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A note on locally noetherian pairs

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All rings are assumed to be commutative integral domains with unity $1\neq 0$ in what follows. In order to generalize Gilmer's work [7], Wardsworth [10] defined a noetherian pair (N.P. in short) as follows:

"Let A and B be two integral domains such that A is a subring of B. Then (A, B) is said to be a noetherian pair if all the rings intermediate between A and B are noetherian."

If A is a subring of B and $\mathfrak{p}$ a prime ideal of A then $B_{\mathfrak{p}}$ denotes the ring $A_{\mathfrak{p}}\otimes_A B$ where as usual $A_{\mathfrak{p}}$ is the localization of A at $\mathfrak{p}$. In Lemma 1 of his paper Wardsworth proves that if A is quasi-semi-local such that $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ is N.P. for any maximal ideal $\mathfrak{m}$ of A, then (A, B) itself is N.P. and then goes on to ask if the condition that A has finitely many maximal ideals can be removed. Clearly, if A is an almost Dedekind domain (equivalently, $A_{\mathfrak{m}}$ is a rank one discrete valuation ring for any maximal ideal $\mathfrak{m}$ of A) which is not a Dedekind domain (for an example of such a domain see Appendix 3 [6]) and Q is the quotient field of A, then (A, Q) is not an N.P. However, $(A_{\mathfrak{m}}, Q_{\mathfrak{m}})$ is N.P. for any maximal ideal $\mathfrak{m}$ of A. In this note we study pairs (A, B) such that $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ is an N.P. for any maximal ideal $\mathfrak{m}$ of A and find in the sequel that many properties of the noetherian pairs generalize to such pairs. The notations and terminology are in general that of Nagata [9] unless stated otherwise.

1. Locally noetherian pairs

For the sake of convenience we make the following definitions.

Definition 1. A ring A is said to be locally noetherian if $A_{\mathfrak{m}}$ is noetherian for any maximal ideal $\mathfrak{m}$ of A.

Definition 2. Let A and B be two integral domains such that A is a subring of B (we shall henceforth say that (A, B) is a pair). The pair (A, B) is said to be a locally noetherian pair, locally N.P. in short, if $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ is a noetherian pair for any maximal ideal $\mathfrak{m}$ of A.

Thus an N.P. is locally N.P. but the converse is not true as pointed out already. The following Lemma is immediate.

Lemma 1.1. *Let (A, B) be a pair of domains. The following are equivalent:*
(i) $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ *is N.P. for any maximal ideal* $\mathfrak{m}$ *of* A.
(ii) $(A_{\mathfrak{p}}, B_{\mathfrak{p}})$ *is N.P. for any prime ideal* $\mathfrak{p}$ *of* A.

Lemma 1.2. *(A, B) is an N.P. if and only if A is noetherian and for each proper ideal I of each ring C intermediate between A and B, C/I is a finitely generated A-module.*

Proof. See Theorem 2 [10].

The above lemma immediately yields the following characterization of locally noetherian pairs.

Proposition 1.3. *Let (A, B) be a pair. (A, B) is locally N.P. if and only if (i) A is locally noetherian and (ii) for any intermediate ring C and a proper ideal I of C, $C/I \otimes_A A_{\mathfrak{m}}$ is a finitely generated $A_{\mathfrak{m}}$-module for any maximal ideal $\mathfrak{m}$ of A.*

Proof. (A, B) is locally N.P. if and only if for any maximal ideal $\mathfrak{m}$ of A, $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ is a noetherian pair. Also, C' is a ring intermediate between $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}}$ if and only if $C' = C_{\mathfrak{m}}$ for a subring C intermediate between A and B. Now the result follows immediately on applying Lemma 1.2.

Remark 1. If (A, B) is locally N.P. then for any multiplicative subset S of A not containing zero (A_S, B_S) is also locally N.P.

It is well known that a locally noetherian ring A is noetherian if each nonzero element of A is contained in finitely many maximal ideals only. Following is an analogus result for locally N.P.'s.

