_k(\gamma) - \lambda\gamma_k = 0\}$. For the rest, the same argument as in 1) can be applied.

We have completed all the proof of the lemma 5.

Now we proceed to the main theorem of this paper. But, before that, note that non-sectional homogeneous transformations might also have eigen-vectors with zero-elements. (See the example of non-sectional transformation in the section 1.) In this respect, they are very different from indecomposable transformations.

Theorem 8. *The maximal eigen-vector of a non-sectional homogeneous transformation is positive and is unique up to the scalar multiplication.*

Proof of the positivity. After the necessary scalar multiplication, we may assume, about the maximal eigen-vector γ,

$$\min\{\gamma_j;\ \gamma_j > 0\} = 1. \tag{18}$$

In order to prove the statement by contradiction, we assume

$$\phi \neq J_0 = \{j;\ \gamma_j > 0\} \subsetneqq \{1, \dots, n\}.$$

We construct the partition J_0, $J_1, \dots$ in the following way. Denote by J_1 all $\bar{\omega} \notin J_0$ of the definition 3, regarding J_0 as θ, $\{1, \dots, n\} - J_0$ as Ω. If $J_0 \cup J_1 \subsetneqq \{1, \dots, n\}$, we construct further J_2, the set of all the $\bar{\omega} \in \{1, \dots, n\} - J_0 - J_1$, regarding $J_0 \cup J_1$ as θ, $\{1, \dots, n\} - J_0 - J_1$ as Ω. We repeat this procedure if necessary. After a finite number of steps, we obtain the partition:

$$J_0 \cup J_1 \cup \cdots \cup J_\nu = \{1, \dots, n\}. \tag{19}$$

Let us now construct positive vectors $\bar{\gamma}(k)$ $(k = 1, \dots, \nu)$ inductively, in the following way.

First, we set

$$\bar{\gamma}_j(\nu) = M_\nu = 1 \qquad \text{for all} \quad j \in \{1, \dots, n\}.$$

Then we choose a sufficiently large number $M_{\nu-1}$ such that the vector $\bar{\gamma}(\nu-1)$:

$$\bar{\gamma}_j(\nu-1)\begin{cases} =M_{\nu-1} & \text{for all } \; j\in J_0\cdots J_{\nu-1} \\ =M_\nu & \text{for all } \; j\in J_\nu \end{cases}.$$

satisfies

$$H_j[\bar{\gamma}(\nu-1)]-\lambda_0\bar{\gamma}_j(\nu-1)>0 \qquad \text{for all } \; j\in J_\nu.$$

This can be done because every $H_j(x)(j\in J_\nu)$ goes to infinity as all $x_k(k\in J_0\cup\cdots\cup J_{\nu-1})$ go to infinity at the same time. (Recall the way of constructing this partition.)

Next we define $\bar{\gamma}(\nu-2)$ and $M_{\nu-2}>M_{\nu-1}$ such that

$$\bar{\gamma}_j(\nu-2)\begin{cases} =M_{\nu-2} & \text{for all } \; j\in J_0\cup\cdots\cup J_{\nu-2} \\ =M_{\nu-1} & \text{for all } \; j\in J_{\nu-1} \\ =M_\nu & \text{for all } \; j\in J_\nu \end{cases}$$

satisfies

$$H_j[\bar{\gamma}(\nu-2)]-\lambda_0\bar{\gamma}_j(\nu-2)>0 \qquad \text{for all } \; j\in J_{\nu-1}.$$

Note that for each $j\in J_\nu$, the same inequality still holds. In fact, for such a j,

$$\begin{aligned} &H_j[\bar{\gamma}(\nu-2)]-\lambda_0\bar{\gamma}_j(\nu-2) \\ &\geqq H_j[\bar{\gamma}(\nu-1)]-\lambda_0\bar{\gamma}_j(\nu-1)>0 \end{aligned}$$

(Remind that $\bar{\gamma}(\nu-2)\geq\bar{\gamma}(\nu-1)$ and $\bar{\gamma}_j(\nu-2)=\bar{\gamma}_j(\nu-1)$ for all $j\in J_\nu$.)

After ν steps, we are led to $\bar{\gamma}(0)$ such that

$$\text{(20)} \qquad \bar{\gamma}_j(0)\begin{cases} =M_0 & \text{for all } \; j\in J_0 \\ =M_1 & \text{for all } \; j\in J_1 \\ \vdots & \qquad \vdots \\ =M_{\nu-1} & \text{for all } \; j\in J_{\nu-1} \\ =M_\nu & \text{for all } \; j\in J_\nu \end{cases}$$

where

$$\text{(21)} \qquad H_j[\bar{\gamma}(0)]-\lambda_0\bar{\gamma}_j(0)>0 \qquad \text{for all } \; j\in J_1\cup\cdots\cup J_\nu.$$

Now we define $\bar{\gamma}$ as follows.

$$\text{(22)} \qquad \bar{\gamma}_j\begin{cases} =M_0\gamma_j & \text{for all } \; j\in J_0 \\ =M_1 & \text{for all } \; j\in J_1 \\ \vdots & \qquad \vdots \\ =M_{\nu-1} & \text{for all } \; j\in J_{\nu-1} \\ =M_\nu & \text{for all } \; j\in J_\nu \end{cases}$$

Reminding that $\gamma_j\geqq 1$ for all $j\in J_0$ and consequently $\bar{\gamma}\geqq\bar{\gamma}(0)$, for all $j\in J_0$, we get

(23) $$H_j(\bar{\gamma})-\lambda_0\bar{\gamma}_j \geqq H_j(M_0\bar{\gamma})-\lambda_0 M_0\bar{\gamma}_j$$
$$=M_0[H_j(\bar{\gamma})-\lambda_0\bar{\gamma}_j]=0.$$

With (20) and (22) in mind, we know (21) remains unchanged, replacing $\bar{\gamma}(0)$ with $\bar{\gamma}$, that is,

(24) $$H_j(\bar{\gamma})-\lambda_0\bar{\gamma}_j \geqq H_j(\bar{\gamma}(0))-\lambda_0\bar{\gamma}_j(0)>0$$

for all $j \in J_1 \cup \cdots \cup J_\nu$.
Combining (23) and (24), we get

$$H(\bar{\gamma})-\lambda_0\bar{\gamma} \geq 0 \quad \text{and} \quad \bar{\gamma}>0.$$

Using the lemma 5, we are led to the next contradiction,

$$\lambda_0<\lambda_0.$$

Proof of the uniqueness. We repeat the way written in Morishima [5] which is used for indecomposable transformations. Assume there were two positive maximal eigen-vectors γ and $\bar{\gamma}$. As usual, we redefine as γ the vector $\rho\gamma$ where

$$\rho=\min_{1 \leqq j \leqq n} \frac{\bar{\gamma}_j}{\gamma_j}$$

Then

$$0<\gamma \leq \bar{\gamma}$$

We set

(25) $$\phi \neq J=\{j;\ \bar{\gamma}_j>\gamma_j\}, \quad \phi \neq K=\{k;\ \bar{\gamma}_k=\gamma_k\}.$$

From the non-sectionality, there exists $\bar{k} \in K$ such that

$$H_{\bar{k}}(\bar{\gamma})>H_{\bar{k}}(\gamma).$$

Since $H(\gamma)=\lambda_0\gamma$ and $H(\bar{\gamma})=\lambda_0\bar{\gamma}$,

$$\lambda_0\gamma_k=H_{\bar{k}}(\gamma)<H_{\bar{k}}(\bar{\gamma})=\lambda_0\bar{\gamma}_{\bar{k}}.$$

But $\gamma_k=\bar{\gamma}_{\bar{k}}$ because $\bar{k} \in K$. This is a contradiction.

