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On invariant tensors of β-changes of Finsler metrics

By

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Let M^n be an n-dimensional differentiable manifold and $F^n=(M^n, L)$ be a Finsler space equipped with a fundamental function $L(x, y)(y^i=\dot{x}^i)$ on M^n. For a differential one-form $\beta(x, dx)=b_i(x)dx^i$ on M^n, we shall deal with a change of Finsler metric which is defined by

$$L(x, y) \longrightarrow \bar{L}(x, y)=f(L(x, y), \beta(x, y)), \tag{0.1}$$

where $f(L, \beta)$ is a positively homogeneous function of L and β of degree one. This is called a β-change of the metric. We have specially interesting example of β-change of the metric, for instance,

$$\begin{aligned} &(1) \quad \bar{L}(x, y)=L(x, y)+\beta(x, y), \\ (0.2) \qquad &(2) \quad \bar{L}(x, y)=L^2(x, y)/\beta(x, y), \\ &(3) \quad \bar{L}(x, y)=L^3(x, y)/\beta^2(x, y). \end{aligned}$$

The change (0.2) (1) has been introduced by Matsumoto [12]*. Hashiguchi and Ichijyō [7] named it a *Randers change* and proved a theorem which shows a relation between a Randers change and a projective change.

Next, the change (0.2) (2) is called a *Kropina change*. For a β-change $L\to\bar{L}=f(L, \beta)$, if L is a Riemannian metric $\alpha(x, dx)=(a_{ij}(x)dx^i dx^j)^{1/2}$, then $\bar{L}=f(L, \beta)$ becomes well-known (α, β)-metric ([5], [6]). In particular $\bar{L}=\alpha+\beta$ is a Randers metric ([3], [9]) and $\bar{L}=\alpha^2/\beta$ is a Kropina metric ([11]). Both of them are closely related to physics and so Finsler spaces with these metrics have been studied by many authors, from various standpoint in the physical and mathematical aspect ([3], [9], [22], [23], [26]).

In §1, we shall study how the fundamental and the torsion tensors change by a β-change of the metric. §2 is devoted to giving transformation formulas of the torsion and the curvature by a β-change of the metric. In §3, we consider Randers changes and give some invariant tensors under these changes, and in §4 we shall study some geometrical properties of these invariant tensors. In §§5 and 6, we are concerned with projective Randers changes and also give a characterization of the vanishing Douglas tensor which is invariant under a

* Number in brackets refer to the references at the end of the paper.

projective Randers change. §7 is devoted to a study of decomposable tensor. In the final section we give another example of projective change besides the Randers change.

The terminology and notations are referred to well-known Rund's book [21] and Matsumoto's monograph [16].

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§1. Changes of connections.

Let $F^n=(M^n, L)$ be an n-dimensional Finsler space with a fundamental function $L(x, y)$. We consider a change of Finsler metric which is defined by $L\to\bar{L}=f(L, \beta)$, and have another Finsler space $\bar{F}^n=(M^n, \bar{L})$ with $\bar{L}=f(L, \beta)$.

Throughout the present paper we shall use the following notations:

$$f_1=\partial f/\partial L, \qquad f_2=\partial f/\partial\beta, \qquad f_{11}=\partial^2 f/\partial L\partial L, \qquad \text{etc.},$$
$$\partial_i=\partial/\partial x^i, \qquad \dot{\partial}_i=\partial/\partial y^i.$$

Since $\bar{L}=f$ is a positively homogeneous function of L and β of degree one, we have

$$f=f_1L+f_2\beta, \qquad Lf_{12}+\beta f_{22}=0, \qquad Lf_{11}+\beta f_{12}=0. \tag{1.1}$$

For the later use we put

$$p=ff_1/L, \qquad q=ff_2, \qquad q_0=ff_{22}. \tag{1.2}$$

Paying attention to $l_i=\dot{\partial}_i L$, from (1.1) we have

$$\bar{l}_i=f_1l_1+f_2b_i. \tag{1.3}$$

Differentiating this by y^j, we have the angular metric tensor $\bar{h}_{ij}=\bar{L}\dot{\partial}_i\dot{\partial}_j\bar{F}$ of $\bar{F}^n$:

$$\bar{h}_{ij}=ph_{ij}+q_0m_im_j, \tag{1.4}$$

where the covariant vector m_i is defined by

(1.4) 1) $$m_i=b_i-\beta y_i/L^2.$$

It is noted that m_i is a non-zero vector orthogonal to y^i. In fact $m_i=0$ gives $L^2b_i-\beta y_i=0$. We differentiate this by y^j and get $\beta g_{ij}-2Ll_jb_i+b_jy_i=0$, which leads to a contradiction $g_{ij}-l_il_j=0$.

