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On G-linebundles and $K_G^{\cdot}(X)$

By

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In this paper we study the G-sheaves on an algebraic variety X endowed with the action of a finite reductive group G. We are mainly interested in the case where X is a smooth projective curve, but we start with a study of the group $\mathrm{Pic}_G X$ of G-linebundles for a general X, and its relation to $K_G^{\cdot}(X)$.

When X is a curve we prove an equivariant Riemann-Roch formula of the classical type (Thm. 5.8), with values in the representation ring of G. It is not valid for all G-linebundles, except if G operates freely on X. The same obstruction prevents the existence in general of a good Chern class for all G-linebundles with values in $K_G^{\cdot}(X)$.

This explains why the classical formula for $K^{\cdot}(X)$ does not have a straight forward analogue in the equivariant case, except when G operates freely (Prop. 5.1). For a general curve X we give an approximate description of $K_G^{\cdot}(X)$ (Thm. 5.12), which becomes more precise when the quotient curve X/G is $\boldsymbol{P}^1$ (Thm. 5.16). In order to bring about the precision we have to introduce the higher equivariant K-groups. Until that point the important technical ingredients were a careful study of the case of a finite variety and a thorough use of the Lefschetz trace formulas from [E-L].

At last we study two examples and give exact formulas for $\mathrm{Pic}_G X$ and $K_G^{\cdot}(X)$. The first one was suggested by P. Baum and is the origin of the paper. It deals with $X=\boldsymbol{P}^1$, G cyclic. The second one treats a hyperelliptic curve X and $G\simeq \boldsymbol{Z}/2$ is the group generated by the canonical involution. This case forced the study of K_1^G upon us, when all other tricks failed to provide the answers.

The first section recalls some general facts on G-actions, some of which may also be found in [Moo].

Section 2 studies the relationship between G-invariant divisors and G-line bundles, the main result being that there is a natural surjection $(\mathrm{Div}\, X)^G \to \mathrm{Pic}_G X$ under rather general assumptions (Prop. 2.9).

In section 3 we introduce equivariant determinants by copying the usual definition as in [M], but it turns out as mentioned above that there are no good equivariant first Chern classes for all G-line bundles with values in $K_G^{\cdot}(X)$. However, we have a reasonable additive map $\bar{c}_1\colon (\mathrm{Div}\, X)^G \to K_{\cdot}^G(X)$.

In section 4 we study $K_{\cdot}^{G}(Z)$ for a finite variety Z. We give an exact description of the category of G-sheaves when Z is reduced and identify in all cases $K_{\cdot}^{G}(Z)$ with a direct sum of representation rings of subgroups of G. It turns out that the Lefschetz trace in this case is described via induced representations.

Section 5 is devoted to smooth projective curves X. Set $Y=X/G$ and let $B\subset X$ and $\Delta\subset Y$ denote respectively the branch locus and the ramification locus for the map $X\to Y$ Let $\mathrm{Div}^{B}(X)^{G}$ denote the group of G-invariant divisors with support outside of B. They generate a subgroup $\mathrm{Pic}_{G}^{B}(X)$ of $\mathrm{Pic}_{G}(X)$.

When $B=\emptyset$ we show that $K_{G}^{\cdot}(X)\simeq \boldsymbol{Z}\oplus\mathrm{Pic}_{G}(X)$ (Prop. 5.1), the proof of which is standard and requires none of the machinery developed here. Then we turn to the case where $B\neq\emptyset$. First we prove a number of formulas in $K_{G}^{\cdot}(X)$, some of which are well-known when $G=(1)$, but only true under certain restrictions in our case. From these formulas we deduce the existence of an equivariant first Chern class $c_{1}^{B}\colon \mathrm{Pic}_{G}^{B}(X)\to K_{G}^{\cdot}(X)$ (Cor. 5.7), and an equivariant Riemann-Roch formula for G-linebundles in $\mathrm{Pic}_{G}^{B}(X)$ (Thm. 5.8).

The G-sheaves on X are almost pull-backs of coherent sheaves on Y. In Prop. 5.9 we show that the defect of being a pull-back is a torsion-sheaf with support in B which is a $\pi^{-1}(\Delta)$-Module. This settles the problem of the correct schemestructure on B for our purposes.

With these preparations we are ready to consider $K_{G}^{\cdot}(X)$. In Thm. 5.12 we define an additive surjection

$$\boldsymbol{Z}\oplus\mathrm{Pic}_{G}^{B}(X)\oplus K_{\cdot}^{G}(B)\longrightarrow K_{G}^{\cdot}(X),$$

where $K_{\cdot}^{G}(B)$ is split into a sum of representation rings of inertia subgroups of G. The map is an isomorphism if $B=\emptyset$. When $Y=X/G\simeq \boldsymbol{P}^{1}$ we show that the factor $\mathrm{Pic}_{G}^{B}(X)$ may be suppressed. Moreover, we prove that $K_{G}^{\cdot}(X)$ first into an exact sequence (Thm 5.16)

$$\boldsymbol{Z}^{r-1}\longrightarrow K_{\cdot}^{G}(B)\longrightarrow K_{\cdot}^{G}(X)\longrightarrow \boldsymbol{Z}\longrightarrow 0,$$

where $r=\mathrm{card}(\Delta)$, which is part of the localization sequence for the higher equivariant K-groups K_{i}^{G}. We do not go into any details with these. They should be defined the way done by Quillen and the localization sequence comes by categorical arguments just as in the usual case.

In the last section we study two kinds of examples. At first we assume that $X=\boldsymbol{P}^{1}$ and G is cyclic. Then we have isomorphisms

$$\mathrm{Pic}_{G}(\boldsymbol{P}^{1})\simeq G\oplus\boldsymbol{Z},\ K_{G}^{\cdot}(\boldsymbol{P}^{1})\simeq R_{k}(G)\oplus R_{k}(G),$$

where the last isomorphism may be chosen $R_{k}(G)$-linear.

Next we let X be hyperelliptic of genus g and $G\simeq \boldsymbol{Z}/2$ generated by the canonical involution. We obtain isomorphisms

$$\mathrm{Pic}_{G}(X)\simeq \boldsymbol{Z}\oplus\hat{G}\oplus{}_{2}J(X)\simeq\boldsymbol{Z}\oplus(\boldsymbol{Z}/2)^{2g+1},$$

$$K_{G}^{\cdot}(X)\simeq\boldsymbol{Z}^{2g+4},$$

where ${}_2J(X)$ denotes the group of points of order two on the jacobian of X.

Both types of examples show that the classical formula in Prop. 5.1 does not hold, when $B \neq \emptyset$.

Notations are standard and compatible with the ones in [E–L] with the following exceptions. Here $\hat{G}$ will denote the dual group of G, and the regular representation of G is denoted by χ_{reg} rather than $k[G]$. Since the greek letter χ is used in connection with representations, we have prefered to write rank-map instead of Euler-characteristic.

At this place we should like to thank a number of collegues in Japan for talks and discussions about these and related topics. Special thanks go to M. Maruyama for a correction to the original treatment of torsion sheaves, and also to the Carlsberg Science Foundation for making the visit to Japan financially possible.

1. *G*-linearization

Assume first that X is a noetherian scheme, G a finite group of automorphisms of X, and $\mathscr{F}$ a coherent sheaf on X. A *G-linearization of* $\mathscr{F}$ is given by a collection of $\mathcal{O}_X$-isomorphisms

$$\bar{g}: g_*\mathscr{F} \xrightarrow{\sim} \mathscr{F}, \quad g \in G,$$

subject to the conditions

$$\bar{1}_G = id_{\mathscr{F}};$$

$$\overline{hg} = \bar{h}_*(\bar{g}),\ g,\ h \in G.$$

(An equivalent definition comes by the consideration of the adjoint isomorphisms $\bar{\bar{g}}: \mathscr{F} \xrightarrow{\sim} g^*\mathscr{F}$, and the conditions $\bar{\bar{1}} = id$, $\overline{\overline{hg}} = g^*(\bar{\bar{h}})\bar{\bar{g}}$.)

We shall use the short hand *G-sheaf* for a G-linearized coherent sheaf. and the sentence "$\mathscr{F}$ is a G-sheaf" in the statements (1.1)–(1.7) below shall mean that $\mathscr{F}$ is provided with a natured G-linearization.

An $\mathcal{O}_X$-linear morphism $\mathscr{F} \to \mathscr{G}$ between G-sheaves is called a *G-morphism*, if it commutes with the action of G. Similarly we use notions such as *G-resolutions*, *G-exact sequences*, etc., to mean resolutions, exact sequences, etc., in which all involved morphisms are G-morphisms.