Thus we have completed all the proof of the theorem 8.

Remark 1. Suppose $H(x)$ is non-sectional. For any λ (possibly $\lambda>\lambda_0$) and any proper subset of suffixes $J \neq \phi$, we can construct a positive vector $\gamma>0$ such that

$$H_k(\gamma)-\lambda\gamma_k>0 \qquad \text{for all} \quad k \in \{1,\dots,n\}-J.$$

(But not necessarily $H_j(\gamma)-\lambda\gamma_j>0$ for all $j \in J$.) We have only to partition the suffixes and construct $\bar{\gamma}(0)$ in the same way as in the proof of this theorem.

Remark 2. From the theorem 3, we know that any positive eigen-vector is associated with the maximal eigen-value. Therefore this theorem also guarantees the uniqueness of the positive eigen-vector for each non-sectional homogeneous transformation.

Remark 3. It might be felt that 2) of the definition 3 is redundant. Indeed, there is room in loosening it. But it cannot be removed. Let us illustrate this situation with an example.

Consider the following two-dimensional homogeneous transformation with a parameter $\mu>0$:

$$H_1(x_1, x_2)=\mu(x_1^2+x_2^2)^{1/2} \tag{26}$$
$$H_2(x_1, x_2)=(x_1^2+x_2^2)^{1/2}\left(\frac{\pi}{2}-\operatorname{Arctan}\frac{x_2}{x_1}\right)\operatorname{Arctan}\frac{x_2}{x_1}.$$

It is easy to show, for any $x_1>0$ and $x_2>0$,

$$\frac{\partial H_1}{\partial x_2}=\mu\frac{\partial r}{\partial x_2}=\mu\sin\theta>0 \tag{27}$$
$$\frac{\partial H_2}{\partial x_1}=\frac{\partial}{\partial x_1}r\left(\frac{\pi}{2}\theta-\theta^2\right)$$
$$=\left(\frac{\pi}{2}\theta-\theta^2\right)\cos\theta-\left(\frac{\pi}{2}-2\theta\right)\sin\theta>0$$

where (r, θ) is the usual polar coordinates.
There are two partitions of suffixes, that, is $\theta=\{1\}$, $\Omega=\{2\}$ and $\theta=\{2\}$, $\Omega=\{1\}$. Using (27), $\bar{\omega}=2$ or $\bar{\omega}=1$ satisfies 1) of the definition 3, respectively. But, in the former case, $\bar{\omega}=2$ does not satisfy 2) of the definition 3. This is the case because

$$H_2(x_1, x_2)<\frac{\pi}{2}x_2 \quad \text{when} \quad x_1>0 \quad \text{and} \quad x_2>0. \tag{28}$$

This inequality follows from (27) and $\lim_{x_1\to\infty} H_2(x_1, x_2)=\frac{\pi}{2}x_2$.

This homogeneous transformation $H(x)$ can have no positive eigen-vector when $\mu\geqq\frac{\pi}{2}$. In fact, assume $H(x_1, x_2)=\lambda_0(x_1, x_2)>0$. With (28) in mind,

$$\lambda_0 x_1=H_1(x_1, x_2)>\mu x_1$$
$$\lambda_0 x_2=H_2(x_1, x_2)<\frac{\pi}{2}x_2\leqq\mu x_2.$$

The first inequality means $\lambda_0>\mu$ while the second $\lambda_0<\mu$. This is a contradiction.
Q. E. D.

One can show similarly that $H(x)$ has a positive eigen-vector when $0<\mu<\frac{\pi}{2}$. Thus we know that the condition 2) may be a little too stringent but is not redundant.

We can extend the latter half of the theorem 5 to the case where $H(x)$ is non-sectional.

Theorem 9. *If $H(x)\leq\bar{H}(x)$ for all $x>0$ and $\bar{H}(x)$ is non-sectional, then*

$$\lambda_0(H)<\lambda_0(\bar{H}).$$

Proof. First we consider the case where $H(x)$ is also non-sectional. Let $\gamma>0$ be the maximal eigen-vector of $H(x)$.

$$\bar{H}(\gamma)\geq H(\gamma)=\lambda_0(H)\gamma.$$

From the positivity, we can apply the lemma 5, and

$$\lambda_0(H)<\lambda_0(\bar{H}).$$

Next, we go on to the general case. We set

$$\tilde{H}(x)=\frac{1}{2}(H(x)+\bar{H}(x)). \tag{29}$$

This $\tilde{H}(x)$ is clearly non-sectional and $H(x)\leq\tilde{H}(x)\leq\bar{H}(x)$ for all $x>0$. From the result just obtained,

$$\lambda_0(\tilde{H})<\lambda_0(\bar{H}). \tag{30}$$

From the general result of the theorem 5,

$$\lambda_0(H)\leqq\lambda_0(\tilde{H}). \tag{31}$$

Combining (30) and (31), we get the final inequality,

$$\lambda_0(H)<\lambda_0(\bar{H}).$$

The proof is completed.

Using the theorem 8, we can sharpen the theorem 6 when $H(x)$ is non-sectional.

Theorem 10. *Let $H(x)$ be non-sectional. Then the following hold.*

1) *The exists a positive $\gamma>0$ such that*

$$H(\gamma)-\lambda\gamma\leq 0$$

if and only if $\lambda>\lambda_0(H)$.

2) *There exists a non-negative $\gamma\geq 0$ such that*

$$H(\gamma)-\lambda\gamma\geq 0$$

if and only if $\lambda<\lambda_0(H)$.

Proof of the sufficiency. Clear. We have only to adopt the maximal eigen-vector as γ.

Proof of the necessity of 1). Clear from the lemma 5.

Proof of the necessity of 2). If $\gamma>0$, we can use the lemma 5 directly. So we may assume that $\gamma\geq 0$ has zero-elements. We set J and a matrix A as following.

$$J=\{j;\ \gamma_j>0\}.$$

$$A_{jj}=1 \qquad \text{for all} \quad j\in J$$

$$A_{ij}=0 \qquad \text{elsewhere.}$$

Setting $\bar{H}(x)$ as

(32) $$\bar{H}(x)=H(Ax)\leq H(x).$$

The last inequality follows from the definition of the non-sectionality.
Then, since $A\gamma=\gamma$,

$$\bar{H}(\gamma)-\lambda\gamma=H(\gamma)-\lambda\gamma\geq 0.$$

From the lemma 4,

(33) $$\lambda\leqq\lambda_0(\bar{H}).$$

From (32), using the theorem 7,

(34) $$\lambda_0(\bar{H})<\lambda_0(H).$$

Combining (33) and (34), we get

$$\lambda<\lambda_0(H).$$

The proof of the theorem 10 is completed.

