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On the existence of transversal homoclinic points of some real analytic plane transformation

By

Masaru MORINAKA

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0. Introduction

In this paper we shall consider the following real analytic plane transformation f:

$$f(x, y)=(y+\xi(x), x),$$

$$f^{-1}(x, y)=(y, x-\xi(y)),$$

where ξ is a real analytic function belonging to X which will be define in Section 5 below. X is not vacant. For example $\xi(x)=cx(1-x)(c>0)$ is in X.

The differential of f is

$$df_{(x,y)}=\begin{pmatrix} \xi'(x) & 1 \\ 1 & 0 \end{pmatrix}$$

(x_0, y_0) is a fixed point if and only if $x_0=y_0$ and $\xi(x_0)=0$. Also, it is hyperbolic if and only if $\xi'(x_0)\neq 0$.

By the parallel transformation along the line $x=y$, its form is invariant ($\xi(x)$ tends to $\xi(x-a)$).

We assume $0=(0, 0)$ to be a hyperbolic fixed point. The eigenvalue λ of df_0 satisfies following equation:

$$\lambda^2-\xi'(0)\lambda-1=0.$$

Let s be the reflection with respect to the line $x+y=1$:

$$s(x, y)=(1-y, 1-x).$$

Conjugating f by s, we get

$$sfs^{-1}=(y, x-\xi(1-y)).$$

The identity $sfs^{-1}=f^{-1}$ holds if and only if ξ satisfies the following functional equation:

$$\xi(1-x)=\xi(x).$$

Under this symmetricity, we consider the heteroclinic point $g(t_0^+)$ (resp. $g(t_0^-)$) on the line $x+y=1$ between unstable (resp. stable) manifold g through (0, 0) and stable (resp. unstable) manifold $s\cdot g$ through (1, 1). (See S. Ushiki [2].)

In the case $\xi(x)=cx(1-x)$, we shall show that each of two heteroclinic points $g(t_0^+)$ and $g(t_0^-)$ is transversal, if $|c|>2\sqrt{2}$. Two transversal heteroclinic points imply transversal homoclinic points. The heteroclinic point $g(t_0^+)$ is always transversal except for isolated values of $c\neq 0$.

Remark. This transformation f is obtained by central difference scheme of the following ordinary differential equation:

$$\frac{dx}{dt}=\xi(x).$$

1. Stable (unstable)manifold theorem

Next is a definition of stable and unstable manifold (See S. Smale [3], C. L. Siegel [5] and S. Ushiki [2]).

Definition 1-1. There exists a real analytic embedding $g=(\alpha, \beta)$ from line to plane such that

$$g(0)=0,$$

$$g(\lambda t)=f(g(t)),$$

where λ is an eigenvalue of df_0. And g is uniquely determined by $\alpha'(0)$. We say that g is stable (resp. unstable) if $|\lambda|<1$ (resp. $|\lambda|>1$).

The pair (α, β) satisfies the following equations:

$$\alpha(0)=0,$$

$$\alpha(\lambda t)-\alpha(t/\lambda)=\xi(\alpha(t)),$$

$$\beta(t)=\alpha(t/\lambda).$$

Let expand α and ξ into the power series:

$$\alpha(t)=\sum_{n=1}^{\infty}\alpha_n t^n,$$

$$\xi(x)=\sum_{n=1}^{\infty}\xi_n x^n.$$

Combining the previous relations and comparing the coefficients of t, we obtain

$$(\lambda^2-\xi_1\lambda-1)\alpha_1=0.$$

Looking at the coefficient of t^n ($n\geq 2$), we get

$$\alpha_n = \frac{\lambda}{\lambda^{2n} - \xi_1 \lambda^n - 1} \sum_{2 \le m \le n} \xi_m \Big(\sum_{i_1 + \cdots + i_m = n} \alpha_{i_1} \cdots \alpha_{i_m} \Big).$$

It is clear that α_n is uniquely determined by $\alpha_1, \ldots, \alpha_{n-1}$.

It is easy to check the convergency of α when ξ is convergent.

Local existence of g implies global existence of g, because of the functional equation. It is trivial that g is embedding.