Proposition 1.4. *Let (A, B) be a locally N.P. If each nonzero element of A is contained in only finitely many maximal ideals of A then (A, B) is an N.P.*

Proof. First of all as remarked above, A is noetherian. If A is a field there is nothing to prove. So we assume that A is not a field. Let C be a ring intermediate between A and B. Consider an ideal I of C. Let $J = A \cap I$. Then by Theorem 4 [10], $J \neq 0$. Now $C_{\mathfrak{m}}/(JC)_{\mathfrak{m}}$ is a finitely generated $A_{\mathfrak{m}}$-module. Since J is contained in only finitely many maximal ideals of A, there is a finitely generated ideal I' of C such that $I \supset I' \supset JC$ and $(I/JC)_{\mathfrak{m}} = (I'/JC)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of A. Thus I is finitely generated so that C is noetherian. Hence (A, B) is N.P.

It is easily observed that if (A, B) is locally N.P. then every ring C intermediate between A and B is locally noetherian. Now we prove the converse of this statement. The following lemma is immediate.

Lemma 1.5. *Let (A, B) be a pair. Let I be an ideal of A such that $IB \neq B$ and let $C = A + IB$ and $S = 1 + IB$. Then*
(i) *S is a multiplicative closed set of the ring C.*
(ii) $C_S = A + IB_S$
(iii) $IB_S \subset \operatorname{rad} C_S$.

In the following $Q(D)$ denotes the quotient field of the integral domain D.

Proposition 1.6. *Let $(A, \mathfrak{m})$ be a local domain with* $\dim A>0$. *Let (A, B) be a pair. Suppose B is not noetherian. Then there exists a ring C intermediate between A and B such that $C_{\mathfrak{n}}$ is not noetherian for some maximal ideal $\mathfrak{n}$ of C.*

Proof. Case I. When $Q(B)$ is algebraic over $Q(A)$, let J be an ideal of B which is not finitely generated. Put $I=J\cap A$. Clearly, $I\neq 0$. Let a be a nonzero element of I. Put $C=A+aB$ and $S=1+aB$. B_S is not noetherian. Hence by Lemma 1.5, C_S $(=A+aB_S)$ is a non-noetherian quasi-local ring.

Case II. When $Q(B)$ is not algebraic over $Q(A)$, take a transcendental element x of B. Let a be a nonzero element of $\mathfrak{m}$. Put $C=A+aA[x]$ and $S=1+aA[x]$. Then Lemma 1.5 and the arguement used in the proof of Theorem 2 in [10] imply that C_S $(=A+aA[x]_S)$ is a non-noetherian quasi-local ring.

Theorem 1.7. *Let (A, B) be a pair. Suppose every C intermediate between A and B is locally noetherian. Then (A, B) is a locally noetherian pair.*

Proof. Case I. When A is not a field, it is sufficient to show that, for any maximal ideal $\mathfrak{m}$ of A, $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ is a noetherian pair. Now any ring intermediate between $A_{\mathfrak{m}}$ and $B_{\mathfrak{m}}$ is also locally noetherian. Thus $A_{\mathfrak{m}}$ is a local ring and $(A_{\mathfrak{m}}, B_{\mathfrak{m}})$ satisfies the assumption of the theorem. Hence, Proposition 1.6 implies the conclusion.

Case II. Let A be a field. The conclusion is clear if B is algebraic over A. We, therefore, assume that $Q(B)$ is not algebraic over A. Take a transcendental element x of B and let $A'=A[x]$. Then (A', B) is locally noetherian pair by Case I. Thus, $Q(B)$ is a finite algebraic extension of $Q(A')$ (Lemma 3, [10]). Krull-Akizuki Theorem implies that (A', B) is a noetherian pair. Also (A, A') is a noetherian pair (Corollary 5, [10]). Therefore, (A, B) is a noetherian pair. This completes the proof of the theorem.

An example of a locally noetherian pair which is not a noetherian pair was mentioned in the introduction. We end this section with another such example.

Example. Let R denote a noetherian integral domain whose derived normal ring is not a finite R-module and that for each prime ideal $\mathfrak{p}$ the derived normal ring of $R_{\mathfrak{p}}$ is a finite $R_{\mathfrak{p}}$-module (Example 8, p. 211, [9]). Set $S=1+xR[x]$ and $A=R[x]_S$. Let $B=\bar{A}$, the derived normal ring of A.