Now, from (1.1), (1.3), (1.4) and (1.4) 1) the fundamental tensor $\bar{g}_{ij}=\dot{\partial}_i\dot{\partial}_j(\bar{L}^2/2)$ of $\bar{F}^n$ is given by

$$\bar{g}_{ij}=pg_{ij}+p_0b_ib_j+p_{-1}(b_iy_j+b_jy_i)+p_{-2}y_iy_j, \tag{1.5}$$

where we put

(1.5) 1)
$$p_0=q_0+f_2^2,$$
$$q_{-1}=ff_{12}/L, \qquad p_{-1}=q_{-1}+pf_2/f,$$
$$q_{-2}=f(f_{11}-f_1/L)/L^2, \qquad p_{-2}=q_{-2}+p^2/f^2.$$

The reciprocal tensor $\bar{g}^{ij}$ of $\bar{g}_{ij}$ can be written as

$$\bar{g}^{ij}=(1/p)g^{ij}-s_0b^ib^j-s_{-1}(b^iy^j+b^jy^i)-s_{-2}y^iy^j, \tag{1.6}$$

where we put

$$\begin{aligned} &b^i=g^{ij}b_j, \qquad b^2=g^{ij}b_ib_j, \qquad \nu=b^2-\beta^2/L^2, \\ &s_0=\bar{L}^2q_0/\tau pL^2, \qquad s_{-1}=p_{-1}\bar{L}^2/p\tau L^2, \\ &s_{-2}=p_{-1}(\nu pL^2-b^2\bar{L}^2)/\tau p\beta L^2, \qquad \tau=\bar{L}^2(p+\nu q_0)/L^2. \end{aligned} \tag{1.6) 1}$$

From the homogeneity it follows that these quantities satisfy

$$\begin{aligned} &q_0\beta+q_{-1}L^2=0, \qquad q_{-1}\beta+q_{-2}L^2=-p, \\ &p_0\beta+p_{-1}L^2=q, \qquad q\beta+pL^2=f^2, \\ &p_{-1}\beta+p_{-2}L^2=0, \qquad s_0\beta+s_{-1}L^2=q/\tau, \\ &s_{-1}b^2+s_{-2}\beta=p_{-1}\nu/\tau. \end{aligned} \tag{1.7}$$

As to the torsion tensor $\bar{C}_{ijk}=\dot{\partial}_k(\bar{g}_{ij}/2)$ of $\bar{F}^n$, from (1.5) and (1.7) we get

$$\bar{C}_{ijk}=pC_{ijk}+p_{-1}(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)/2+p_{02}m_im_jm_k/2, \tag{1.8}$$

where we put $p_{02}=\partial p_0/\partial\beta$. Contracting this by $\bar{g}^{hk}$, we have

$$\bar{C}_i{}^h{}_j=C_i{}^h{}_j-V_i{}^h{}_j, \tag{1.9}$$

where we put

$$\begin{aligned} V_i{}^h{}_j=&Q^h(pC_{imj}b^m-p_{-1}m_im_j)-(m^h/p-\nu Q^h)(p_{02}m_im_j+p_{-1}h_{ij})/2 \\ &-p_{-1}(h^h{}_im_j+h^h{}_jm_i)/2p, \\ Q^h=&s_0b^h+s_{-1}y^h, \qquad h^h{}_i=g^{hr}h_{ir}, \qquad m^h=g^{hr}m_r. \end{aligned} \tag{1.9) 1}$$

We denote by the symbol ($|$) the h-covariant differentiation with respect to the Cartan connection $C\Gamma$ and put

$$2E_{jk}=b_{j|k}+b_{k|j}, \qquad 2F_{jk}=b_{j|k}-b_{k|j}. \tag{1.10}$$

Now we deal with well-known functions $G^i(x, y)$ which are (2)p-homogeneous in y^i and are written as $G^i=\gamma_j{}^i{}_ky^jy^k/2$, by putting

$$\gamma_j{}^i{}_k=g^{ir}(\partial_kg_{jr}+\partial_jg_{kr}-\partial_rg_{jk})/2.$$

Owing to (1.5) and (1.6), a straightforward calculation leads to

$$\bar{G}^i(x, y):=(\bar{\gamma}_j{}^i{}_ky^jy^k)/2=G^i+D^i, \tag{1.11}$$

where the vector D^i is given by

$$\begin{aligned} &D^i=(q/p)F^i{}_0+(pE_{00}-2qF_{r0}b^r)(s_{-1}y^i+s_0b^i)/2, \\ &F^i{}_j=g^{ir}F_{rj}, \end{aligned} \tag{1.11) 1}$$

and the subscript 0 (excluding s_0) means the contraction by y^i.