From the description of G-linearization the following facts are immediate.

(1.1) If $\mathscr{F}$ and $\mathscr{G}$ are G-sheaves then $\mathscr{F} \oplus \mathscr{G}$, $\mathscr{F} \otimes \mathscr{G}$, $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ and $\wedge^r \mathscr{F}$ are all G-sheaves.

(1.2) If $\mathscr{L}$ is a G-line bundle, then so is $\mathscr{L}^{-1}$.

(1.3) Let $\mathscr{F}$ be a G-sheaf and $\mathscr{G} \subset \mathscr{F}$ a coherent subsheef. Then $\mathscr{G}$ is a G-sheaf and the inclusion map $\mathscr{G} \subset \mathscr{F}$ a G-morphism if and only if $\mathscr{G}$ is stable under the action of G. Moreover, if this holds, then $\mathscr{F}/\mathscr{G}$ is a G-sheaf and the natural morphism $\mathscr{F} \to \mathscr{F}/\mathscr{G}$ is a G-morphism.

(1.4) Conversely, if $\varphi: \mathscr{F} \to \mathscr{H}$ is an (epimorphic) G-morphism between G-sheaves, then Ker φ is G-stable. (In this way the category of G-sheaves becomes abelian, and the kernels, images etc. are the usual ones endowed with canonical G-linearization.)

(1.5) In particular, a closed subscheme $Y \subset X$ admits a G-action such that the inclusion map is G-equivariant if and only if the ideal sheaf of Y is G-stable in $\mathcal{O}_X$.

(1.6) Let $\mathcal{E}$ and $\mathcal{F}$ denote locally free G-sheaves of rank r and s. Then the usual formulas

$$\wedge^n(\mathcal{E} \oplus \mathcal{F}) = \bigoplus_{i=0}^{n} (\wedge_i \mathcal{E} \otimes \wedge^{n-i} \mathcal{F});$$

$$\wedge^{rs}(\mathcal{E} \otimes \mathcal{F}) = (\wedge^r \mathcal{E})^s \otimes (\wedge^s \mathcal{F})^r$$

hold between G-sheaves. In particular, if $r=1$ then

$$\wedge^s(\mathcal{E} \otimes \mathcal{F}) = \mathcal{E}^{\otimes s} \otimes \wedge^s \mathcal{F},$$

which also holds when $\mathcal{F}$ is just coherent.

(1.7) Let $0 \to \mathcal{E}_r \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to 0$ be a G-exact sequence of locally free G-sheaves. Set $d_i = rk\, \mathcal{E}_i$. Then one has a canonical G-isomorphism

$$\bigotimes_{i=1}^{r} (\wedge^{d_i} \mathcal{E}_i)^{(-1)^i} \simeq \mathcal{O}_X.$$

(The proofs of (1.6) and (1.7) follow the patterns in the case without a G-action, cf. [M, p. 104].)

Assume now that X is quasi-projective over a field k and that $G \subseteq \mathrm{Aut}_k X$. As mentioned in [E–L] it follows from Kambayashi's theorem that there exists a very ample G-line bundle $\mathcal{O}_X(1)$ on X. In this case every G-sheaf $\mathcal{F}$ has a G-resolution by locally free G-sheaves. Namely, set $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$, for all $n \in \boldsymbol{Z}$, with the induced G-action. For $n \gg 0$, $\mathcal{F}(n)$ is generated by its global sections, and the exact sequence

$$\mathcal{O}_X \otimes_k \Gamma(X, \mathcal{F}(n)) \longrightarrow \mathcal{F}(n) \longrightarrow 0$$

is automatically a G-sequence, if $\Gamma(X, \mathcal{F}(n))$ is endowed with the canonical G-action. Twisting by $\mathcal{O}_X(-n)$ yields an exact G-sequence

$$\mathcal{O}_X(-n) \otimes_k \Gamma(X, \mathcal{F}(n)) \xrightarrow{\varphi} \mathcal{F} \longrightarrow 0,$$

and $\mathrm{Ker}\, \varphi$ is a G-sheaf by (1.4). We replace $\mathcal{F}$ by $\mathrm{Ker}(\varphi)$ and go on as above, which produces the claimed G-resolution of $\mathcal{F}$.

When X is also regular, the resolution may be chosen of length $\leqq \dim X$, by (1.4).

If instead we impose the condition on X that it be a regular, affine noetherian scheme, then we have a similar situation with $\mathcal{O}_X(1)$ replaced by (the ample sheaf) $\mathcal{O}_X$ itself. This may be applied to, say, the case where $\Gamma(X, \mathcal{O}_X)$ is a (localization of a) ring of integers in a number field. We intend to come back to this case at another occasion.

Just as in the case without a group G one has the exact sequence

$$K_\cdot^G(Y) \longrightarrow K_\cdot^G(X) \longrightarrow K_\cdot^G(U) \longrightarrow 0, \tag{1.8}$$

valid for a closed G-invariant subscheme Y of a noetherian scheme X with complement U, and a finite reductive group G. The usual proof carries over.

Here one may choose the reduced structure on Y, because if J is any G-invariant nilpotent ideal of $\mathcal{O}_Y$, defining a subscheme Y', then the G-morphism $Y' \to Y$ induces a bijection

$$K_\cdot^G(Y) \xrightarrow{\sim} K_\cdot^G(Y'). \tag{1.9}$$

Namely, every G-$\mathcal{O}_Y$-sheaf has a finite filtration by G-$\mathcal{O}_Y$-subsheaves, whose successive quotients are G-$\mathcal{O}_{Y'}$-sheaves. One concludes by general non-sense.

2. *G*-divisors and *G*-line bundles

At first we need no particular assumptions on X nor on G, so let X be a noetherian scheme. It follows from (1.1) and (1.2) that the set of isomorphisms classes of G-line bundles on X forms an abelian group that we shall denote by $\mathrm{Pic}_G(X)$. This makes Pic_G into a contravariant functor for G-equivariant morphisms. One has the forgetful homomorphism

$$\beta\colon \mathrm{Pic}_G X \longrightarrow \mathrm{Pic}\, X, \tag{2.1}$$

whose image consists of the line bundles that admit some G-linearization. $\mathrm{Ker}(\beta)$ may be interpreted as the group of G-linearizations of $\mathcal{O}_X$. One has a canonical isomorphism

$$\mathrm{Ker}\,\beta \simeq \mathrm{Hom}_{gr}(G, \Gamma(X, \mathcal{O}_X)^*). \tag{2.2}$$

So, if X is proper and geometrically connected over a field k and $G \subseteq \mathrm{Aut}_k X$, then one has

$$\mathrm{Ker}\,\beta \simeq \hat{G}, \tag{2.3}$$

where $\hat{G} \simeq \mathrm{Hom}_{gr}(G, k^*)$ is the character group (or dual group). In particular, in this case any two G-linearizations of a given line bundle on X differ by a character on G.

Let us return to the general assumptions on X and G. The group acts on the group of Cartier divisors on X, $\mathrm{Div}(X)$, and on $\mathrm{Pic}\, X$. Denote the invariant subgroups by $(\mathrm{Div}\, X)^G$ and $(\mathrm{Pic}\, X)^G$. The elements of $(\mathrm{Div}\, X)^G$ will be called G-divisors.

Let $D \in \mathrm{Div}\, X$ have the local equation l_x at a point $x \in X$. Then $D \in (\mathrm{Div}\, X)^G$ if and only if $g_*(l_x)$ is a local equation for D at $g(x)$, for all $x \in X$ and all $g \in G$. Now, G acts on the (non-coherent) sheaf of rational functions K_X, and realizing $\mathcal{O}(D)$ as a subsheaf of K_X we see that if $D \in (\mathrm{Div}\,(X))^G$, then $\mathcal{O}(D)$ is G-invariant. Therefore $\mathcal{O}(D)$ has a natural G-linearization. In this way we obtain a homomorphism

$$\alpha\colon (\mathrm{Div}\, X)^G \longrightarrow \mathrm{Pic}_G X \tag{2.4}$$

that, composed with β in (2.1) yields a homomorphism

$$\gamma\colon (\mathrm{Div}\ X)^G \longrightarrow (\mathrm{Pic}\ X)^G. \tag{2.5}$$

For a positive G-divisor D associated with a codimension one subscheme D of X, $\alpha(D)=\mathcal{O}(D)$, with the linearization compatible with the inclusion $\mathcal{O}(-D)\subset\mathcal{O}_X$. We shall now study the maps α and γ.