Remark 1. If $\lambda > 0$, we have a majorant function:

$$\bar{\alpha}(t) = \frac{t}{1 - rt} \text{ and}$$

$$\bar{\xi}(x) = \frac{x}{1 - r(\lambda - 1)x} - \frac{1}{\lambda} \frac{x}{1 - r(1/\lambda - 1)x},$$

where r is sufficiently large.
(Since $\bar{\alpha}$ is a linear fractional function, $\bar{\xi}$ exists.)

Remark 2. In the case $\xi(x) = cx(1 - x)$, we can show that the function $\alpha(t)$ ($t < 0$, $\lambda > 0$) is monotone by using induction on α_n.

2. Function α.

λ and α have been defined in Section 1.

Theorem 2–1. *Let I be an interval including 0 on which ξ is monotone and convex (resp. concave). If $\lambda > 1$, then α is monotone and convex (resp. concave) on $\lambda \cdot J$ and if $0 < \lambda < 1$, then α is monotone concave (resp. convex) on $\frac{1}{\lambda} J$ where J is the connected component of 0 in $\alpha^{-1}(I)$. If λ is negative, then α is monotone on K where K is some connected interval including 0 such that I does not contain $\alpha(K)$.*

Remark. In this paper, we say that v is monotone (resp. convex, concave) if v is strictly monotone (resp. convex, concave), namely $v''(x) \ne 0$ (resp. $v''(x) > 0$, $v''(x) < 0$) on some interval.

Proof. We can derive the next equation from the functional equation of α in the proof of theorem 1–1,

$$\lambda \alpha'(\lambda t) - \frac{1}{\lambda} \alpha'(t/\lambda) = \alpha'(t) \xi'(\alpha(t)).$$

If $\lambda > 1$, then $\xi'(x) > 0$ on I. From the above functional equation, $\alpha'(t/\lambda) \gtrless 0$ and $\alpha'(t) \gtrless 0$ imply $\alpha'(\lambda t) \gtrless 0$. Then α is monotone on λJ.

We can prove the monotonicity of α in the cases $0 < \lambda < 1$, $-1 < \lambda < 0$ and $\lambda < -1$ by using similar methods.

From the functional equation of α', we obtain

$$\lambda^2 \alpha''(\lambda t) - \frac{1}{\lambda^2} \alpha''(t/\lambda) = \alpha''(t) \xi'(\alpha(t)) + (\alpha'(t))^2 \xi''(\alpha(t)).$$

Let $t=0$ and $\xi'(0)=\lambda-\frac{1}{\lambda}$, then

$$\{(\lambda-1/2)^2-(1/\lambda-1/2)^2\}\alpha''(0)=(\alpha'(0))^2\xi''(0).$$

If $\lambda>1$, then $\xi'(x)>0$ on I and $\alpha''(0)$ has the same sign with $\xi''(0)$. From the above functional equation of α'', we conclude that $\alpha''(t/\lambda)\gtrless 0$, $\alpha''(t)\gtrless 0$ and $\xi''(x)\gtrless 0$ imply $\alpha''(\lambda t)\gtrless 0$. Then $\alpha''(t)\gtrless 0$ on λJ. We can prove the same things in the case $0<\lambda<1$ by the same way.

3. Zero of γ

From now on, we suppose that $\alpha'(0)=1$.

We define a function γ follows:

$$\gamma(t)=\alpha(t)+\beta(t)-1=\alpha(t)+\alpha(t/\lambda)-1.$$

Remark 1. α, β and γ are dependent on ξ and on sign of λ.

Theorem 3–1. *We assume λ to be positive. If $\xi(x)$ is positive on* (0, 1), *then $\gamma(t)$ has a zero on* $(0, \infty)$.

Remark 2. Considering f^{-1} instead of f, we can obtain another sufficient condition.

Proof. See [2]. Consider the region:

$$\{(x, y)|x>0,\ y>0 \quad \text{and} \quad x+y<1\},$$

and the function $x+y$ with respect of f, then $g(t>0)$ intersects the line $x+y=1$.