We shall examine how the Cartan connection $C\Gamma$ changes by a β-change of the metric. Let $C\bar{\Gamma}=(\bar{F}_j{}^i{}_k, \bar{N}^i{}_j, \bar{C}_j{}^i{}_k)$ be the Cartan connection on the space $\bar{F}^n=(M^n, \bar{L})$. For coefficients $N^i{}_j=\dot{\partial}_j G^i$ of the non-linear connection, we differentiate (1.11) by y^i and get

(1.12) $$\bar{N}^i{}_j=N^i{}_j+D^i{}_j,$$

where the tensor $D^i{}_j=\dot{\partial}_j D^i$ is given by

(1.12) 1) $$\begin{aligned} &D^i{}_j:=(1/p)A^i{}_j-Q^iA_{rj}b^r-qb_{0|j}(s_{-1}b^i+s_{-2}y^i),\\ &A_{ij}:=(1/2)E_{00}B_{ij}+qF_{ij}+F_{i0}Q_j-(pC_{ijm}+V_{ijm})D^m,\\ &A^i{}_j=g^{ir}A_{rj}, \qquad V_{ijm}=g_{sj}V_i{}^s{}_m, \qquad Q_i=p_{-1}y_i+p_0b_i,\\ &B_{jk}=(p_{-1}h_{jk}+p_{02}m_jm_k)/2. \end{aligned}$$

Here, for the covariant vector Q_i it is noted that

(1.13) $$Q_0=q, \qquad \dot{\partial}_kQ_j=B_{jk}/2.$$

Let $B\bar{\Gamma}=(\bar{G}_j{}^i{}_k, \bar{N}^i{}_j)$ be the Berwald connection on $\bar{F}^n$. Differentiating (1.12) by y^k, we have connection coefficients $\bar{G}_j{}^i{}_k=\dot{\partial}_k\bar{N}^i{}_j$ of $B\bar{\Gamma}$ which are given by

(1.14) $$\bar{G}_j{}^i{}_k:=G_j{}^i{}_k+B_j{}^i{}_k, \qquad B_j{}^i{}_k=\dot{\partial}_kD^i{}_j,$$

where $G_j{}^i{}_k$ are connection coefficients of $B\Gamma$ on F^n. Therefore from (1.6), (1.8), (1.9) and (1.12) we obtain connection coefficients $\bar{F}_j{}^i{}_k$ of the Cartan connection $C\bar{\Gamma}$ on $\bar{F}^n$ as follows:

(1.15) $$\begin{aligned} \bar{F}_j{}^i{}_k:&=\bar{\gamma}_j{}^i{}_k+\bar{C}_{jkr}\bar{N}^r{}_m\bar{g}^{im}-\bar{C}_k{}^i{}_r\bar{N}^r{}_j-\bar{C}_j{}^i{}_r\bar{N}^r{}_k\\ &=F_j{}^i{}_k+D_j{}^i{}_k, \end{aligned}$$

where we put

(1.15) 1) $$\begin{aligned} D_j{}^i{}_k=&\{g^{is}/p-Q^ib^s-y^s(s_{-1}b^i+s_{-2}y^i)\}\\ &\cdot(B_{sj}b_{0|k}+B_{sk}b_{0|j}-B_{kj}b_{0|s}+F_{sj}Q_k+F_{sk}Q_j+E_{kj}Q_s+pC_{jkr}D^r{}_s\\ &+V_{jkr}D^r{}_s-pC_{skm}D^m{}_j-V_{sjm}D^m{}_k-pC_{sjm}D^m{}_k-V_{skm}D^m{}_j). \end{aligned}$$

The tensor $D_j{}^i{}_k$, called the *difference tensor*, has the following properties:

(1.16) $$(1)\quad D_j{}^i{}_0=B_j{}^i{}_0=D^i{}_j, \qquad (2)\quad D_0{}^i{}_0=2D^i.$$

Theorem 1.1. *The covariant vector, the components $b_i(x)$ of which are coefficients of the one-form β, is parallel with respect to the Cartan connection $C\Gamma$ on F^n if and only if the difference tensor $D_j{}^i{}_k$ of* (1.15) 1) *vanishes.*

Proof. Assume that the vector $b_i(x)$ is parallel with respect to $C\Gamma$. Then (1.10) shows $E_{ij}=F_{ij}=0$, and so (1.11) 1) implies $D^i=0$ and $D^i{}_j=0$. Consequently (1.15) 1) leads to $D_j{}^i{}_k=0$.