Proposition 2.6. *Assume that X is proper and irreducible over an algebraically closed field k and that $G\to\mathrm{Aut}_k X$ is reductive and has order n. Then one has*

(i) $\mathrm{Coker}(\gamma)$ *is a finite group with n-torsion.*

(ii) *If G is cyclic, then γ is surjective.*

Proof. Let K denote the field of rational functions on X. We have an exact sequence

$$(\mathrm{Div}\ X)^G \xrightarrow[\gamma]{} (\mathrm{Pic}\ X)^G \xrightarrow[\delta]{} H^1(G, K^*/k^*),$$

which is part of long exact sequence of group cohomology associated with the exact sequence of G-modules, $0\to K^*/k^*\to\mathrm{Div}\ X\to\mathrm{Pic}\ X\to 0$. From the exact sequence $1\to k^*\to K^*\to K^*/k^*\to 1$ we obtain $H^1(G, K^*/k^*)\subset H^2(G, k^*)$, since $H^1(G, k^*)=0$ by Hilbert's Satz 90. As n is prime to char (k), we have another exact sequence $0\to\mu_n\to k^*\to k^*\to 0$, where $\mu_n\simeq Z/n$ is the group of n'th root of unity in k^*. This implies that $H^2(G, k^*)$ is a subgroup of $H^3(G, Z/n)$. Multiplication by n on the latter group is zero, and it is finite since Z/n is finitely generated abelian. This proves (i).

If G is cyclic, that is, $G\simeq Z/n$, then $H^2(G, k^*)\simeq k^*/(k^*)^n=(1)$, so $H^1(G, K^*/k^*)=0$. This proves (ii).

Remark 2.7. We do not know if γ is always surjective, but A. Thorup has shown us that $H^3(Z/p\times Z/p, Z)\neq 0$. In this way one may produce an example with $H^1(G, K^*/k^*)\neq 0$.

We shall now turn to the map α.

Lemma 2.8. *Assume that X is proper and geometrically irreducible over a field k. Then $\mathrm{Im}(\alpha)$ contains $\hat{G}$.*

Proof. Let K denote the field of rational functions on X, and let V be a 1-dimensional representation of G over k. Then $\mathcal{O}_X\otimes_k V$ belongs to $\mathrm{Im}(\alpha)$, if we can find a rational function $f\in K\backslash k$ such that $k\cdot f$ is a G-invariant k-subspace of K isomorphic to V. From the exact sequence of G-modules $0\to k^*\to K^*\to K^*/k^*\to 0$ we deduce an exact sequence

$$0\longrightarrow k^*\longrightarrow (K^*)^G\longrightarrow (K^*/k^*)^G\xrightarrow[\delta]{}\hat{G}\longrightarrow 0,$$

where we have used that $(k^*)^G=k^*$, $\hat{G}=H^1(G, k^*)$ and $H^1(G, K^*)=0$. Since δ is surjective we may pick an $f\in K^*$ whose image in $(K^*/k^*)^G$ maps onto V, and this f can be used above.

It is plausible that α is surjective under the hypothesis in Lemma 2.8. However, we have only been able to show the following somewhat weaker result.

Proposition 2.9. *Assume that X is a projective irreducible and reduced variety over an algebraically closed field k, and that $G \subseteq \mathrm{Aut}_k X$ is reductive. Then the map α in (2.4) is surjective. Moreover,* $\mathrm{Ker}\,\alpha = (K^*)^G/k^*$.

Proof. Let $Y = X/G$ denote the quotient variety and $\pi\colon X \to Y$ the canonical morphism. Notice first, that under the sole assumption that X be quasi-projective, there exists a very ample G-line bundle in $\mathrm{Im}(\alpha)$. Namely, pick any very ample G-line bundle $\mathscr{L}$ on X. Then $\bigotimes_{g\in G} g^*\mathscr{L}$ will do. A different way to see this is to pick any very ample line bundle $\mathcal{O}_Y(1)$ on Y. Then $\pi^*\mathcal{O}_Y(n)$ can be used for $n \gg 0$. Since $\mathcal{O}_Y(n)$ is very ample on Y for $n > 0$, we may thus assume that $\mathcal{O}_X(1) = \pi^*\mathcal{O}_Y(1) \in \mathrm{Im}(\alpha)$.

For an arbitrary $\mathscr{L} \in \mathrm{Pic}_G X$ we claim that $\mathscr{L}(n)$ has a G-invariant global section $s \neq 0$ if $n \gg 0$. Assume this for a moment. Then we obtain a G-injection $\mathcal{O}_X \to \mathscr{L}(n)$ which, tensored by $\mathscr{L}^{-1}(-n)$ yields an exact G-sequence

$$0 \longrightarrow \mathscr{L}^{-1}(-n) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

where $D \in (\mathrm{Div}\,X)^G$. Hence, $\mathscr{L}(n) = \alpha(D)$. Since $\mathrm{Im}(\alpha)$ is a subgroup of $\mathrm{Pic}_G X$ we conclude that $\mathscr{L} \in \mathrm{Im}(\alpha)$.

The existence of $s \in H^0(X, \mathscr{L}(n))\backslash\{0\}$ follows from the formulas for the Lefschetz trace in [E–L]. When X is a smooth curve, we may use the formula in [E–L, Thm. 3.8] which gives

$$L_{G,X}(\mathscr{L}) = \frac{1}{|G|}\cdot \deg \mathscr{L} \cdot \chi_{reg} + A,$$

where χ_{reg} denotes the regular representation of G and A is a universally *bounded* combinations of the irreducible representations of G. Replacing $\mathscr{L}$ by $\mathscr{L}(n)$ gives us $\deg \mathscr{L} + n\cdot\deg \mathcal{O}_X(1)$ instead of $\deg \mathscr{L}$. Hence, the trivial 1-dimensional representation of G must occur for $n \gg 0$.

For general X there is no similar ready formula in [E–L], but we shall depart from the much cruder one, ([E–L, Lemma 2.2]). It tells the following. Let $\mathscr{F}$ be a G-sheaf on X. Define the functor π_*^G by $\pi_*^G(\mathscr{F}) = \pi_*(\mathscr{F})^G$. Let V denote an irreducible G-representation. The V occurs $\chi(\pi_*^G(\mathscr{F} \otimes_k \check{V}))$ times in $L_{G,X}(\mathscr{F})$. Here χ is the usual Euler-Poincré characteristic. Our V is the trivial 1-dimensional representation, $\mathscr{F} = \mathscr{L}(n)$ and we have $\pi_*^G(\mathscr{L}(n)) = \pi_*(\mathscr{L})(n)^G = \pi_*^G(\mathscr{L})(n)$. Since $\mathcal{O}_X(1)$ and $\mathcal{O}_Y(1)$ are very ample. $L_{G,X}(\mathscr{L}(n))$ reduces to $[H^0(X, \mathscr{L}(n))]$ and $\chi(\pi_*^G\mathscr{L}(n))$ to $\dim_k H^0(Y, \pi_*^G(\mathscr{L})(n))$, for $n \gg 0$. So, we have to show that $H^0(Y, \pi_*^G(\mathscr{L})(n)) \neq 0$ for $n \gg 0$. Since $\pi_*^G(\mathscr{L}) \neq 0$, this again follows from the ampleness of $\mathcal{O}_Y(1)$.

The assertion about $\mathrm{Ker}\,\alpha$ is left as an exercise.

The following general lemma is useful in computations.

Lemma 2.10. *Assume that X is an integral k-scheme of finite type and that $G \subset \mathrm{Aut}_k X$ acts admissibly on X. Denote the quotient scheme X/G by Y and let*

$\pi: X \to Y$ be the canonical morphism. Then one has a commutative diagram

$$\begin{array}{ccc} \operatorname{Div} Y & \longrightarrow & (\operatorname{Div} X)^G \\ \downarrow & & \downarrow \alpha \\ \operatorname{Pic} Y & \longrightarrow & \operatorname{Pic}_G X, \end{array}$$

where the non-defined maps are the obvious ones.

Proof. The diagram is first of all commutative up to the action of G on the elements in $\operatorname{Pic}_G X$. Now, two divisors on Y are linearly equivalent if they differ by $\operatorname{div}(f)$, where f is a rational function on Y. Their inverse images in $(\operatorname{Div} X)^G$ then differ by $\operatorname{div}(\pi^* f)$, and $\pi^* f$ is a G-invariant rational function on X. Therefore they have the same image in $\operatorname{Pig}_G X$.