Theorem 3–2. *We assume λ to be negative. If $\xi(x)$ is positive on* (0, 1) *and $\xi(x)$ is negative on* $(-\infty, 0)$ *and* $\xi'(x)<1$ *on* $[1, \infty)$, *then $\gamma(t)$ has zero's on* $(-\infty, 0)$ *and on* $(0, \infty)$.

Proof. First consider the region:

$$\{(x, y)\,|\,x<0 \quad \text{and} \quad 0<y<1\}$$

and the functions x and y with respect to f^{-2}:

$$f^{-2}=(x-\xi(y),\ y-\xi(x-\xi(y))).$$

Then $g(t<0)$ intersects the line $y=1$. Next consider the region:

$$\{(x, y)\,|\,x+y-\xi(y)<1 \quad \text{and} \quad y>1\}$$

and the functions x and y with respect to f^{-2}. Then $g(t<0)$ intersects the curve $x+y-\xi(y)=1$. This curve is the image of the line $x+y=1$ by f. So $g(t>0)$ intersects the line $x+y=1$.

We can show that $g(t<0)$ intersects the line $x+y=1$ in the same way.

Remark. Let t_0 be a zero of γ. If ξ satisfies the functional equation $\xi(x)=\xi(1-x)$, then $g(t_0)$ is a heteroclinic point between (0, 0) and (1, 1).

4. The value $\delta(t_0^+)$ and $\delta(t_0^-)$.

Let t_0^+ (resp. t_0^-) be the minimum (resp. maximum) zero of γ on $(0, \infty)$ (resp. $-\infty, 0)$). This is well defined because $\gamma(0)=-1$.

We introduce a function δ as follows:

$$\delta(t)=\frac{\beta'(t)}{\alpha'(t)}=\frac{1}{\lambda}\frac{\alpha'(t/\lambda)}{\alpha'(t)},$$

where δ means the inclination of stable (resp. unstable) curve at point $g(t)$. From the functional equation of α', δ satisfies

$$\frac{1}{\delta(\lambda t)}-\delta(t)=\xi'(\alpha(t)).$$

Theorem 4–1. *We assume λ to be positive. If ξ is a concave monotone increasing function on $I=[0, 1/2)$ and there exists a point (d, e) such that*

$$d+e+\xi(d)=1,$$

$$e=\frac{1}{\lambda}d,$$

$$0<d<\frac{1}{2} \quad \text{and}$$

$$\frac{1}{\lambda}+\xi'(d)>1,$$

then $1/\lambda<\delta(t_0^+)<1$.

Remark. Point (d, e) is unique if exists.

Proof. From theorem 2–1 and $\alpha'(0)=1$, α is a concave monotone increasing function on $\lambda\cdot J$.

$$\delta(t)=\frac{1}{\lambda}\frac{\alpha'(t/\lambda)}{\alpha'(t)}>\frac{1}{\lambda} \qquad \text{on} \quad \lambda\cdot J.$$

Integrating this inequality, we can show that $g(\lambda\cdot J)$ lies above the line $y=\frac{1}{\lambda}x$. $g(J)$ intersects the curve $x+y+\xi(x)=1$ because $d<1/2$ and $-1-\xi'(x)<0$ on I. And this intersection is really $g(t_0^+/\lambda)$ because α is monotone on J. t_0^+ is on $\lambda\cdot J$.

$$\delta(t_0^+)>\frac{1}{\lambda}.$$

ξ is concave on I and $\alpha(t_0^+/\lambda)<d<1/2$.

$$\delta(t_0^+/\lambda)+\xi'(\alpha(t_0^+/\lambda))>\frac{1}{\lambda}+\xi'(d)>1.$$

From the functional equation of δ, we get

$$\delta(t_0^+) = \frac{1}{\delta(t_0^+/\lambda) + \xi'(\alpha(t_0^+/\lambda))} < 1.$$

Theorem 4–2. *We assume λ to be negative. If ξ is concave monotone increasing function on $I = (-\infty, 1/2)$ and there exists a point (d, e) such that*

$$d + e + \xi(d) = 1,$$

$$d + e - \xi(e) = 1,$$

$$0 < d < 1/2 \quad \text{and}$$

$$e < 0,$$

then $\delta(t_0^-) < 1/\lambda < 1$.