Conversely if $D_j{}^i{}_k$ vanishes, (1.16) immediately gives $D^i=0$. Contracting (1.11) 1) by y_i, we have $pE_{00}-2qF_{r0}b^r=0$, because of $s_0\beta+s_{-1}L^2\neq 0$ in (1.7).

Thus (1.11) 1) reduces to $F_{i0}=0$ and $E_{00}=0$, and so (1.10) gives $b_{i|j}=0$.

A Finsler space F^n is called a *Berwald space* if the Berwald connection of F^n is a linear connection, that is, connection coefficients $G_j{}^i{}_k$ are functions of the position (x^i) only. As an immediate consequence of Theorem 1.1, and (1.14) we have

Theorem 1.2. *Assume that the original space F^n is a Berwald space and the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$ on F^n. Then the space $\bar{F}^n$ obtained from F^n by the β-change is also a Berwald space.*

Corollary 1.1. *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Riemannian connection on Riemannian space $F^n=(M^n, L=\alpha)$. Then $\bar{F}^n=(M^n, \bar{L}=f(\alpha, \beta))$, obtained from F^n by the β-change, is a Berwald space.*

§2. Change of the torsion and the curvature tensors.

In this section, we shall consider how the torsion and the curvature tensors change by a β-change of the metric.

Let $F\Gamma=(F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ be a Finsler connection on the space F^n and let K be a Finsler tensor field, for instance, of type (1, 1). Then the h- and the v-covariant derivatives of K are respectively defined by

$$K^i{}_{j|k}=\delta K^i{}_j/\partial x^k+K^m{}_jF_m{}^i{}_k-K^i{}_mF_j{}^m{}_k\,,$$

$$K^i{}_j|_k=\dot{\partial}_kK^i{}_j-K^i{}_mC_j{}^m{}_k+K^m{}_jC_m{}^i{}_k\,,$$

$$\delta/\partial x^k=\partial/\partial x^k-N^r{}_k\partial/\partial y^r.$$

The torsion and the curvature tensors of $F\Gamma$ are written as follows:

$$R_j{}^i{}_k=\mathfrak{A}_{(jk)}\{\delta N^i{}_j/\partial x^k\},\qquad P_j{}^i{}_k=\dot{\partial}_kN^i{}_j-F_k{}^i{}_j\,,$$

$$R_h{}^i{}_{jk}=\mathfrak{A}_{(jk)}\{\delta F_h{}^i{}_j/\partial x^k+F_h{}^m{}_jF_m{}^i{}_k\}+C_h{}^i{}_mP_j{}^m{}_k\,,$$

$$P_h{}^i{}_{jk}=\dot{\partial}_kF_h{}^i{}_j-C_h{}^i{}_{k|j}+C_h{}^i{}_mP_j{}^m{}_k\,,$$

$$S_h{}^i{}_{jk}=\mathfrak{A}_{(jk)}\{\dot{\partial}_kC_h{}^i{}_j+C_h{}^m{}_kC_m{}^i{}_j\}.$$

Throughout the paper, for the sake of brevity, we shall adopt the notations $\mathfrak{S}_{(ijk)}$ and $\mathfrak{A}_{(ij)}$ such that

$$\mathfrak{S}_{(ijk)}\{X_{ir}X_j{}^r{}_k\}=X_{ir}Y_j{}^r{}_k+X_{jr}Y_k{}^r{}_i+X_{kr}Y_i{}^r{}_j\,,$$

$$\mathfrak{A}_{(ij)}\{X_{ir}Y_j{}^r{}_k\}=X_{ir}Y_j{}^r{}_k-X_{jr}Y_i{}^r{}_k\,.$$

Let $F\bar{\Gamma}$ be a Finsler connection on $\bar{F}^n$, obtained from F^n by a β-change. Then the torsion and the curvature tensors change as follows ([4]).