3. Equivariant determinants and Chern classes

In this section X will be a smooth projective connected variety defined over an algebraically closed field k, and we shall assume that $G \subset \operatorname{Aut}_k X$. Set $d = \dim X$.

Lemma 3.1. *Every G-sheaf $\mathscr{F}$ on X has a G-resolution*

$$0 \longrightarrow \mathscr{E}_r \longrightarrow \cdots \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{E}_0 \longrightarrow \mathscr{F} \longrightarrow 0,$$

where the $\mathscr{E}_i$ are locally free G-sheaves and $r \leq d$.

Proof. See section 1.

This lemma is the main point in the proof of the isomorphism $K^{\cdot}_G(X) \xrightarrow{\sim} K^G_{\cdot}(X)$, providing the inverse map.

We shall now construct a homomorphism

$$\operatorname{det}_G : K^{\cdot}_G(X) \longrightarrow \operatorname{Pic}_G(X), \tag{3.2}$$

that we call the *equivariant determinant.*

Let $\mathscr{F}$ be a G-sheaf and let

$$0 \longrightarrow \mathscr{E}_r \longrightarrow \cdots \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{E}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$

be a G-resolution of $\mathscr{F}$ by locally free G-sheaves $\mathscr{E}_i$ of rank d_i. We set

$$\operatorname{det}_G(\mathscr{F}) = (\wedge^{d_0}\mathscr{E}_0) \otimes (\wedge^{d_1}\mathscr{E}_1)^{-1} \otimes \cdots \otimes (\wedge^{d_r}\mathscr{E}_r)^{(-1)^r},$$

where the linearization is provided by (1.1). The independence of the choice of the resolution is proved easily. Let us indicate the case where one has $r = 1$. Let $0 \to \mathscr{E}'_1 \to \mathscr{E}'_0 \to \mathscr{F} \to 0$ be another G-resolution of $\mathscr{F}$. Construct the following G-diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathscr{E}_0 & = & \mathscr{E}_0 & \longrightarrow & 0 & & \\
 & & \uparrow{\scriptstyle\beta} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{E}_2 & \longrightarrow & \mathscr{E}_0 \oplus \mathscr{E}_0' & \xrightarrow{\alpha} & \mathscr{E} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{E}_1' & \longrightarrow & \mathscr{E}_0' & \longrightarrow & \mathscr{E} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The unexplained maps are the obvious ones. The map α is the sum of the given ones. $\mathscr{E}_2=\operatorname{Ker}(\alpha)$ and β is defined as a composition. It follows from the diagram that one has G-isomorphisms $\Lambda^{\max}(\mathscr{E}_2)=\Lambda^{\max}(\mathscr{E}_0)\otimes\Lambda^{\max}(\mathscr{E}_1')$. $\Lambda^{\max}(\mathscr{E}_0\oplus\mathscr{E}_0')\simeq\Lambda^{\max}\mathscr{E}_0\otimes\Lambda^{\max}\mathscr{E}_0'$ (use 1.6) and (1.7). So, finally

$$\Lambda^{\max}(\mathscr{E}_0\oplus\mathscr{E}_0')\otimes\Lambda^{\max}(\mathscr{E}_2)^{-1}\simeq(\Lambda^{\max}\mathscr{E}_0')\otimes(\Lambda^{\max}\mathscr{E}_1')^{-1}$$

In this formula we may excange $\mathscr{E}_0'$ and $\mathscr{E}_1'$ by $\mathscr{E}_0$ and $\mathscr{E}_1$, with the same $\mathscr{E}_2$, which proves the assertion about $\det_G(\mathscr{F})$.

Similar arguments show that $\det_G$ is additive on short exact G-sequences. Consequently it extends to a homomorphism as written in (3.2).

We shall now pass to the first Chern class. Since X is smooth every $D\in\operatorname{Div}X$ has a unique representation as a Weil divisor $D=\sum n_iW_i$. For $g\in G$ one has $g^*D=\sum n_ig^*W_i$, so $D\in(\operatorname{Div}X)^G$ if and only if D may be written

$$D=\sum n_jD_j,\ n_j\in Z, \tag{3.3}$$

where each $D_j=\sum_{g\in G/H}g^*W_i$, for some closed irreducible subvariety W_i of codimension 1, with decomposition subgroup H. Call a divisor of the type D_j an *orbit divisor.*

The decomposition in (3.3) is unique. The support of an orbit divisor D_j will be denoted by the same letter. It is a G-invariant subscheme of X. Hence, the structure sheaf $\mathcal{O}_{D_j}$ is a G-sheaf. We define a map

$$\bar{c}_1:(\operatorname{Div}X)^G\longrightarrow K_G^{\cdot}(X) \tag{3.4}$$

by setting $\bar{c}_1(D)=\sum n_j[\mathcal{O}_{D_j}]$. It follows that $\bar{c}_1$ is a homomorphism.

Lemma 3.5. *The composition* $\det_G\circ\bar{c}_1:(\operatorname{Div}X)^G\to\operatorname{Pic}_G X$ *coincides with the canonical map* α.

Proof. It suffices to prove $\alpha(D)=\det_G(\bar{c}_1(D))$ for an orbit divisor D. This follows immediately from the definitions and the exact sequence $0\to\mathcal{O}(-D)\to\mathcal{O}_X\to\mathcal{O}_D\to 0$.

In case $G=(1)$ the map $\bar{c}_1$ vanishes on $\operatorname{Ker}\alpha$, hence factors through a homomorphism $c_1:\operatorname{Pic}X\to K^{\cdot}(X)$. Furthermore, for all effective divisors D and D', one has $\bar{c}_1(D+D')=[\mathcal{O}_{D+D'}]$. This is no longer true for general G. The last equality only holds for orbit divisors whose support is disjoint from the fixed point loci of the elements in G.

We shall pursue the study of this problem only when X is a curve, in the following section. Here we finish by noting that one has an additive rank-homomorphism $rk\colon K_G^{\bullet}(X)\to \boldsymbol{Z}$ with an inverse that takes 1 onto $[\mathcal{O}_X]$. In this way $\boldsymbol{Z}$ embeds into $K_G^{\bullet}(X)$, and we have

$$(3.6)\qquad \boldsymbol{Z}\cap \bar{c}_1((\operatorname{Div} X)^G)=(0).$$

4. $K_{\bullet}^G(Z)$ for a finite k-variety Z.

Let k be an algebraically closed field and let Z be a k-variety whose underlying topological space $|Z|$ is a nonempty finite discrete set of points. Assume that the subgroup $G\subseteq \operatorname{Aut}_k Z$ is reductive.

Now, $|Z|$ may be written as a disjoint union of G-orbits

$$|Z|=|Z_1|\cup\cdots\cup|Z_r|,$$

where each Z_i denotes a closed subset of Z with the reduced scheme-structure. Then Z_i has a natural G-action. Pick a point $P_i\in Z_i$ in each orbit and denote the inertia subgroup of G at P_i by H_i.

Theorem 4.1. *Notations as above.*

a) *The category of G-$\mathcal{O}_{Z_i}$-sheaves is naturally equivalent to the category of $k[H_i]$-modules. In particular, one has a canonical isomorphism*

$$K_{\bullet}^G(Z_i)\xrightarrow{\sim} R_k(H_i).$$

b) *If $H_i=(1)$, then every G-$\mathcal{O}_{Z_i}$-sheaf is of the form $V\otimes_k\mathcal{O}_{Z_i}$, where V is a finite-dimensional trivial k-representation of G. In this case the isomorphism in (a) becomes $K_{\bullet}^G(Z_i)\xrightarrow{\sim}\boldsymbol{Z}$, and it is induced by the map $V\to\dim V$.*

c) *Under the isomorphism in (a) the Lefschetz trace on $K_{\bullet}^G(Z_i)$ turns into the induced representation map*

$$R_k(H_i)\xrightarrow[\text{ind}]{} R_k(G).$$

d) *The embedding $Z_1\cup\cdots\cup Z_r\subset Z$ induces an isomorphism*

$$K_{\bullet}^G(Z)\xrightarrow{\sim} R_k(H_1)\oplus\cdots\oplus R_k(H_r).$$

Proof. In the proof of (a) we assume $Z=Z_i$ in order to simplify the notation. Set $P=P_i$ and $H=H_i$. The choice of P defines an identification $|Z|=G/H$ and Z is the spectrum of a k-algebra A, isomorphic to $k^{G/H}$ as a G-k-algebra, where G acts trivially on k and by translation on the index set G/H.