Corollary. *Let $H(x)$ be non-sectional. Then, the following hold.*

1) *If $H(\gamma)-\lambda_0\gamma\leqq 0$ for a positive vector $\gamma>0$, then*

$$H(\gamma)=\lambda_0\gamma.$$

2) *If $H(\gamma)-\lambda_0\gamma\geqq 0$ for a non-negative vector $\gamma\geq 0$, then*

$$H(\gamma)=\lambda_0\gamma \quad \text{and} \quad \gamma>0.$$

Proof of 1). Assume that $H(\gamma)-\lambda_0\gamma\leq 0$. Then, from the theorem 10, $\lambda_0>\lambda_0$. This is a contradiction.

Proof of 2). Assume that $H(\gamma)-\lambda_0\gamma\geq 0$. Then, from the theorem 10, $\lambda_0<\lambda_0$. This contradiction shows $H(\gamma)=\lambda_0\gamma$. The maximal eigen-vector must be positive. So $\gamma>0$. Q. E. D.

Remark. We can not replace the positivity of the vector γ with the non-negativity in 1)'s of both the present theorem and corollary. Let $H(x_1, x_2)$ be the example of non-sectional transformation in the section 2. In this case, $\lambda_0(H)=3^{1/2}$. And $H(1,0)-3^{1/2}(1,0)=(2^{1/2}-3^{1/2})(1,0)\leqq 0$. But $H(1,0)\neq 3^{1/2}(1,0)$.

5. Resolvent Problem

In this section, we treat the resolvent problem, that is, we investigate the non-negative solutions of the resolvent equation:

(35) $$\lambda x-H(x)=c \quad \text{where} \quad c\geqq 0.$$

First, we look into the general case where no further conditions are imposed on the homogeneous transformation $H(x)$. We formulate and prove the next theorem in somewhat different way from Morishima [5].

Theorem 11. *Let $H(x)$ be a homogeneous transformation.*

1) *Let* $\lambda>\lambda_0(H)$ *be fixed. Then* (35) *is solvable for all* $c\geqq 0$. *The solution is unique when* $c>0$ *and* $c=0$, *but it is not necessarily unique for the other* $c\geq 0$. *Choosing an appropriate one among the solutions for each* $c\geqq 0$ *if not unique, the solution* $R_\lambda(c)$ *forms a homogeneous transformation with respect to* $c\geqq 0$. *Moreover*

1.1) $R_\lambda(c)>0$ *when* $c>0$.

1.2) $R_\lambda(c)\leq R_\lambda(\bar{c})$ *when* $0\leqq c\leq \bar{c}$.

1.3) $R_\lambda(c)<R_\lambda(\bar{c})$ *when* $0\leqq c<\bar{c}$.

Conversely

2) *If* (35) *has a non-negative solution for some* $c>0$, *then* $\lambda>\lambda_0(H)$.

Proof of 1). We begin with the solvability. Since $\lambda>\lambda_0(H)$, we can apply the theorem 6. We have, therefore, $\gamma>0$ such that

$$\frac{1}{\lambda}H(\gamma)<\gamma. \tag{36}$$

Remark $\lambda>\lambda_0(H)\geqq 0$. Let $c\geqq 0$ be fixed. Then, for a sufficiently large $K>0$,

$$\frac{1}{\lambda}(H(K\gamma)+c)<K\gamma. \tag{37}$$

We define the map

$$F(x)=\frac{1}{\lambda}(H(x)+c). \tag{38}$$

Note that $[0, K\gamma]\times\cdots\times[0, K\gamma]$ is invariant through this map. In fact, for all $0\leqq x\leqq K\gamma$, using (37),

$$0\leqq\frac{c}{\lambda}\leqq F(x)\leqq\frac{1}{\lambda}(H(K\gamma)+c)<K\gamma.$$

By virtue of Brouwer's fixed point theorem, $F(x)$ has a fixed point in $[0, K\gamma]\times\cdots\times[0, K\gamma]$:

$$x=F(x)=\frac{1}{\lambda}(H(x)+c).$$

This $x\geqq 0$ is clearly a non-negative solution of (35) for $c\geqq 0$. And note that $x=\frac{1}{\lambda}(H(x)+c)\geqq\frac{c}{\lambda}>0$ when $c>0$ which proves 1.1).

Next we prove the fact:

$$x\leqq\bar{x} \quad \text{if} \quad \lambda x-H(x)=c,\ \lambda\bar{x}-H(\bar{x})=\bar{c} \quad \text{and} \quad 0<c\leqq\bar{c}. \tag{39}$$

We follow the ingeneous way of Morishima [5]. First, we set

$$\rho=\min_{1\leqq j\leqq n}\frac{\bar{x}_j}{x_j}. \tag{40}$$

Assume $x\nleqq\bar{x}$, that is, $\rho<1$. Then, since $c>0$ and $\rho x\leqq\bar{x}$,

(41)
$$\begin{aligned}\rho x &= \frac{\rho}{\lambda}(H(x)+c)\\ &= \frac{1}{\lambda}(H(\rho x)+\rho c)\\ &< \frac{1}{\lambda}(H(\bar{x})+\bar{c})\\ &= \bar{x}.\end{aligned}$$

(41) contradicts (40).

(39) shows the uniqueness of the solution of (35) for $c>0$. In fact, $\lambda x - H(x) = c$ and $\lambda\bar{x} - H(\bar{x}) = c$ imply that $x \leqq \bar{x}$ and $x \geqq \bar{x}$, i.e., $x = \bar{x}$. As for $c=0$, (35) has only the trivial solution because it must be an eigen-vector of $H(x)$ with λ which is greater than the maximal eigen-value $\lambda_0(H)$.

Let $\lambda x - H(x) = c$ with $c>0$. Then $\tilde{x} = \rho x$ is the unique solution of $\lambda\tilde{x} - H(\tilde{x}) = \rho c$, which proves

(42) $$R_\lambda(\rho c) = \rho R_\lambda(c) \quad \text{when} \quad c>0 \quad \text{and} \quad \rho \geqq 0.$$

In the sequel, we shall define $R_\lambda(c)$ for $c \geq 0$ with some zero-elements as a limit of $R_\lambda(c')$ where $c'>0$ and $c' \downarrow c$. It gurantees the non-decrease and the homogeneity of $R_\lambda(c)$ with respect to c by virtue of (39) and (42). It also guarantees $R_\lambda(c)$ for $c \geq 0$ with zero-elements is still a solution of (35) since $H(x)$ is continuous.

We now define $R_\lambda(c)$ for $c \geq 0$ with some zero-elements. For that purpose, we take a strictly decreasing positive sequence ($c(\nu) > c(\nu+1) > 0$ for all ν) which converges to $c \geq 0$. Since $R_\lambda(c(\nu))$ is non-increasing from (39), we can define

(43) $$R_\lambda(c) = \lim_{\nu\to\infty} R_\lambda(c(\nu)).$$

This limit does not depend on the choice of $\{c(\nu)\}$. In fact, if we have two strictly decreasing $c(\nu) \downarrow c$ and $c'(\nu) \downarrow c$, we can combine their subsequences to form another strictly decreasing $c''(\nu) \downarrow c$. Then we have

$$\lim_{\nu\to\infty} R_\lambda(c(\nu)) = \lim_{\nu\to\infty} R_\lambda(c''(\nu)) = \lim_{\nu\to\infty} R_\lambda(c'(\nu)).$$