$$
(2.1)\quad
\begin{aligned}
&(1)\quad \bar{C}_j{}^i{}_k = C_j{}^i{}_k - V_j{}^i{}_k\,,\\
&(2)\quad \bar{R}_j{}^i{}_k = R_j{}^i{}_k + \mathfrak{A}_{(jk)}\{D^i{}_{j|k} - (B_j{}^i{}_r + P_j{}^i{}_r)D^r{}_k\}\,,\\
&(3)\quad \bar{P}_j{}^i{}_k = P_j{}^i{}_k + (A_k{}^i{}_j + C_k{}^i{}_m D^m{}_j) + B_j{}^i{}_k\,,\\
&(4)\quad \bar{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} + S_h{}^i{}_{mn} D^m{}_j D^n{}_k - \mathfrak{A}_{(jk)}\\
&\qquad\qquad \cdot\{A_h{}^i{}_{j|k} - A_h{}^i{}_{jn} D^n{}_k + (A_n{}^i{}_j + C_n{}^i{}_m D^m{}_j) A_h{}^n{}_k + A_m{}^i{}_j C_h{}^m{}_n D^n{}_k\\
&\qquad\qquad + P_h{}^i{}_{jn} D^n{}_k - V_r{}^i{}_h P_j{}^r{}_n D^n{}_k + V_m{}^i{}_h D_j{}^m{}_k - V_h{}^i{}_m B_j{}^m{}_r D^r{}_k\}\\
&\qquad - V_h{}^i{}_m R_j{}^m{}_k\,,\\
&(5)\quad \bar{P}_h{}^i{}_{jk} = P_h{}^i{}_{jk} - S_h{}^i{}_{mk} D^m{}_j - A_h{}^i{}_{jk} - C_m{}^i{}_k A_h{}^m{}_j + C_h{}^m{}_k A_m{}^i{}_j\\
&\qquad - V_m{}^i{}_h P_j{}^m{}_k - A_j{}^i{}_m V_h{}^m{}_k + V_m{}^i{}_k A_h{}^m{}_j + V_j{}^i{}_{k|h} + V_m{}^i{}_h B_j{}^m{}_k\\
&\qquad - V_k{}^i{}_{mh} D^m{}_j + C_r{}^i{}_h V_k{}^r{}_m D^m{}_j - V_h{}^i{}_r C_m{}^r{}_k D^m{}_j\,,\\
&(6)\quad \bar{S}_h{}^i{}_{jk} = S_h{}^i{}_{jk} + \mathfrak{A}_{(jk)}\{C_m{}^i{}_k V_h{}^m{}_j - C_h{}^m{}_k V_m{}^i{}_j - V_m{}^i{}_k V_h{}^m{}_j\}\,,
\end{aligned}
$$

where tensors $D^i{}_j$, $B_j{}^i{}_k$, $A_j{}^i{}_k$, $V_j{}^i{}_k$, $A_h{}^i{}_{jk}$ and $V_h{}^i{}_{jk}$ are respectively given by

$$
(2.1)\ 1)\quad
\begin{aligned}
&D^i{}_j = \bar{N}^i{}_j - N^i{}_j\,, \qquad B_j{}^i{}_k = \dot{\partial}_k D^i{}_j\,,\\
&A_j{}^i{}_k = F_j{}^i{}_k - \bar{F}_j{}^i{}_k + C_j{}^i{}_m (N^m{}_k - \bar{N}^m{}_k)\,,\\
&V_j{}^i{}_k = C_j{}^i{}_k - \bar{C}_j{}^i{}_k\,, \qquad A_j{}^i{}_{kh} = \dot{\partial}_h A_j{}^i{}_k\,, \qquad V_j{}^i{}_{kh} = \dot{\partial}_h V_j{}^i{}_k\,.
\end{aligned}
$$

Moreover, for the tensor $A_j{}^i{}_k$ in (2.1) 1) we get

$$
(2.2)\quad
\begin{aligned}
&(C\Gamma)\quad A_j{}^i{}_k = -D_j{}^i{}_k - C_j{}^i{}_m D^m{}_k\,, &&(B\Gamma)\quad A_j{}^i{}_k = B_j{}^i{}_k\,,\\
&(H\Gamma)\quad A_j{}^i{}_k = H_j{}^i{}_k\ (= -B_j{}^i{}_k - C_j{}^i{}_m D^m{}_k)\,, &&(R\Gamma)\quad A_j{}^i{}_k = -D_j{}^i{}_k\,,
\end{aligned}
$$

where $R\Gamma$ and $H\Gamma$ are respectively the Rund connection and the Hashiguchi connection ([4], [16]). Therefore, on account of (2.2), we can derive the torsion and the curvature tensors of each connection in the concrete form, for example,