To any given G-$\mathcal{O}_Z$-sheaf we may associate its fiber V over P. Then V is a $k[H]$-module. This is one way of the equivalence. Conversely, let V be any finite-dimensional $k[H]$-module. We define an equivalence relation $\sim$ in $V\times G$ by setting

$$(\xi, g) \sim (h(\xi), gh^{-1}), \quad \text{all} \quad h \in H.$$

Set $\tilde{V} = (V \times G)/\sim$. Then we define a map $\varphi: \tilde{V} \to G/H$ by $cl(\xi, g) \to cl(g)$. Let E denote the set of sections to φ. Then E has a natural G-A-module structure; so it corresponds to a G-$\mathcal{O}_Z$-sheaf. This yields the other way of the equivalence.

If $H = (1)$, then $\tilde{V} = V \times G/H$ above, with the obvious G-action. This implies (b).

The Lefschetz trace of a G-$\mathcal{O}_Z$-sheaf corresponding to a G-A-module E, is E itself viewed as a $k[G]$-module, which is just the induced representation $ind_H^G(V)$ in the notation above. This proves (c).

Finally, (d) is an immediate consequence of (a) and the isomorphism (1.9).

Note that (a) in the theorem above is a discrete analogue of the characterization of homogeneous vector bundles over a homogeneous space. (We are greatful for a convincing discussion with N. Vigand Petersen on this point.)

It may also be viewed as a special semi-local version of the computation of $K_{\cdot}^G$ for a regular local k-algebra, which follows from [E–L, Lemma 3.4].

5. On the structure of $K_G^{\cdot}$ for curves

Throughout this section we assume that X is a smooth connected projective curve defined over an algebraically closed field k, and that $G \subseteq \mathrm{Aut}_k(X)$ is reductive. Then we let $Y = X/G$ be the quotient curve and $\pi: X \to Y$ be the canonical morphism. The branch locus of π is denoted by B. Here $B \subset X$ is a finite set of points. The scheme structure will be settled later.

First consider the case when B is empty.

Proposition 5.1. *Assume that G operates freely on X. Then one has an isomorphism*

$$K_G^{\cdot}(X) \simeq Z \oplus \mathrm{Pic}_G X.$$

Proof. We shall not need the previous computations. It is a general fact, that when G operates freely, then the morphism π is étale. Hence it induces an equivalence between the category of all coherent sheaves on Y and the category of coherent G-sheaves on X. In particular one gets an isomorphism $K_G^{\cdot}(X) \simeq K^{\cdot}(Y)$. Since we have $K.(Y) \simeq Z \oplus \mathrm{Pic}\, Y$ and $\mathrm{Pic}\, Y \simeq \mathrm{Pic}_G X$ by the equivalence just mentioned, the assertion follows.

By inspection of a few commutative diagrams it follows that the isomorphism above is given by rk $\times \det_G$, and the inverse is based upon an equivariant Chern class $c_1: \mathrm{Pic}_G X \to K_G^{\cdot}(X)$.

Let us return to the case when B may be non-empty. Then we have a disjoint decomposition

$$B = B_1 \cup \cdots \cup B_r,$$

where the B_i are the G-orbits on X consisting of strictly less than $|G|$ points.

Proposition 5.2. *Notations as above. Let B have any schemestructure which makes it a closed G-invariant k-subscheme of X. Pick points $P_1 \in B_1, \ldots, P_r \in B_r$ and denote the inertia groups at these points by $H_1, \ldots, H_r$. They are all non-trivial cyclic groups and we have a canonical isomorphism*

$$K^G_\cdot(B) \xrightarrow{\sim} R_k(H_1) \oplus \cdots \oplus R_k(H_r).$$

Proof. B satisfies the conditions of Z in Thm. 4.1.

Definition 5.3. Define the subgroups $\mathrm{Div}^B(X) \subset \mathrm{Div}(X)$ and $\mathrm{Pic}^B_G(X) \subset \mathrm{Pic}_G(X)$ as

$$\mathrm{Div}^B(X) = \{D \in \mathrm{Div}(X) \mid \mathrm{supp}(D) \cap B = \emptyset\};$$

$$\mathrm{Pic}^B_G|(X) = \alpha(\mathrm{Div}^B(X)^G),$$

and denote the restrictions of $\bar{c}_1$ and α to $\mathrm{Div}^B(X)$ by $\bar{c}^B_1$ and α^B.

We proceed with some technical lemmas on computations in $K^\cdot_G(X)$. We identify Z with $Z[\mathcal{O}_X]$ in $K^\cdot_G(X)$.

Lemma 5.4. *Let D denote a G-orbit on X, endowed with the reduced scheme-structure, and also the associated G-divisor, and let H be the inertiagroup of some point P in D. Let χ be the representation of H on the tangentline to X at P. Then we have*

(1) *The map of $K^G_\cdot(D) \simeq R_k(H)$ into $K^\cdot_G(X)$ induced by the inclusion $D \subset X$ is injective.*

(2) *For any natural number n the equality*

$$[\mathcal{O}_{nD}] = ind^G_H\Big(\sum_{l=0}^{n-1} \chi^l\Big) [\mathcal{O}_D]$$

holds in $K^\cdot_G(X)$.

(3) *If $H = (1)$ then $[\mathcal{O}_{nD}] = n[\mathcal{O}_D]$.*

Proof. As $D \subset X$ is a closed embedding, the Lefschetz trace L_G comments with the induced map $K^G_\cdot(D) \to K^G_\cdot(X)$. By Thm. 4.1(c), L_G is injective on $K^G_\cdot(D)$. This implies (1).

Let $D_{(n)}$ denote D with the scheme-structure defined by the ideal $\mathcal{O}_X(-nD)$. We may replace D by $D_{(n)}$ in the considerations above. Thus, it suffices to prove the equality in (2) after applying L_G to both sides. Hence, the equality follows if we can show the equality

$$[\mathcal{O}_{nD}] = \Big(\sum_{l=0}^{n-1} \chi^l\Big) \cdot [\mathcal{O}_D]$$

in $K^G_\cdot(D_{(n)})$. This one is an easy computation based on the filtration $\mathcal{O}(-nD) \subset \mathcal{O}(-(n-1)D) \subset \cdots \subset \mathcal{O}(-D) \subset \mathcal{O}_X$.

The assertion (3) is a special case of (2).

Lemma 5.5. *Let E, $F \in \mathrm{Div}(X)^G$ be two effective divisors with* $\operatorname{supp} E \cap \operatorname{supp} F = \emptyset$. *Then one has a G-isomorphism*

$$\mathcal{O}_{D+E} \simeq \mathcal{O}_D \oplus \mathcal{O}_E.$$

Proof. Localizing around $\operatorname{supp}(E+F)$ we may replace X by a semi-local principal ideal domain A with a G-action. The G-$\mathcal{O}_X$-ideals $\mathcal{O}(-E)$, $\mathcal{O}(-F)$ and $\mathcal{O}(-E-F)$ correspond to G-invariant ideals I, J and $I\cdot J$ in A. The claim follows because the canonical map

$$A \xrightarrow[\varphi]{} A/I \oplus A/J$$

is surjective and has kernel $I\cdot J = I \cap J$.

Lemma 5.6. *Let $\mathcal{L}$, $\mathcal{M} \in \mathrm{Pic}_G(X)$ and let $D \in \mathrm{Div}^B(X)^G$ be an effective divisor. Let $l = [\mathcal{L}]$, $m = [\mathcal{M}]$ in $K_G^{\cdot}(X)$. Then we have*

$$l\cdot[\mathcal{O}_D] = [\mathcal{O}_D]. \tag{1}$$

Assume furthermore that either $\mathcal{M} \in \mathrm{Pic}_G^B(X)$ or else that $\mathcal{L} = \mathcal{O}_X(E)$, $\mathcal{M} = \mathcal{O}_X(F)$, where E, $F \in \mathrm{Div}(X)^G$ satisfy $\operatorname{supp} E \cap \operatorname{supp} F = \emptyset$. *Then*

$$(l-1)(m-1) = 0, \tag{2}$$

Moreover, if $E \in \mathrm{Div}^B(X)^G$, we have

$$\tilde{c}_1^B(E) = [\alpha^B(E)] - 1. \tag{3}$$

Proof. By lemma 5.5 and lemma 5.4 (3) we may assume that D is an orbit divisor in (1). Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ is a free G-$\mathcal{O}_D$-module of rank one and by Thm. 4.1 (b) we have $[\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_D] = [\mathcal{O}_D]$ in $K_{\cdot}^G(D)$. Now push forward to $K_G^{\cdot}(X)$.