We prove the continuity of $R_\lambda(c)$. First, we assume $c>0$ or $c=0$. We take a sequence $\{c(\nu)\}$ converging to c (not always monotonously). $R_\lambda(c(\nu))$ is clearly bounded. Any accumulating point of $R_\lambda(c(\nu))$ must be the unique solution of (35) for c since $H(x)$ is continuous. This means

$$\lim_{\nu\to\infty} R_\lambda(c(\nu)) = R_\lambda(c)$$

which shows the continuity of $R_\lambda(c)$ at $c>0$ and $c=0$. Next, we assume $c \geq 0$ has some zero-elements. If we take a sufficiently small $\varepsilon>0$, $R_\lambda(\tilde{c})$ where $\tilde{c}_j = c_j + \varepsilon$ for all j is near enough to $[R_\lambda(c)]_j$ from (43). In this way, using the non-decrease and the homogeneity of $R_\lambda(c)$,

$$(1-\varepsilon)R_\lambda(c)\leqq R_\lambda(\gamma)\leqq R_\lambda(\tilde{c}) \quad \text{when} \quad (1-\varepsilon)c\leqq\gamma\leqq\tilde{c}.$$

Because $\varepsilon>0$ is arbitray, this means the continuity of $R_\lambda(c)$ at $c\geq 0$. Thus we have proved $R_\lambda(c)$ is continuous, therefore, a homogeneous transformation.

We prove now 1.2) and 1.3). It is clear that each $x\geqq 0$ cannot be solutions of (35) for different $c\geqq 0$ and $\bar{c}\geqq 0$. This means $R_\lambda(c)\neq R_\lambda(\bar{c})$ when $c\neq\bar{c}$. This proves 1.2). Let now $0\leqq c<\bar{c}$. For a sufficiently small $\delta>0$,

$$0\leqq c<(1-\delta)\bar{c}<\bar{c}.$$

This proves, from 1.1),

$$R_\lambda(c)\leqq(1-\delta)R_\lambda(\bar{c})<R_\lambda(\bar{c})(>0).$$

1.3) is proved.

Proof of 2). Let the condition be satisfied:

$$\lambda x-H(x)=c>0.$$

So x must be clearly positive. We can apply the theorem 6, so

$$\lambda>\lambda_0(H).$$

We have completed all the proof of the theorem 11.

Remark 1. The solution of (35) for $c\geq 0$ with some zero-elements is not always unique. Let $H(x)$ be the example of the non-sectional transformation in the section 2. Consider the following resolvent equation:

$$2x-H(x)=(1,0) \quad \text{where} \quad 2>3^{1/2}=\lambda_0(H).$$

This equation has two solutions:

$$\left(\frac{2+2^{1/2}}{2},\ 0\right) \quad \text{and} \quad \left(\frac{4+10^{1/2}}{3},\ \frac{4+10^{1/2}}{6}\right)$$

Remark 2. 2) of the present theorem 11 becomes untrue if we replace $c>0$ by $c\geq 0$. Let $H(x)$ be the same as in the remark 1.

$$\frac{3}{2}x-H(x)=\left(\frac{3}{2}-2^{1/2},\ 0\right)$$

has a non-negative solution (1, 0), but $\frac{3}{2}<2^{1/2}=\lambda_0(H)$.

Now we consider the case where $H(x)$ is non-sectional. Note that the remarks of the preceding theorem are also valid for the next theorem.

Theorem 12. *Let $H(x)$ be non-sectional.*

1) *Let $\lambda>\lambda_0(H)$ be fixed. Then* (35) *is solvable for all $c\geqq 0$. The solution is unique when $c>0$ and $c=0$, but it is not always unique for the other $c\geq 0$. However, there exists only one positive solution for each $c\geq 0$. Defining $R_\lambda(c)$ as the positive solution when $c\geq 0$ and $R_\lambda(0)=0$, it forms an indecomposable homogeneous transformation with respect to c. Moreover*

1.1) $R_\lambda(c) < R_\lambda(\bar{c})$ *when* $0 \leqq c \leq \bar{c}$.

Conversely

2) *If* (35) *has a positive solution for some* $c \geq 0$, *then* $\lambda > \lambda_0(H)$.

Proof of 1). We begin with the existence of positive solutions for $c \geq 0$. If $c > 0$, it immediately follows from the preceding theorem 11. Assume $c \geq 0$ has some zero-elements. From the remark 1 of the theorem 8, we have a positive vector $\tilde{\gamma} > 0$ such that

$$H_j(\tilde{\gamma}) - \lambda\tilde{\gamma}_j > 0 \qquad \text{for all } j \in J = \{j;\, c_j = 0\}$$

Then, with a sufficiently small $\tilde{K} > 0$, we have

$$\tilde{K}\tilde{\gamma} < \frac{1}{\lambda}(c + H(\tilde{K}\tilde{\gamma})). \tag{44}$$

Let $\gamma > 0$ be the maximal eigen-vector of the non-sectional $H(x)$. Then, with a sufficiently large $K > 0$,

$$K\gamma > \frac{1}{\lambda}(c + H(K\gamma)). \tag{45}$$

We set a map:

$$F(x) = \frac{1}{\lambda}(c + H(x)).$$

Through this map, the interval $[\tilde{K}\tilde{\gamma}_1, K\gamma_1] \times \cdots \times [\tilde{K}\tilde{\gamma}_n, K\gamma_n]$ is invariant. In fact, for x such that $0 < \tilde{K}\tilde{\gamma} \leqq x \leqq K\gamma$,

$$\tilde{K}\tilde{\gamma} < \frac{1}{\lambda}(c + H(\tilde{K}\tilde{\gamma})) \leqq F(x) \leqq \frac{1}{\lambda}(c + H(K\gamma)) < K\gamma.$$

The fixed point of $F(x)$ is a solution of (35) which is positive because $x = F(x) > \tilde{K}\tilde{\gamma} > 0$.

We shall show the positive solution for $c \geq 0$ with zero-elements is unique and the same one as $R_\lambda(c)$ in the theorem 11. Let $\gamma > 0$ be the positive solution of (35) where $c \geq 0$ has zero-elements. We set γ' and γ'' as follows.

$$\gamma' = (1 - \varepsilon)\gamma \quad \text{and} \quad \gamma'' = (1 + \varepsilon)\gamma.$$

γ' satisfies

$$\lambda\gamma'_j - H_j(\gamma') - c_j = (1 - \varepsilon)(\lambda\gamma_j - H_j(\gamma)) - c_j < 0$$

for all $j \in \{j;\, c_j > 0\}$,

$$\lambda\gamma'_k - H_k(\gamma') - c_k = (1 - \varepsilon)(\lambda\gamma_k - H_k(\gamma)) - c_k = 0$$

for all $k \in \{k;\, c_k = 0\}$.