Proof. From the functional equation of δ:

$$\delta(t) = \frac{1}{\dfrac{1}{\delta(\lambda^2 t)} - \xi'(\alpha(\lambda t))} - \xi'(\alpha(t)),$$

$\delta(0) = 1/\lambda < \delta(\lambda^2 t) < 0$ implies $1/\lambda < \delta(t) < 0$ on L where L is an interval in K and $\alpha(L) = (0, 1/2)$. Because $I = (-\infty, 1/2)$ $\alpha(K)$ contains $(0, 1/2)$. Inequality $1/\lambda < \delta(t)$ holds in some neighbourhood of origin because $\xi'(0) < 0$.
Consider the following region:

$$\{(x, y) \mid x + y + \xi(x) < 1,\ x + y - \xi(y) < 1,\ x > 0 \text{ and } y < 0\}.$$

$g(L)$ intersects the curve $x + y - \xi(y) = 1$ because $d < 1/2$. And this intersection is really $g(\lambda t_0^-)$ because α is monotone on L. So,

$$\delta(t_0^-) = \frac{1}{\delta(\lambda t_0^-)} - \xi'(\alpha(t_0^-)) < \frac{1}{\delta(0)} - \xi'(0) = \frac{1}{\lambda}.$$

Remark. Let t_0 be a zero of γ such that $|\delta(t_0)| \neq 1$. If ξ satisfies the functional equation $\xi(x) = \xi(1 - x)$, then $f(t_0)$ is a transversal heteroclinic point.

Example. In the case $\xi(x) = cx(1 - x)$, f has a transversal homoclinic point if $|c| > 2\sqrt{2}$. (Check the sufficient conditions of Theorem 4–1 and 4–2.)

5. Shift of ξ_0

Let X be the set of all real analytic functions ξ that satisfy the following four conditions:

$$\xi(x) = \xi(1 - x),$$

$$\xi(0) = 0$$

$$\xi'(x) > 0 \quad \text{on} \quad (-\infty, 1/2) \quad \text{and}$$

$$\xi''(x)<0 \quad \text{on} \quad (-\infty, 1/2).$$

We introduce a shift ξ for a given function ξ_0 in X by the following way.

$$\xi(u, x)=u\xi_0(x) \qquad (u>0).$$

In this section the letter u express a positive number. For any u, $\xi(u, x)$ belongs to X be cause X is a cone.

We consider the following parameterized plane transformation:

$$f(u, x, y)=(y+\xi(u, x), x).$$

f has a heteroclinic point for any u (See Section 3), and f has a transversal heteroclinic point for large u (See Section 4).

We can recognized α, β, γ and δ as a real analytic function of t and u (See Section 1). And we can also recognize $t_0(t_0^+, t_0^-)$ as a real analytic variety defined by γ (subvariety of t_0) in t-u plane.

Conjecture. f has a transversal heteroclinic point except for isolated value of u.

If there exists a union of irreducible components of t_0 such that projection from it to u line is surjective and if δ is not constant on each of them, then the conjecture is true because of the theorem of identity.

Moreover, if $\delta\neq -1$ everywhere and $\delta\not\equiv 1$ on it, then it is irreducible and the conjecture is true.

Example. In the case $\xi=\lambda x(1-x)$ $(\lambda>0)$, the conjecture is true. Projection from t_0^+ to u line is surjective (Theorem 3–1). $\delta\neq -1$ on t_0^+ everywhere. Under the same aspect with theorem 4–1, we only have to check that there exists a point (d, e) such that

$$d+e+\xi(d)=1,$$

$$e=\frac{1}{\lambda}d \quad \text{and}$$

$$0<d<\frac{1}{2}$$

for any u. This is an absolute inequality. And $\delta\not\equiv 1$ on t_0^+ (Theorem 4–1).

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$$f(x, y)=(y+cx(1-x), x),$$

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Department of Mathematics
Kyoto University

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