(Case of $C\Gamma$)

$$
(2.3)\quad
\begin{aligned}
&(1)\quad \bar{R}_j{}^i{}_k = R_j{}^i{}_k + \mathfrak{A}_{(jk)}\{D^i{}_{j|k} - (B_j{}^i{}_r + P_j{}^i{}_r)D^r{}_k\}\,,\\
&(2)\quad \bar{P}_j{}^i{}_k = P_j{}^i{}_k - D_j{}^i{}_k + B_j{}^i{}_k\,,\\
&(3)\quad \bar{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} + 2S_h{}^i{}_{mn} D^m{}_j D^n{}_k - \mathfrak{A}_{(jk)}\\
&\qquad\qquad \cdot\{A_h{}^i{}_{j|k} - A_h{}^i{}_{jn} D^n{}_k + D_n{}^i{}_j D_h{}^n{}_k + P_h{}^i{}_{jn} D^n{}_k - V_r{}^i{}_h P_j{}^r{}_n D^n{}_k\\
&\qquad\qquad + V_m{}^i{}_h D^m{}_{j|k} - V_h{}^i{}_m B_j{}^m{}_r D^r{}_k\} - V_h{}^i{}_m R_j{}^m{}_k\,,\\
&(4)\quad \bar{P}_h{}^i{}_{jk} = P_h{}^i{}_{jk} - 2S_h{}^i{}_{mk} D^m{}_j - A_h{}^i{}_{jk} + C_m{}^i{}_k D_h{}^m{}_j - C_h{}^m{}_k D_m{}^i{}_j\\
&\qquad - V_m{}^i{}_h P_j{}^m{}_k - A_j{}^i{}_m V_h{}^m{}_k + V_m{}^i{}_k A_h{}^m{}_j + V_j{}^i{}_{k|h} - V_m{}^i{}_h B_j{}^m{}_h\\
&\qquad - V_k{}^i{}_{mh} D^m{}_j - C_r{}^i{}_h V_k{}^r{}_m D^m{}_j + V_h{}^i{}_s C_m{}^s{}_k D^m{}_j\,,\\
&(5)\quad \bar{S}_h{}^i{}_{jk} = S_h{}^i{}_{jk} + \mathfrak{A}_{(jk)}\{C_m{}^i{}_k V_h{}^m{}_j - C_h{}^m{}_k V_m{}^i{}_j - V_m{}^i{}_k V_h{}^m{}_j\}\,,
\end{aligned}
$$

....

Assume that the vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$. Then Theorem 1.1 asserts that the difference tensor $D_j{}^i{}_k$ vanishes. Therefore, (2.2) and (2.3) imply

$$\bar{R}_h{}^i{}_{jk}=R_h{}^i{}_{jk}-V_h{}^i{}_m R_j{}^m{}_k . \tag{2.4}$$

Further we contract (2.4) by y^h and get $\bar{R}_0{}^i{}_{jk}=R_0{}^i{}_{jk}-V_0{}^i{}_m R_j{}^m{}_k$. Because of $V_j{}^i{}_k=C_j{}^i{}_k-\bar{C}_j{}^i{}_k$, it is clear that $V_0{}^i{}_m=0$. Consequently $\bar{R}_h{}^i{}_{jk}=0$ and $R_h{}^i{}_{jk}=0$ are mutually equivalent. Thus we have

Theorem 2.1. *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$. Then the h-curvature tensor $\bar{R}_h{}^i{}_{jk}$ of $\bar{F}^n$, obtained from F^n by the β-change, vanishes if and only if the h-curvature tensor $R_h{}^i{}_{jk}$ of F^n vanishes.*

Now the space F^n is called a *locally Minkowski space* if F^n is a Berwald space and the h-curvature tensor $R_h{}^i{}_{jk}$ vanishes. From Theorem 1.2 and Theorem 2.1 we have

Theorem 2.2. *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$ on F^n. If F^n is a locally Minkowski space, then the space $\bar{F}^n$, obtained from F^n by the β-change, is also locally Minkowskian.*

Next, we restrict ourselves to a Riemannian space F^n with a Riemannian metric α. Then $\bar{F}^n$, obtained from F^n by a β-change, becomes a Finsler space with a so-called (α, β)-metric. Since $C_{hij}=0$ and $S_{hijk}=0$ in F^n, the h-, the hv-, and the v-curvature tensors of $\bar{F}^n$ are respectively given by