We modify γ' very slightly in the same way as in the proof of the lemma 5 so that we can find $0 < \bar{\gamma} < \gamma$ satisfying

$$\lambda\bar{\gamma} - H(\bar{\gamma}) - c < 0. \tag{46}$$

We also modify γ'' to find $\tilde{\gamma}>\gamma$

$$\lambda\tilde{\gamma}-H(\tilde{\gamma})-c>0. \tag{47}$$

In a usual way, from (46) and (47),

$$G(x)=\frac{1}{\lambda}(c+H(x))$$

maps $[\bar{\gamma}_1, \tilde{\gamma}_1]\times\cdots\times[\bar{\gamma}_n, \tilde{\gamma}_n]$ into itself.
A slight perturbation of c leaves (46) and (47) unchanged. Therefore, the solution of (35) for c' near enough to c can be found in $[\bar{\gamma}_1, \tilde{\gamma}_1]\times\cdots\times[\bar{\gamma}_n, \tilde{\gamma}_n]$. Recalling that the solution for a positive c' is unique (see the theorem 11) and $\bar{\gamma}$, $\tilde{\gamma}$ can be taken arbitrarily near to γ (take a small ε), we can assert

$$\gamma=\lim_{c'\downarrow c} R_\lambda(c').$$

This shows γ is the same one as $R_\lambda(c)$ in the theorem 11, consequently unique. From the fact just proved and the theorem 11, $R_\lambda(c)$ as a positive solution of (35) for $c\geq 0$ is clearly a homogeneous transformation with respect to c.

Now we prove 1.1) which is stricter than to say $R_\lambda(c)$ is indecomposable. Assume 1.1) did not hold. We denote

$$J=\{j;\ 0\leqq\gamma_j<\bar{\gamma}_j\}$$

$$K=\{k;\ 0<\gamma_k=\bar{\gamma}_k\}$$

where $\gamma=R_\lambda(c)$ and $\bar{\gamma}=R_\lambda(\bar{c})>0$ ($\bar{c}\geq 0$). The non-emptiness of J follows from 1.2) of the theorem 11. Since $H(x)$ is non-sectional, there exists $\bar{k}\in K$ such that

$$H_{\bar{k}}(\gamma)<H_{\bar{k}}(\bar{\gamma})$$

But, we have

$$c_{\bar{k}}=\lambda\gamma_{\bar{k}}-H_{\bar{k}}(\gamma)\leqq\lambda\bar{\gamma}_{\bar{k}}-H_{\bar{k}}(\bar{\gamma})=\bar{c}_{\bar{k}}.$$

This contradicts the definition of K.

Proof of 2). Let the condition be satisfied.

$$\lambda x-H(x)=c\geq 0 \quad\text{and}\quad x>0.$$

From the theorem 10,

$$\lambda>\lambda_0(H).$$

We have proved all the statements of the theorem 12.

For the sake of comparison, we write down the results about indecomposable homogeneous transformations. We omit its proof. See Morishima [5] for the proof and the original formulation.

Theorem 13. *Let $H(x)$ be indecomposable.*

1) *Let $\lambda>\lambda_0(H)$ be fixed. Then* (35) *has a unique solution for all $c\geqq 0$. The*

solution $R_\lambda(c)$ for each $c \geqq 0$ forms an indecomposable transformation with respect to c. Moreover

1.1) $R_\lambda(c) > 0$ *when* $c \geq 0$.

1.2) $R_\lambda(c) < R_\lambda(\bar{c})$ *when* $0 \leqq c \leq \bar{c}$.

Conversely

2) *If* (35) *has a non-negative solution for some* $c \geq 0$, *then* $\lambda > \lambda_0(H)$.

6. Cannonical Expansion of Sectional Homogeneous Transformations

It is known that there is a unique decomposition of any decomposable (reducible) matrix to indecomposable (irreducible) submatrices. (See Gantmacher [7].) For homogeneous transformations, a parallel argument is possible. In this section, we investigate this problem, restricting our attention to docile transformations which we shall define later. First, we introduce some abbreviations.

Definition 4. Let $\Theta \cup \Omega = \{1,\dots, n\}$ be a proper partition. We denote by x_Θ, $H_\Theta(x)$ the projections to the coordinates x_θ (for all $\theta \in \Theta$), $H_\theta(x)$ (for all $\theta \in \Theta$), respectively.

Definition 5. A homogeneous transformation $H(x)$ is called docile if it satisfies the following.

1) $H(x)$ is real-analytic in $(0, \infty) \times \cdots \times (0, \infty)$.

2) If there exist a proper partition $\Theta \cup \Omega = \{1,\dots, n\}$, $\varpi \in \Omega$, vectors $0 \leqq \gamma_\Theta < \tilde{\gamma}_\Theta$, $0 < \gamma_\Omega$ such that

$$H_\varpi(\gamma_\Theta, \gamma_\Omega) < H_\varpi(\tilde{\gamma}_\Theta, \gamma_\Omega),$$

then

$$H_\varpi(x_\Theta, x_\Omega) \longrightarrow \infty$$

when $x_\theta \to \infty$ for all $\theta \in \Theta$ and $x_\Omega > 0$ is fixed.

Remark. An equivalent of 2) is the following: Let $\gamma_\Omega > 0$ be fixed. If $H_\Omega(x_\Theta, \gamma_\Omega)$ is bounded, then $H_\Omega(x_\Theta, \gamma_\Omega) \equiv \text{const}$.

Example of docile transformation:

$$H_j(x) \equiv [P_j(x_1,\dots, x_n)]^{1/m_j} \qquad \text{for all } j$$

where each $P_j(x)$ is a homogeneous polynomial of degree m_j with non-negative coefficients.

We begin with a lemma.

Lemma 6. *Let $H(x)$ be docile and $\Theta \cup \Omega = \{1,\dots, n\}$ be a proper partition. If there exist $0 \leqq \gamma_\Theta < \tilde{\gamma}_\Theta$ and $0 < \gamma_\Omega$ such that*

$$H_\Omega(\gamma_\Theta, \gamma_\Omega) = H_\Omega(\tilde{\gamma}_\Theta, \gamma_\Omega)$$

then

$$H_\Omega(x_\Theta, x_\Omega) \equiv H(0, x_\Omega)$$

Proof. First, we note

$$(48) \qquad H_\Omega(x_\Theta, \gamma_\Omega) \equiv \text{const.} \quad \text{when} \quad \gamma_\Theta \leqq x_\Theta \leqq \tilde{\gamma}_\Theta.$$

In fact, for such x_Θ,

$$H_\Omega(\gamma_\Theta, \gamma_\Omega) \leqq H_\Omega(x_\Theta, \gamma_\Omega) \leqq H_\Omega(\tilde{\gamma}_\Theta, \gamma_\Omega) = H_\Omega(\gamma_\Theta, \gamma_\Omega).$$

Then, from its real-analyticity in $(0, \infty) \times \cdots \times (0, \infty)$ and continuity in $[0, \infty) \times \cdots \times [0, \infty)$,

$$(49) \qquad H_\Omega(x_\Theta, \gamma_\Omega) \equiv \text{const.} \equiv H_\Omega(0, \gamma_\Omega) \qquad \text{for all} \quad x_\Theta \geqq 0.$$

Let an arbitrary $y_\Omega \geqq 0$ be fixed. Then there exists some $C > 1$ such that $y_\Omega \leqq C\gamma_\Omega$. Thus,

$$\begin{aligned} H_\Omega(x_\Theta, y_\Omega) &\leqq H_\Omega(Cx_\Theta, C\gamma_\Omega) \\ &= CH_\Omega(x_\Theta, \gamma_\Omega) \\ &= CH_\Omega(0, \gamma_\Omega). \end{aligned}$$

From the remark of the definition 5,

$$H_\Omega(x_\Theta, y_\Omega) \equiv H_\Omega(0, y_\Omega).$$

Since $y_\Omega \geqq 0$ is arbitrary, the lemma 6 is proved.