Clearly (A, B) is locally N.P. Now A is noetherian and x is in rad A. Thus, B/xB is not a finitely generated A-module. Hence (A, B) is not an N.P.

2. Construction of locally noetherian pairs

Several of the results proved in [10] can be generalized suitably to the case of locally noetherian pairs. The following remark is immediate (cf. Corollary 5, [10]).

Remark 2. Let A be an integral domain and x an indeterminate over A. The following are equivalent:

(i) A is a field.

(ii) $(A, A[x])$ is a noetherian pair.

(iii) $(A, A[x])$ is a locally noetherian pair.

The following generalizes Corollary 6 in [10] and Gilmer's classification of domains all of whose subrings are noetherian [7].

Proposition 2.1. *Let B be a domain all of whose subrings are locally noetherian. Let P be the prime ring of B. Then (P, B) is a noetherian pair. Further,*

(i) *If $ch\ B=0$ then B is contained in a finite algebraic extension of the field of rational numbers.*

(ii) *If $ch\ B\neq 0$ then either B is algebraic over P or B is contained in a finitely generated field F such that* $\operatorname{tr\,deg}_P F=1$.

Proof. By Theorem 1.7 (P, B) is locally N.P. Then Proposition 1.4 implies that (P, B) is in fact a noetherian pair. The rest follows on applying Corollary 6 in [10] itself.

Next we wish to construct locally noetherian pairs. This construction is on lines similar to that of the construction of noetherian pairs. Let us recall the following from [10].

Definition 3. A maximal ideal $\mathfrak{m}$ of a ring R is said to be low (resp. high) maximal ideal according as $\operatorname{ht}\mathfrak{m}=1$ (resp. $\operatorname{ht}\mathfrak{m}>1$).

Notation Denote $\tilde{A}=\cap\{A_{\mathfrak{m}}: \mathfrak{m}$ is high maxiaml ideal of $A\}$.

Lemma 2.2. *If A is a noetherian ring, then $(A, \tilde{A})$ is a noetherian pair.*

Proof. See Theorem 8 [10].

Lemma 2.3. *Let A be a locally noetherian ring. Then $(A, \tilde{A})$ is locally noetherian pair.*

Proof. If $\mathfrak{m}$ is a high maximal ideal of A, then $\tilde{A}_{\mathfrak{m}}=A_{\mathfrak{m}}$. If, however, $\mathfrak{m}$ is a low maximal ideal then $A_{\mathfrak{m}}$ is a one dimensional noetherian ring. Thus it follows that $(A_{\mathfrak{m}}, \tilde{A}_{\mathfrak{m}})$ is an N.P. Hence $(A, \tilde{A})$ is a locally noetherian pair.

Proposition 2.4. *Let (A, B) be a locally noetherian pair. Let C be a ring intermediate between A and B. Then (C, B) is also a locally noetherian pair.*

Proof. C is locally noetherian. Let $C\subset T\subset B$ and $\mathfrak{m}$ a maximal ideal of C. Put $\mathfrak{p}=\mathfrak{m}\cap A$. Then $T_{\mathfrak{m}}$ is a ring of quotients of $T_{\mathfrak{p}}$ and therefore noetherian. Hence (C, B) is locally N.P.

Remark 3. Theorem 10 in [10] is also generalized as follows:

"Suppose (A, B) is locally N.P. Let T be the integral closure of A and B. Then $B\subset\tilde{T}$. If, however, A is noetherian then $\dim B=\dim A$. In case A has no low maximal ideals, B is integral over A."

The following which is a generalization of Theorem 13 in [10], helps construct new locally N.P.'s from known ones.

Proposition 2.5. *Let R be locally noetherian and $R \subset A \subset T$, where A is a finite integral extension of R. If (A, T) is locally noetherian pair then (R, T) is also a locally noetherian pair.*

Proof. Let B be a ring intermediate between R and T. Since (A, T) is locally noetherian pair, $C = B[A]$ is locally noetherian and a finite integral extension of B. Using Theorem 2 in [5], B must be locally noetherian. An application of Theorem 1.7 now completes the proof of the proposition.

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