$$\begin{aligned}
(1)\quad \bar{R}_h{}^i{}_{jk}&=R_h{}^i{}_{jk}+\bar{C}_h{}^i{}_m R_i{}^m{}_k+\mathfrak{A}_{(jk)}\\
&\quad\cdot\{D_j{}^i{}_{h|k}+D_r{}^i{}_j D_h{}^r{}_k+\bar{C}_h{}^i{}_m(D_j{}^m{}_{|k}+D_s{}^m{}_j D^s{}_k)\},\\
(2)\quad \bar{P}_h{}^i{}_{jk}&=\dot{\partial}_k D_h{}^i{}_j+D_j{}^i{}_m\bar{C}_h{}^m{}_k-D_h{}^m{}_j\bar{C}_m{}^i{}_k-\bar{C}_m{}^i{}_h D_j{}^m{}_k\\
&\quad-\dot{\partial}_h\bar{C}_k{}^i{}_m D^m{}_j+\bar{C}_j{}^i{}_{k|h},\\
(3)\quad \bar{S}_h{}^i{}_{jk}&=\bar{h}_{hk}M_{ij}-\bar{h}_{hj}M_{ik}+\bar{h}_{ij}M_{hk}-\bar{h}_{ik}M_{hj},
\end{aligned}$$

where $|$ is the h-covariant differentiation with respect to the Cartan connection $C\bar{\Gamma}$ on $\bar{F}^n$, and $M_{ij}=p_{-1}\{\bar{L}^2 p_{-1}\nu\bar{h}_{ij}/2\tau L^2+(p_{-1}+\bar{L}^2\nu(pp_{02}-3p_{-1}q_0)/L^2\tau)m_i m_j\}/2$.

Owing to Corollary 1.1 and Theorem 2.2, we obtain

Proposition 2.1 (Kikuchi ([10]). *A Finsler space $\bar{F}^n$ with an (α, β)-metric is locally Minkowskian if and only if $\nabla_j b_i=0$ and $R_h{}^i{}_{jk}=0$ hold good, where $R_h{}^i{}_{jk}$ is the Riemannian curvature tensor of $F^n=(M^n, \alpha)$ and ∇_j is the covariant differentiation with respect to the Riemannian connection.*

Next, we consider the hv-curvature tensor $\bar{P}_{hijk}$ of $\bar{F}^n=(M^n, (L, \beta))$. Then (2.3) and Theorem 1.1 show that $\bar{P}_j{}^i{}_k=0$ is equivalent to $P_j{}^i{}_k=0$ if the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$. Here we shall recall the concept of Landsberg space:

Definition 2.1. A Finsler space F^n is called a *Landsberg space* if the *hv*-curvature tensor P_{hijk} of $C\Gamma$ vanishes, or equivalently $P_j{}^i{}_k=0$.

As the *hv*-curvature tensor P_{hijk} of a Riemannian space $F^n=(M^n, \alpha)$ vanishes identically, we can state

Theorem 2.3. *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Cartan connection $C\Gamma$. Then a Landsberg space remains to be a Landsberg space by the β-change of the metric.*

Corollary 2.1. *Assume that the covariant vector $b_i(x)$ is parallel with respect to the Riemannian connection constructed from a Riemannian metric α. Then a Finsler space with the (α, β)-metric is a Landsberg space.*

§ 3. Randers change.

We consider a special β-change called a *Randers one* which is defined by $L\to\bar{L}=L+\beta$. As a special case where $L(x, dx)$ is Riemannian, we have a Randers metric. Moreover if L is a Randers metric and the covariant vector $b_i(x)$ of one-form β is gradient, then the Randers change is regarded as a gauge change which is important in the quantum electrodynamics [27]. Recently, Hashiguchi and Ichijyō [7] considered some properties that remain invariant under a Randers change, and proved that any geodesic remains to be a geodesic if and only if the covariant vector $b_i(x)$ is gradient.

We shall first introduce certain tensors which remain invariant by a Randers change. In this case, it follows from (1.1), (1.2) and (1.5) 1) that $f_1=f_2=1$, $p=\bar{L}/L$ and $q=\bar{L}$, so that (1.3) yields

$$\bar{l}_i=l_i+b_i\,. \tag{3.1}$$

Putting

$$L_{ij}=h_{ij}/L\,, \tag{3.2}$$

we observe from (3.1) that the tensor L_{ij} is invariant by the Randers change. From (3.1) and (3.2) we have the fundamental theorem of a Randers change:

Theorem 3.1. *The following statements are equivalent.*

(1) *A β-change $L\to\bar{L}$ of the Finsler metric is a Randers change,*

(2) $\partial^2(\bar{L}-L)/\partial y^i\partial y^j=0$,

(3) $L_{ij}=\bar{L}_{ij}$.

We shall call L_{ij} in (3.2) the *first fundamental tensor of the Randers change*. From now on, we shall call a tensor which is invariant under a Randers change an *R-invariant tensor*.