Assume $H(x)$ is docile and sectional. From the definition 3, we have a proper partition $I \cup J = \{1, \ldots, n\}$, $0 \leqq \gamma_J < \tilde{\gamma}_J$ and $\gamma_I > 0$ such that

$$H_I(\gamma_I, \gamma_J) = H_I(\gamma_I, \tilde{\gamma}_J).$$

We used the docileness of $H(x)$. Then, applying the lemma 6, we get the following.

$$(50) \qquad \begin{aligned} &H_I(x_I, x_J) \equiv H_I(x_I, 0). \\ &H_J(x_I, x_J) \equiv Z_J(x_I, x_J) + H_J(0, x_J) \end{aligned}$$

where $Z_J(x_I, x_J) \equiv H_J(x_I, x_J) - H_J(0, x_J)$, consequently $Z_J(x_I, x_J) \geqq 0$ and $Z_J(0, x_J) \equiv 0$.

If $H_I(x_I, 0)$ or $H_J(0, x_J)$ is sectional, we refine the partition $I \cup J$. For instance, if both are sectional, we get the partition $I_1 \cup I_2 = I$, $J_1 \cup J_2 = J$ where

$$\begin{aligned} &H_{I_1}(x_{I_1}, x_{I_2}, 0, 0) \equiv H_{I_1}(x_{I_1}, 0, 0, 0) \\ &H_{I_2}(x_{I_1}, x_{I_2}, 0, 0) \equiv \tilde{Z}_{I_2}(x_{I_1}, x_{I_2}, 0, 0) + H_{I_2}(0, x_{I_2}, 0, 0) \\ &H_{J_1}(0, 0, x_{J_1}, x_{J_2}) \equiv H_{J_1}(0, 0, x_{J_1}, 0) \\ &H_{J_2}(0, 0, x_{J_1}, x_{J_2}) \equiv \tilde{Z}_{J_2}(0, 0, x_{J_1}, x_{J_2}) + H_{J_2}(0, 0, 0, x_{J_2}) \end{aligned}$$

and

$$\tilde{Z}_{I_2}(x) \geqq 0, \quad \tilde{Z}_{I_2}(0, x_{I_2}, 0, 0) \equiv 0$$

$$\tilde{Z}_{J_2}(x) \geqq 0, \quad \tilde{Z}_{J_2}(0, 0, 0, x_{J_2}) \equiv 0$$

Therefore we have got a further expansion of $H(x)$, recalling (50)

$$\begin{aligned} H_{I_1}(x) &\equiv H_{I_1}(x_{I_1}, x_{I_2}, 0, 0) \\ &\equiv H_{I_1}(x_{I_1}, 0, 0, 0) \\ H_{I_2}(x) &\equiv H_{I_2}(x_{I_1}, x_{I_2}, 0, 0) \\ &\equiv \tilde{Z}_{I_2}(x_{I_1}, x_{I_2}, 0, 0) + H_{I_2}(0, x_{I_2}, 0, 0) \\ H_{J_1}(x) &\equiv Z_{J_1}(x_I, x_J) + H_{J_1}(0, 0, x_{J_1}, x_{J_2}) \\ &\equiv Z_{J_1}(x_I, x_J) + H_{J_1}(0, 0, x_{J_1}, 0) \\ H_{J_2}(x) &\equiv Z_{J_2}(x_I, x_J) + H_{J_2}(0, 0, x_{J_1}, x_{J_2}) \\ &\equiv Z_{J_2}(x_I, x_J) + \tilde{Z}_{J_2}(0, 0, x_{J_1}, x_{J_2}) + H_{J_2}(0, 0, 0, x_{J_2}) \end{aligned}$$

We redefine $\tilde{Z}_{I_2}(x_I, 0)$ as $Z_{I_2}(x)$, $Z_{J_2}(x) + \tilde{Z}_{J_2}(0, x_J)$ as $Z_{J_2}(x)$. Thus we have obtained

$$\begin{aligned} H_{I_1}(x) &\equiv H_{I_1}(x_{I_1}, 0, 0, 0) \\ H_{I_2}(x) &\equiv Z_{I_2}(x) + H_{I_2}(0, x_{I_2}, 0, 0) \\ H_{J_1}(x) &\equiv Z_{J_1}(x) + H_{J_1}(0, 0, x_{J_1}, 0) \\ H_{J_2}(x) &\equiv Z_{J_2}(x) + H_{J_2}(0, 0, 0, x_{J_2}) \end{aligned}$$

where $Z_{I_2}(x) \geqq 0$, $Z_{J_1}(x) \geqq 0$, $Z_{J_2}(x) \geqq 0$ and

$$\begin{aligned} &Z_{I_2}(0, x_{I_2}, x_{J_1}, x_{J_2}) \equiv 0 \\ &Z_{J_1}(0, 0, x_{J_1}, x_{J_2}) \equiv 0 \\ &Z_{J_2}(0, 0, 0, x_{J_2}) \equiv 0 \end{aligned}$$

We continue this procedure unless every $H_{J_k}(0,\dots, 0, x_{J_k}, 0,\dots, 0)$ is non-sectional. Thus the former part of the next theorem is proved. For the simplicity, we denote $H_\theta(0,\dots, 0, x_\theta, 0,\dots, 0)$, $(0,\dots, 0, x_\theta, 0,\dots, 0)$ by $H_\theta(x_\theta, 0)$ and $(x_\theta, 0)$ from now on.

Theorem 14. *Any docile homogeneous transformation has a cannonical expansion with the partition of suffixes*

$$J_1 \cup \cdots \cup J_\mu \cup J_{\mu+1} \cup \cdots \cup J_\nu = \{1,\dots, n\}$$

such that

$$H_{J_k}(x) \equiv H_{J_k}(x_{J_k}, 0) \qquad \text{for all} \quad k = 1,\dots, \mu$$

$$H_{J_k}(x) \equiv Z_{J_k}(x) + H_{J_k}(x_{J_k}, 0) \qquad \text{for all} \quad k = \mu+1,\dots, \nu$$

where each $H_{J_k}(x_{J_k}, 0)$ $(k = 1,\dots, \nu)$ *is non-sectional and* $Z_{J_k}(x) \geqq 0$, $Z_{J_k}(x) \not\equiv 0$, $Z_{J_k}(0,\dots, 0, x_{J_k}, x_{J_{k+1}},\dots, x_{J_\nu}) \equiv 0$ *for all* $k = \mu+1,\dots, \nu$.

Moreover, if we do not take the order of $\{J_k\}$ *into account, the partition* $J_1 \cup \cdots \cup J_\nu = \{1, \dots, n\}$ *is unique.*

Proof. Only the proof of the uniqueness is left. Note that $\mu \geqq 1$, in the first place.

We assume there is another cannonical partition

$$I_1 \cup \cdots \cup I_{\mu'} \cup I_{\mu'+1} \cup \cdots \cup I_{\nu'} = \{1, \dots, n\}$$

First we show any $J_k (1 \leqq k \leqq \mu)$ and $I_{k'} (1 \leqq k' \leqq \mu')$ are identical or disjoint. We prove this by contradiction. Assume $\phi \neq J_k \cap I_{k'} \subsetneqq J_k$ or $\phi \neq J_k \cap I_{k'} \subsetneqq I_{k'}$. For the definiteness, we assume the first. Since $H_{J_k}(x) \equiv H_{J_k}(x_{J_k}, 0)$ and $H_{I_{k'}}(x) \equiv \equiv H_{I_{k'}}(x_{I_{k'}}, 0)$, $H_{J_k \cap I_{k'}}(x)$ is a function of only $x_{J_k \cap I_{k'}}$. This means $H_{J_k}(x) \equiv H_{J_k}(x_{J_k}, 0)$ is sectional, a contradiction.