Assume now that (2) has been proved when $\mathcal{L}$ and $\mathcal{M}$ are both associated with effective divisors. In the general case, we then have $l = l_1 l_2^{-1}$, $m = m_1 m_2^{-1}$, where $(l_i - 1)(m_j - 1) = 0$. Then

$$\begin{aligned}(l-1)(m-1) &= (l_1 - l_2)(m_1 - m_2) l_2^{-1} m_2^{-1} \\ &= ((l_1 - 1) - (l_2 - 1))((m_1 - 1) - (m_2 - 1)) l_2^{-1} m_2^{-1} \\ &= 0.\end{aligned}$$

Remains the effective case. We have $\mathcal{L} = \mathcal{O}_X(E)$, $\mathcal{M} = \mathcal{O}_X(F)$, with effective G-divisors E and F. Assume first that $F \in \mathrm{Div}_G^B(X)$. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{O}_F \longrightarrow 0$$

which, tensorized with $\mathcal{L}$, becomes

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{M} \longrightarrow \mathcal{L} \otimes \mathcal{M} \otimes \mathcal{O}_F \longrightarrow 0.$$

This implies, by (1),

$$l\cdot m = l + [\mathcal{O}_F] = l + m - 1,$$

which is equivalent to (2).

Assume next that $\operatorname{supp} E \cap \operatorname{supp} F = \emptyset$. By lemma 5.5 we get $[\mathcal{O}_{D+E}] = [\mathcal{O}_D] + [\mathcal{O}_E]$. Exact sequences, similar to the ones above, translate this into $1 - l^{-1}m^{-1} = 1 - l^{-1} + 1 - m^{-1}$, which again is equivalent to (2).

In order to prove (3) we write $E = E_+ - E_-$ with $E_+, E_- \in \operatorname{Div}^B(X)^G$ both effective. From lemma 5.5 and lemma 5.4 (3) we get

$$\bar{c}_1^B(E) = [\mathcal{O}_{E_+}] - [\mathcal{O}_{E_-}] = (1 - l_+^{-1}) - (1 - l_-^{-1}) = l_-^{-1} - l_+^{-1}$$

where we have set $l_\pm = [\alpha^B(E_\pm)]$. So, we must show that $l_-^{-1} - l_+^{-1} = l_+ l_-^{-1} - 1$. This is equivalent to $(l_+ - 1)(l_-^{-1} - l_+^{-1}) = 0$, which follows from (2) above.

Corollary 5.7. *The map $\bar{c}_1^B$ in Def. 5.3 factors through $\operatorname{Pic}_G(X)$ and gives rise to an injective homomorphism*

$$c_1^B : \operatorname{Pic}_G^B(X) \longrightarrow K_G^{\cdot}(X),$$

which may be thought of as an equivariant Chern class.

Proof. It follows from lemma 5.4 (3) and lemma 5.5 that $\det_G \circ \bar{c}_1^B = \alpha^B$. Hence it suffices to prove that $\bar{c}_1^B$ vanishes on $\operatorname{Ker}(\alpha^B)$. This follows from lemma 5.6 (3).

A first application of the computations above will be an equivariant Riemann-Roch formula.

Theorem 5.8. *Let $\mathcal{L} \in \operatorname{Pic}_G^B(X)$. Then*

$$L_G(\mathcal{L}) = L_G(\mathcal{O}_X) + \deg(\mathcal{L}).$$

Proof. Let $\mathcal{L} = \alpha^B(D)$, $D \in \operatorname{Div}^B(X)$. If D is effective we have $[\mathcal{O}_D] = [\mathcal{L}][\mathcal{O}_D] = [\mathcal{L}] - 1$. Now, $L_G(\mathcal{O}_D) = \deg D$, by lemma 5.4 (3), lemma 5.5 and Thm. 4.1 (b). So, the claim follows in this case.

In general, $D = D_+ - D_-$ with D_+ and D_- both effective. Set $l_\pm = \alpha^B(D_\pm)$. From lemma 5.6 (2) we get $0 = (l_+ - 1)(1 - l_-^{-1})$, or, $l_+ l_-^{-1} = l_+ - (1 - l_-^{-1}) = l_+ - [\mathcal{O}_{D_-}]$. Apply L_G and use the previous case.

Proposition 5.9. *For all G-$\mathcal{O}_X$-sheaves $\mathcal{F}$, the quotient $\mathcal{R} = \mathcal{F}/\pi^* \pi_* \mathcal{E}$ is a $\pi^{-1}(\Delta)$-Module, where $\Delta \subset Y$ is the ramification locus for π, with the reduced scheme structure, and $\pi^{-1}(\Delta)$ is the scheme-theoretical inverse image of Δ.*

Proof. $\mathcal{F}$ above has a 2-step G-resolution by locally free G-sheaves. The functor π^* is exact because π is flat, and π_*^G is exact because π is affine and G is reductive. Therefore it suffices to prove the lemma in the case when $\mathcal{F}$ is locally free.

Let $P \in X$ be a branch point for π and set $Q = \pi(P)$. The inertiagroup G_P at P is isomorphic to Z/e, where e is the ramification index at P. The claim in the proposition is that R_P is a $\mathcal{O}_{P,X}/m_{P,X}^e$-module, that is $\operatorname{Ann}(R_P) \supset m_{P,X}^e$. We shall give a proof of this which is very close to the proof of the Lefschetz formula (3.7) in [E–L].

Since R_P has finite lenght, we may complete with respect to the $m_{Q,Y}$-adic topology. So, let $A = \hat{\mathcal{O}}_{Q,Y}$, $B = \hat{\mathcal{O}}_{P,X}$ and $C = \mathcal{O}_X \otimes_{\mathcal{O}_Y} A$. Then C is a complete semi-local ring isomorphic to $\Pi\hat{\mathcal{O}}_{P',X}$, when P' runs through $\pi^{-1}(Q)$. Also, C has a natural G-action, B has a G_P-action, and $B^{G_P}=C^G=A$. Moreover, $\mathscr{F}$ gives rise to a free G-C-module F, $\mathscr{R}$ to a G-C-module R, and the exact sequence

$$0 \longrightarrow \pi^*\pi_*^G\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow \mathscr{R} \longrightarrow 0$$

corresponds to an exact sequence

$$0 \longrightarrow F^G \underset{A}{\otimes} C \longrightarrow F \longrightarrow R \longrightarrow 0. \tag{5.10}$$

Localizing (5.10) at P we get

$$0 \longrightarrow F^G \underset{A}{\otimes} C \underset{C}{\otimes} B \longrightarrow F \underset{C}{\otimes} B \longrightarrow R_B \longrightarrow 0. \tag{5.11}$$

Here $F^G \underset{A}{\otimes} C \underset{C}{\otimes} B = F^G \underset{A}{\otimes} B \simeq (F \underset{C}{\otimes} B)^{G_P} \underset{A}{\otimes} B$. Picking a uniformizing parameter u in B we have an isomorphism $B \simeq k[[u]]$. We choose u so that a generator σ of G_P acts on u by multiplication with an r-th root of unity ζ. The inclusion $A \subset B$ identifies A with $k[[u^e]]$. By Lemma 3.4 in [E–L] the free G_P-B-module $F \underset{C}{\otimes} B$ is isomorphic to $B \otimes V$, for a uniquely determined $k[G_P]$-module V, and the sequence (5.11) may be rewritten

$$0 \longrightarrow (B \underset{k}{\otimes} V)^{G_P} \underset{A}{\otimes} B \longrightarrow B \underset{k}{\otimes} V \longrightarrow R_B \longrightarrow 0.$$

As G_P is cyclic, V decomposes into a direct sum of 1-dimensional G_P-modules, and R_B decomposes accordingly. In our proof we may, and shall, therefore assume that $\dim V = 1$, that is, $B \underset{k}{\otimes} V = k[[u]] \cdot \xi$, where ξ is a k-basis for V. The action of G_P is given by

$$\sigma(u) = \zeta \cdot u, \quad \sigma(\xi) = \zeta^r \cdot \xi, \quad 0 \le r < e.$$

An easy computation shows that $(B \underset{k}{\otimes} V)^{G_P} = u^{e-r} k[[u]] \cdot \xi$. We have to show the inclusion

$$u^e \cdot B \underset{k}{\otimes} V \subset (B \underset{k}{\otimes} V)^{G_P} \underset{A}{\otimes} B.$$

This holds, even with u^e replaced by u^{e-r}, as we see from the above. This ends the proof.