Next, we show any $J_k (1 \leqq k \leqq \mu)$ and $I_{k'} (\mu' + 1 \leqq k' \leqq \nu')$ are disjoint, and any $J_k (\mu + 1 \leqq k \leqq \nu)$ and $I_{k'} (1 \leqq k' \leqq \mu')$ are also disjoint. We prove only the former one by contradiction, because the proof of the latter is the same. We may assume $J_k - I_{k'} \neq \phi$. In fact, if $J_k \subsetneqq I_{k'}$, $H_{I_{k'}}(x_{I_{k'}}, 0)$ must be sectional, a contradiction. And if $J_k = I_{k'}$, $Z_{J_k}(x) \equiv Z_{I_{k'}}(x)$ must satisfy $Z_{J_k}(x) \equiv 0$ and $Z_{I_{k'}}(x) \not\equiv 0$ at the same time, a contradiction. We may assume k' is the smallest that satisfies $J_k - I_{k'} \neq \phi$. (The above argument shows $k' \geqq \mu' + 1$.) The follows

$$J_k - (I_{k'} \cup I_{k'+1} \cup \cdots \cup I_{\nu'}) = \phi.$$

This implies $J_k \subseteqq I_{k'} \cup \cdots \cup I_{\nu'}$, therefore

$$\begin{aligned} H_{I_{k'} \cap J_k}(x_{I_{k'}}, 0) &\equiv H_{I_{k'} \cap J_k}(0, \dots, 0, x_{I_{k'}}, \dots, x_{I_{\nu'}}) \\ &\equiv H_{J_k \cap I_{k'}}(x) \\ &\equiv H_{J_k \cap I_{k'}}(x_{J_k}, 0) \end{aligned}$$

It means $H_{I_{k'} \cap J_k}(x_{I_{k'}}, 0) \equiv H_{I_{k'} \cap J_k}(x_{I_{k'} \cap J_k}, 0)$ which contradicts the non-sectionality of $H_{I_{k'}}(x_{I_{k'}}, 0)$. This contradiction shows each $J_k (1 \leqq k \leqq \mu)$ is disjoint from $I_{\mu'+1} \cup \cdots \cup I_{\nu'}$.

We have proved $J_1 \cup \cdots \cup J_\mu$ and $I_{\mu'+1} \cup \cdots \cup I_{\nu'}$ are disjoint, so are $I_1 \cup \cdots \cup I_{\mu'}$ and $J_{\mu+1} \cup \cdots \cup J_\nu$. This means $J_1 \cup \cdots \cup J_\mu = I_1 \cup \cdots \cup I_{\mu'}$. We proved also any $J_k (1 \leqq k \leqq \mu)$ and $I_{k'} (1 \leqq k' \leqq \mu')$ are identical or disjoint. Therefore $\{I_1, \dots, I_{\mu'}\}$ is only a permutation of $\{J_1, \dots, J_\mu\}$.

We have only to repeat the same procedure for

$$H_{J_{\mu+1} \cup \cdots \cup J_\nu}(0, \dots, 0, x_{J_{\mu+1}}, \dots, x_{J_\nu})$$

and so on. We have completed all the proof of the theorem 14.

Remark. From the theorem 14,

$$H_{J_k}(0, \dots, 0, x_{J_k}, x_{J_{k+1}}, \dots, x_{J_\nu}) \equiv H_{J_k}(x_{J_k}, 0)$$

Especially,

$$H_{J_k}(0, \dots, 0, 0, x_{J_{k+1}}, \dots, x_{J_\nu}) \equiv 0.$$

Example of cannonical expansion:

Consider the following docile homogeneous transformation.

$$H_1(x)=x_1^{1/2}x_2^{1/2}$$

$$H_2(x)=2x_1+x_2$$

$$H_3(x)=3x_1^{1/2}x_5^{1/2}+x_2+x_3$$

$$H_4(x)=(2x_1^2x_5+3x_4x_5^2)^{1/3}$$

$$H_5(x)=(x_3^2x_4+2x_4^3+x_5^3)^{1/3}.$$

Its cannonical partition of suffixes is

$$J_1=J_\mu=\{1, 2\},\quad J_2=J_{\mu+1}=\{3\},\quad J_3=J_\nu=\{4, 5\}.$$

We give two applications of cannonical expansion.

Theorem 15. *Let $H(x)$ be docile and its cannonical partition be*

$$J_1\cup\cdots\cup J_\nu=\{1,\dots, n\}.$$

Then the next formula holds.

$$\lambda_0(H)=\max_{1\leqq k\leqq\nu}\{\lambda_0(H_{J_k}(x_{J_k}, 0))\}.$$

Proof. For simplicity, we use the following notations.

$$\lambda_0(k)=\lambda_0(H_{J_k}(x_{J_k}, 0)),\quad k=1,\dots, \nu.$$

$$\bar\lambda_0=\max_{1\leqq k\leqq\nu}\{\lambda_0(k)\}.$$

Proof of $\lambda_0(H)\geqq\bar\lambda_0$. Let us take an arbitrary $\lambda<\bar\lambda_0$. Then there exists a certain k such that

$$\lambda<\lambda_0(k).$$

We denote by $\gamma_{J_k}>0$ the maximal eigen-vector of $H_{J_k}(x_{J_k}, 0)$. Setting $\gamma=(\gamma_{J_k}, 0)$,

$$H_j(\gamma)-\lambda\gamma_j=(\lambda_0(k)-\lambda)_j>0 \qquad \text{for all}\quad j\in J_k$$

$$H_j(\gamma)-\lambda\gamma_j=H_j(\gamma)\geqq 0 \qquad \text{for all}\quad j\notin J_k$$

From 2) of the theorem 6, we get

$$\lambda<\lambda_0(H).$$

Since λ is an arbitrary number smaller than $\bar\lambda_0$, we obtain

$$\bar\lambda_0\leqq\lambda_0(H).$$

Proof of $\lambda_0(H)\leqq\bar\lambda_0$. Let us take $\lambda>\bar\lambda_0$ arbitrarily. We shall construct a positive vector $\tilde\gamma>0$ such that

$$H(\tilde\gamma)-\lambda\tilde\gamma<0.$$

We denote by γ_{J_k} the maximal eigen-vector of $H_{J_k}(x_{J_k}, 0)$ for all k. In other words,

$$H_{J_k}(\gamma_{J_k}, 0)=\lambda_0(k)\gamma_{J_k} \quad \text{and} \quad \gamma_{J_k}>0. \tag{51}$$

First, we put, with $m_\nu=1$,

$$\bar{\gamma}(\nu)=(0, m_\nu\gamma_{J_\nu}). \tag{52}$$

Since $\lambda>\lambda_0(\nu)$,

$$H_{J_\nu}(\bar{\gamma}(\nu))-\lambda\bar{\gamma}_{J_\nu}(\nu)=m_\nu(\lambda_0(\nu)-\lambda)\gamma_{J_\nu}<0. \tag{53}$$

Recalling the remark of the theorem 14,

$$H_{J_k}(\bar{\gamma}(\nu))-\lambda\bar{\gamma}_{J_k}(\nu)=H_{J_k}(0, m_\nu\gamma_{J_\nu})-0=0. \tag{54}$$

for all $k<\nu$.

We go on to the second step.

$$\bar{\gamma}(\nu-1)=(0,\dots, 0, m_{\nu-1}\gamma_{J_{\nu-1}}, m_\nu\gamma_J) \tag{55}$$

with a sufficiently small $m_{\nu-1}>0$, (53) unchanged:

$$H_{J_\nu}(\bar{\gamma}(\nu-1))-\lambda\bar{\gamma}_{J_\nu}(\nu-1)<0. \tag{56}$$

In the same way as we obtained (53) and (54),

$$H_{J_{\nu-1}}(\bar{\gamma}(\nu-1))-\lambda\bar{\gamma}_{J_{\nu-1}}(\nu-1)<0, \tag{57}$$

$$H_{J_k}(\bar{\gamma}(\nu-1))-\lambda\bar{\gamma}_{J_k}(\nu-1)=0 \qquad \text{for all} \quad k<\nu-1. \tag{58}$$

We define $m_{\nu-2}, m_{\nu-3},\dots$ in the same way. Finally we are led to $\bar{\gamma}(1)=(m_1\gamma_{J_1}, m_2\gamma_{J_2},\dots, m_\nu\gamma_{J_\nu})$ such that

$$H_{J_k}(\bar{\gamma}(1))-\lambda\bar{\gamma}_{J_k}(1)<0 \qquad \text{for all} \quad k=1,\dots, \nu. \tag{59}$$

This means, from the lemma 6,

$$\lambda>\lambda_0(H).$$

Since λ is an arbitrary number greater than $\bar{\lambda}_0$, we get

$$\bar{\lambda}_0\geqq\lambda_0(H).$$

We have proved the theorem 15.

At the end of this paper, we give another application of cannonical expansion.

Theorem 16. *Let $H(x)$ be docile and expanded cannonically with*

$$J_1\cup\cdots\cup J_\mu\cup J_{\mu+1}\cup\cdots\cup J_\nu=\{1,\dots, n\}.$$

There exists a positive vector associated with $\lambda_0(H)$ if and only if

$$\lambda_0(H_{J_k}(x_{J_k}, 0))=\lambda_0(H) \qquad \textit{for all} \quad k=1,\dots, \mu,$$

$$\lambda_0(H_{J_k}(x_{J_k}, 0))<\lambda_0(H) \qquad \textit{for all} \quad k=\mu+1,\dots, \nu.$$

Proof of sufficiency. We use the same procedures as in the proof of the theorem 15 and of the theorem 8. So we write the proof here rather briefly.

Let $\gamma_{J_k}>0$ be the maximal eigen-vector of $H_{J_k}(x_{J_k}, 0)$ for each $k=1,\dots, \nu$. We construct a positive vector

$$\bar{\gamma}=(m_\mu\gamma_{J_1},\dots, m_\mu\gamma_{J_\mu}, m_{\mu+1}\gamma_{J_{\mu+1}},\dots, m_\nu\gamma_{J_\nu})$$

where we determine $m_\nu, m_{\nu-1},\dots, m_\mu$ inductively so small that it satisfies

$$(60)\qquad \begin{aligned} &H_j(\gamma)=\lambda_0(H)\gamma_j \qquad \text{for all} \quad j\in J_1\cup\cdots\cup J_\mu \\ &H_j(\gamma)<\lambda_0(H)\gamma_j \qquad \text{for all} \quad j\in J_{\mu+1}\cup\cdots\cup J_\nu. \end{aligned}$$

Next, we split every $J_k(\mu+1\leqq k\leqq \nu)$ into J'_k and J''_k. J'_k is all the suffixes $j\in J_k$ such that

$$H_j(x_{J_1},\dots, x_{J_{k-1}}, y_{J_k},\dots, y_{J_\nu}) \longrightarrow \infty$$

when every $x_j(j\in J_1\cup\cdots\cup J_{k-1})\to\infty$ and $y_{J_k}>0,\dots, y_{J_\nu}>0$ are fixed. And $J''_k=J_k-J'_k$. $J'_k(\mu+1\leqq k\leqq\nu)$ is not empty because $H_{J_k}(x)\not\equiv H(0,\dots, 0, x_{J_k},\dots, x_{J_\nu})$ means, from 2) of the definition 5, there exists at least one such $j\in J_k$.

Recalling the remark of the theorem 8, there exists $\gamma'_{J_k}>0(\mu+1\leqq k\leqq\nu)$ such that

$$H_{J''_k}(\gamma'_{J_k}, 0)>\lambda_0\gamma'_{J''_k}.$$

We construct a positive vector

$$\tilde{\gamma}=(M_\mu\gamma_{J_1},\dots, M_\mu\gamma_{J_\mu}, M_{\mu+1}\gamma'_{J_{\mu+1}},\dots, M_\nu\gamma'_{J_\nu})$$

where $M_\nu, M_{\nu-1},\dots, M_\mu$ are determined inductively so large that $\tilde{\gamma}$ satisfies

$$(61)\qquad \begin{aligned} &H_j(\tilde{\gamma})=\lambda_0\tilde{\gamma}_j \qquad \text{for all} \quad j\in J_1\cup\cdots\cup J_\mu \\ &H_j(\tilde{\gamma})>\lambda_0\tilde{\gamma}_j \qquad \text{for all} \quad j\in J_{\mu+1}\cup\cdots\cup J_\nu. \end{aligned}$$

After a necessary scalar multiplication, we may assume $\bar{\gamma}<\tilde{\gamma}$. Then, the map $\frac{1}{\lambda_0}H(x)$ leaves the interval

$$[\bar{\gamma}_1, \tilde{\gamma}_1]\times\cdots\times[\bar{\gamma}_n, \tilde{\gamma}_n]$$

invariant. Thus there exists a positive fixed point of this map which is a positive eigen-vector associated with $\lambda_0(H)$.

Proof of necessity. Let γ be a positive eigen-vector associated with $\lambda_0(H)$. Then

$$(62)\qquad H_{J_k}(\gamma_{J_k}, 0)=H_{J_k}(\gamma)=\lambda_0(H)\gamma_{J_k} \qquad \text{for all} \quad k=1,\dots, \mu.$$

$$(63)\qquad \begin{aligned} H_{J_k}(\gamma_{J_k}, 0)&=H_{J_k}(\gamma)+H_{J_k}(\gamma_{J_k}, 0)-H_{J_k}(\gamma) \\ &\leq\lambda_0\gamma_{J_k} \qquad \text{for all} \quad k=\mu+1,\dots, \nu \end{aligned}$$

since $H_{J_k}(\gamma_{J_k}, 0)\leq H_{J_k}(\gamma)(\mu+1\leqq k\leqq\nu)$ follows from the lemma 6.
From (62),

$$\lambda_0(H_{J_k}(x_{J_k}, 0)) = \lambda_0(H) \qquad \text{for all} \quad k = 1,\dots, \mu.$$

From (63), using the theorem 10,

$$\lambda_0(H_{J_k}(x_{J_k}, 0)) < \lambda_0(H) \qquad \text{for all} \quad k = \mu+1,\dots, \nu.$$

We have completed all the proof of the theorem 16.

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