Theorem 5.12. *Notations as above. Let the subgroups $H_1, \dots, H_r$ of G be defined as in* Prop. 5.2. *Then one has an additive surjection*

$$Z \oplus \mathrm{Pic}_G^B(X) \oplus R_k(H_1) \oplus \cdots \oplus R_k(H_r) \underset{\varphi}{\longrightarrow} K_G^{\cdot}(X).$$

Proof. We define ψ by defining its restriction to the various factors. Set $\psi(1) = [\mathcal{O}_X]$, $\psi \mid \mathrm{Pic}_G^B(X) = c_1^B$, and the restriction to the representation rings is defined by the isomorphism in Prop. 5.2 and the covariance of $K_{\cdot}^G$ for the (proper) inclusion $B \subset X$. Here B has the scheme-structure $\pi^{-1}(\Delta)$ as in Prop. 5.9.

For $\mathscr{L} \in \mathrm{Pic}_G^B(X)$ one has $[\mathscr{L}] = c_1^B(\mathscr{L}) + [\mathcal{O}_X]$ by lemma 5.6 (3), so we only need to show that $K_G^{\cdot}(X)$ is generated by the elements of the following two types

(a) $[\mathscr{L}]$, $\mathscr{L}\in \mathrm{Pic}_G^B(X)$.

(b) $[\mathscr{M}]$, $\mathscr{M}$ a G-$\mathscr{O}_{\pi^{-1}(\Delta)}$-sheaf.

First of all $K_G^{\cdot}(X)$ is generated by locally free G-$\mathscr{O}_X$-sheaves, and by Prop. 5.9 it suffices to take sheaves of the form $\pi^*\mathscr{E}$, with $\mathscr{E}$ locally free on Y, together with (b). On the curve Y every $\mathscr{E}$ is a successive extension of invertible sheaves. Therefore, since π is flat, it suffices to take $\pi^*\mathscr{E}$ with $\mathscr{E}$ invertible. Let $D, E\in \mathrm{Div}(Y)$ satisfy $(\mathrm{supp}\, D)\cap \Delta=\emptyset$, $\mathrm{supp}\, E\subseteq \Delta$, $\mathscr{E}=\mathscr{O}_Y(D+E)$. Set $l=[\pi^*(\mathscr{O}_Y(D))]$ and $m=[\pi^*(\mathscr{O}_Y(E)]$. By lemma 2.10 $\pi^*(\mathscr{O}_Y(D))$ belongs to $\mathrm{Pic}_G^B(X)$, so by lemma 5.6 (2) we get $[\pi^*(\mathscr{E})]=l\cdot m=l+m-1$. Write E as a difference between effective divisors, $E=E_+-E_-$, both with support in Δ. Then we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_Y(E) & \longrightarrow & \mathscr{O}_Y(E_+) & \longrightarrow & \mathscr{O}_Y(E_+)\otimes_{\mathscr{O}_Y}\mathscr{O}_{E_-} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathscr{N}_1 & & \\ 0 & \longrightarrow & \mathscr{O}_Y & \longrightarrow & \mathscr{O}_Y(E_+) & \longrightarrow & \mathscr{O}_Y(E_+)\otimes_{\mathscr{O}_Y}\mathscr{O}_{E_+} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & \mathscr{N}_2 & & \end{array}$$

Using the flatness of π we deduce

$$m=1+[\pi^*(\mathscr{N}_2)]-[\pi^*(\mathscr{N}_1)], \tag{5.13}$$

so

$$[\pi^*(\mathscr{E})]=l+[\pi^*(\mathscr{N}_2)]-[\pi^*(\mathscr{N}_1)].$$

Now, $l=[\mathscr{L}]$ for $\mathscr{L}\in \mathrm{Pic}_G^B(X)$ and $\pi^*(\mathscr{N}_i)$ are $\pi^{-1}(\Delta)$-modules. This finishes the proof.

Using the rank-map on $K_G^{\cdot}(X)$ it is easy to see that $\mathrm{Ker}(\psi)\subseteq \mathrm{Pic}_G^B(X)\oplus K_{\cdot}^G(B)$, but we have not been able to write an explicit useful formula for it. When $Y=\boldsymbol{P}^1$ the situation simplifies, and we shall see below that the introduction of the higher K_i^G shed more light on the situation. The reason is that (1.8) becomes right-split, but this does not hold in general.

Proposition 5.14. *Notations as above. Assume that $G\neq(1)$ and that the quotient curve $Y=\boldsymbol{P}^1$. Then one has a surjection*

$$Z\oplus R_k(H_1)\oplus\cdots\oplus R_k(H_r)\twoheadrightarrow K_G^{\cdot}(X).$$

Proof. We just need to show that the factor $\mathrm{Pic}_G^B(X)$ may be suppressed in Thm. 5.12. We saw in the proof of Thm. 5.12 that $K_G^{\cdot}(X)$ is generated by elements of the form $[\pi^*(\mathscr{E})]$, with $\mathscr{E}$ invertible on Y, and $[\mathscr{M}]$, with $\pi^{-1}(\Delta)$-sheaves $\mathscr{M}$. Since $Y=\boldsymbol{P}^1$, $G\neq(1)$ we have $\Delta\neq\emptyset$. So every $\mathscr{E}$ is of the form $\mathscr{O}_Y(E)$, with $\mathrm{supp}\, E\subset\Delta$. Then $[\pi^*(\mathscr{E})]$ takes the form of m in (5.13), which proves our claim.

The result above may be viewed from a different angle. Set $U=X-B$, $V=$

$P^1-\Delta$. Denote the higher equivariant K-groups by K_i^G, defined in Quillen's way. Since the map $U\to V$ is finite étale, we have isomorphisms $K_i(V)\simeq K_i^G(U)$, $i\geq 0$ (cf. the proof of Prop. 5.1). The sequence (1.8) is part of the localization sequence for the K_i^G. The functor π^* gives rise to a commutative digram of abelian groups

$$\begin{array}{ccccccccc} \cdots K_1^G(U) & \longrightarrow & K_\cdot^G(B) & \longrightarrow & K_\cdot^G(X) & \longrightarrow & K_\cdot^G(U) & \longrightarrow & 0 \\ \| & & \uparrow & & \uparrow & & \| & & \\ \cdots K_1(V) & \longrightarrow & K_\cdot(\Delta) & \longrightarrow & K_\cdot(Y) & \longrightarrow & K_\cdot(V) & \longrightarrow & 0 \end{array} \tag{5.15}$$

(This holds for any Y, not just when $Y=P^1$.) Since $\mathrm{Pic}(V)=0$ we have $K_\cdot(V)\simeq Z$, the isomorphism being given by the rank-map. So is the composed map $K_\cdot^G(X)\to Z$ deduced from (5.15). Let π: $K_\cdot^G(U)\to K_\cdot^G(B)$ denote the map in (5.15). Then we have

$$\operatorname{Ker}\psi=\operatorname{Im}(\pi)$$

in Prop. 5.14. The ramification locus Δ consists of r points, so we have $V=A^1-\{Q_1,\dots,Q_{r-1}\}$. Using the localization sequence in usual K-theory one easily proves by induction on r that $K_1(V)\simeq k^*\oplus Z^{r-1}$. The factor k^* is the image by the determinant map and also sits inside $K_1(Y)$. Therefore it goes to zero in $K_\cdot(\Delta)$ and in $K_\cdot^G(B)$. In this way we obtain the following amplification of Prop. 5.14.

Theorem 5.16. *Notations and assumptions as above. There is an exact sequence of abelian groups.*

$$Z^{r-1}\longrightarrow R_k(H_1)\oplus\cdots\oplus R_k(H_r)\longrightarrow K_\cdot^G(X)\longrightarrow Z\longrightarrow 0.$$

6. Examples

Consider first the case $X=P^1$, G a finite cyclic reductive group. We shall prove that the map ψ in Prop. 5.14 fits into a split-exact sequence of abelian groups

$$0\longrightarrow Z\xrightarrow[\nu]{} Z\oplus R(G)\oplus R(G)\xrightarrow[\psi]{} K_G(P^1)\longrightarrow 0,$$

with $\nu(Z)=Z\cdot(\chi_{reg},\ -\chi_{reg})$. Consequently, there is an abstract isomorphism of abelian groups

$$R(G)\oplus R(G)\underset{\psi}{\simeq} K_G^{\cdot}(P^1) \tag{6.1}$$

We shall show that ψ may be chosen (almost canonically) to be $R(G)$-linear.

By the result in section 2 the map β in (2.1) yields an isomorphism $\mathrm{Pic}_G(P^1)\simeq G\oplus Z$. Therefore we only have $K_G^{\cdot}(P^1)\simeq Z\oplus\mathrm{Pic}_G(P^1)$ if $G=(1)$.

In order to prove the assertions above we choose a generator σ for G. It has two fixed points on P^1 and we select coordinates so they become 0 and ∞. Then σ acts on the affine coordinate by multiplication with a primitive n'th root of unity, $n=|G|$. Denote the corresponding character on G by χ_0. The G-line bundles $\mathcal{O}(\{0\})$ and $\mathcal{O}(\{\infty\})$ are both $\mathcal{O}(1)$, but with different G-actions. With the help of the exact sequences

$$0 \longrightarrow \mathcal{O}(-x) \longrightarrow \mathcal{O}_{P^1} \longrightarrow \mathcal{O}_{\{x\}} \longrightarrow 0,$$

for $x=0$ and $x=\infty$, and the Thm. 4.1 (a), one checks that $\mathcal{O}(\{\infty\})=\chi_0\otimes_k\mathcal{O}(\{0\})$

Now let $(\xi, \eta)\in R(G)\oplus R(G)$ and assume that

$$\xi\cdot[\mathcal{O}_{\{0\}}]+\eta\cdot[\mathcal{O}_{\{\infty\}}]=0.$$

Using the above this may be rewritten as

$$\xi+\eta=[\xi+\eta\cdot\chi_0^{-1})\cdot[\mathcal{O}(-\{0\})].$$

Since $\mathcal{O}(-1)$ has no cohomology and $L_G(\xi\cdot 1)=\xi$, we deduce that $\xi+\eta=0$. Furthermore $[\mathcal{O}(-\{0\})]$ is invertible, so $\xi+\eta\cdot\chi_0^{-1}=0$. From this it follows that $\xi=-\eta=m\cdot\chi_{reg}$, for some $m\in Z$. This proves the inclusion Ker $(\psi)\subseteq Z\cdot(\chi_{reg}, -\chi_{reg})$, and the other inclusion is immediate.

Let us now define the map ψ in (6.1) by letting its restriction to the first factor be the ringhomomorphism corresponding to the structure map $P^1\to\{pt\}$, and the other restriction be equal to the restriction of ψ to either of the two copies of $R(G)$. *F.ex.* $\psi(\xi, \eta)=\xi\cdot 1+\eta\cdot[\mathcal{O}_{\{0\}}]$. Then it is easy to see that ψ is surjective and since both sides are finite free Z-modules of the same rank, ψ is an isomorphism.

Next we assume char$(k)\neq 2$ and consider a hyperelliptic curve X of genus $g\geq 2$. Let G be the group of order two generated by the canonical involution σ. Then $Y=P^1$ and the branch points $P_1,\ldots, P_{2g+2}\in X$ for $\pi\colon X\to Y$ are the hyperelliptic points of X (c.f. [L–K]). Let us first compute $\mathrm{Pic}_G X$ and $(\mathrm{Pic}\, X)^G$. Since G is cyclic and $\hat{G}\simeq Z/2$ it follows from section 2 that we have an exact sequence

$$0\longrightarrow Z/2\longrightarrow \mathrm{Pic}_G X\longrightarrow (\mathrm{Pic}\, X)^G\longrightarrow 0.$$

For any $P\in X$ we may consider the divisor $P+\sigma(P)$ on X, which is independent of P. As $\alpha\colon(\mathrm{Div}\, X)^G\to\mathrm{Pic}_G X$ is surjective it follows that $\mathrm{Pic}_G X$ is generated by $\mathcal{O}_X(P_1),\ldots, \mathcal{O}_X(P_{2g+2})$. The same is true for $(\mathrm{Pic}\, X)^G$, forgetting the G-action. Fix a hyperelliptic point, say P_1. Then $(\mathrm{Pic}\, X)^G$ is generated by $\mathcal{O}_X(P_1)$, $\mathcal{O}_X(P_1-P_i)$, $i=2,\ldots, 2g+2$. But the subgroup of Pic X generated by $\mathcal{O}_X(P_1-P_i)$, all i, is isomorphic to the group of points of order 2 on the jacobian of X, ${}_2J(X)$, and one has ${}_2J(X)\simeq(Z/2)^{2g}$. These considerations provide a splitting of the exact sequence

$$0\longrightarrow {}_2J(X)\longrightarrow(\mathrm{Pic}\, X)^G\xrightarrow[\deg]{} Z\longrightarrow 0.$$

Hence $(\mathrm{Pic}\, X)^G\simeq Z\oplus(Z/2)^{2g}$.

The group $\mathrm{Pic}_G(X)$ is an extension of $(\mathrm{Pic}\, X)^G$ by $\hat{G}=Z/2$, which is trivial, since every $\mathcal{L}\in\mathrm{Pic}_G X$ is a G-subsheaf of the sheaf of rational functions on X. Therefore we have $\mathrm{Pic}_G X\simeq Z\oplus(Z/2)^{2g+1}$.

At each point P_i the inertiagroup H_i equals G. We have $R_k(H_i)=Z\cdot e_i\oplus Z\cdot e_i^\sigma$, where e_i (resp. e_i^σ) is the 1-dimensional trivial (resp. non-trivial) representation of H_i. Set $A=R_k(H_1)\oplus\cdots\oplus R_k(H_{2g+2})$. Then e_1, $e_1^\sigma,\ldots, e_{2g+2}$, $e_{2g+2}{}^\sigma$ is a Z-basis for A. Set $\mathcal{L}_0=[\mathcal{O}_X(P+\sigma P)]$, for $P\in X$. Then $[\mathcal{O}_{P+\sigma P}]=1-[\mathcal{L}_0^{-1}]$ is independent of P, and by lemma 5.4 (2) we see that

$$[\mathcal{O}_{2P_i}]=(1+\chi_\sigma)[\mathcal{O}_{P_i}]$$

is independent of the index i. Consequently, $\psi(e_i+e_i^\sigma)=\psi(e_j+e_j^\sigma)$, all i, j. Let $A'\subseteq A$ denote the subgroup generated by the elements

$$e_i+e_i^\sigma-e_j-e_j^\sigma,\ i, j=1,\dots, 2g+2.$$

It is immediate to check that A' is a free direct summand of A of rank $2g+1$. The exact sequence in Thm. 5.16 is

$$Z^{2g+1}\xrightarrow[\rho]{} A \xrightarrow[\psi]{} K^{\cdot}_G(X)\longrightarrow Z\longrightarrow 0,$$

and we have seen that $A'\subseteq \operatorname{Im}(\rho)$. It follows easily that we have an equality, $A'=\operatorname{Im}(\rho)$, so that

$$K^{\cdot}_G(X)\simeq Z^{2g+4} \tag{6.2}$$

As we have seen earlier the map $\psi\colon A\to K^G_{\cdot}(X)$ is $R_k(G)$-linear and so is the one $Z\to K^G_{\cdot}(X)$, $1\to[\mathcal{O}_X]$. However, we do not know whether A' is a $R_k(G)$-submodule of A, so we do not know whether the isomorphism (6.2) can be made $R_k(G)$-linear with a suitable structure on the right-hand side. Of course, this would follow if all the K^G_i were $R_k(G)$-modules and the connecting homomorphism $R_k(G)$-linear.

It seems worth while noting, that because of the G-action, one classical relation has disappeared. Namely, for any numbering of the hyperelliptic points one has a linear equivalence

$$P_1+\cdots+P_{g+1}\sim P_{g+2}+\cdots+P_{2g+2}.$$

In $K_{\cdot}(X)$ this translate into the equality

$$[\mathcal{O}_{P_1}]+\cdots+[\mathcal{O}_{P_{g+1}}]=[\mathcal{O}_{P_{g+2}}]+\cdots+[\mathcal{O}_{P_{2g+2}}].$$

However, in $K^G_{\cdot}(X)$ the relation becomes

$$[\mathcal{O}_{P_1}]+\cdots+[\mathcal{O}_{P_{g+1}}]=\chi_\sigma([\mathcal{O}_{P_{g+2}}]+\cdots+[\mathcal{O}_{P_{2g+2}}])+1-\chi_\sigma,$$

which is nonlinear. The induced linear relation is a consequence of the ones previously considered.

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