Since $\dot{\partial}_k h_{ij}=2C_{ijk}-(L_{ik}l_j+L_{jk}l_i)$, differentiating (3.2) by y^k and putting

(3.3) $$L_{ijk}:=\dot{\partial}_k L_{ij}=\{2C_{ijk}-(L_{ij}l_k+L_{jk}l_i+L_{ki}l_j)\}/L\,,$$

we get $L_{ijk}=\bar{L}_{ijk}$. This invariant tensor L_{ijk} is called the *second fundamental tensor of the Randers change.*

Moreover, putting $L_{hijk}=\dot{\partial}_h L_{ijk}$ and refferring to the T-tensor

$$T_{hijk}=LC_{hij}|_k+l_hC_{ijk}+l_kC_{ijh}+l_iC_{hjk}+l_jC_{hik}\,,$$

it is seen that

(3.4) $$L_{hijk}=2T_{hijk}/L^2+\mathfrak{S}_{(ijk)}\{2C_{hrj}C_i{}^r{}_k/L+(-2C_{hij}+l_hL_{ij})l_k/L^2 \\ -L_{ij}L_{hk}/L\}-4l_hC_{ijk}/L^2$$

satisfies $\bar{L}_{hijk}=L_{hijk}$, therefore we get another R-invariant tensor L_{hijk}, which is called the *third fundamental tensor of the* Randers change.

We shall consider how the fundamental and the torsion tensors change by a Randers change. From (1.5) we have

(3.5) $$\bar{g}_{ij}=\mu g_{ij}+b_ib_j+(b_il_j+b_jl_i)-\beta l_il_j/L\,,\quad \mu=\bar{L}/L\,,$$

and (1.6) gives

(3.6) $$\bar{g}^{ij}=(1/\mu)g^{ij}+\omega l^il^j-(l^ib^j+l^jb^i)/\mu^2,\quad \omega=(Lb^2+\beta)/\bar{L}\mu^2.$$

Next, for the $h(hv)$-torsion tensor C_{ijk}, (1.8) leads to

(3.7) $$\bar{C}_{ijk}=\mu C_{ijk}+(L_{ij}m_k+L_{jk}m_i+L_{ki}m_j)/2\,.$$

Contracting this by $\bar{g}^{hk}$, we have

(3.8) $$\bar{C}_i{}^h{}_j=C_i{}^h{}_j+(h_i^hm_j+h_j^hm_i+h_{ij}m^h)/2\bar{L}-C_{ijr}b^rl^h/\mu \\ -(2m_im_j+m^2h_{ij})l^h/2L\mu^2,\quad m^2=g^{ij}m_im_j\,,\quad m^i=g^{ir}m_r\,.$$

Thus the torsion vector $\bar{C}_i=\bar{C}_i{}^r{}_r$ is given by

(3.9) $$\bar{C}_i=C_i+(n+1)m_i/2\bar{L}\,.$$

On the other hand, paying attention to (3.1), the vector m_i in (1.4) 1) and $\bar{C}_i$ are rewritten in the form

(3.9′) $$m_i=\bar{l}_i-\mu l_i\,,\quad \bar{C}_i=C_i+(n+1)(\bar{l}_i-\mu l_i)/2\bar{L}\,.$$

Contracting (3.8) by $\bar{g}^{ij}$ and then $\bar{C}_h$, we obtain

(3.10) $$\bar{C}^h=(1/\mu)C^h+(n+1)m^h/2\mu^2L-\{C_\beta+(n+1)m^2/2\bar{L}\}l^h/\mu^2,$$

(3.11) $$\bar{C}^2=(1/\mu)C^2+(n+1)\{C_\beta+(n+1)m^2/4\bar{L}\}/L\mu^2,$$

where $C^2=C^iC_i$, $C_\beta=C_ib^i$.

If C_β is eliminated from (3.10) and (3.11), m^h is written as a linear combination of $\bar{C}^h$, C^h and l^h. Substituting this into $m^i=b^i-\beta y^i/L^2$, we get

(3.12) $$b^i=2L\mu^2(\bar{C}^i-C^i/\mu/(n+1)+\{\beta/L+m^2/2\mu+2\mu L^2(\mu\bar{C}^2-C^2)/(n+1)^2\}l^i.$$

Further, substituting from (3.12) into (3.6) and putting

$$(3.13)\qquad {}^*g^{ij}=L\{g^{ij}+2(C^iy^j+C^jy^i)/(n+1)+(4C^2/(n+1)^2-1/L^2)y^iy^j\},$$

we see ${}^*\bar{g}^{ij}={}^*g^{ij}$, that is, ${}^*g^{ij}$ is R-invariant. Therefore, on account of this tensor, we can derive some new R-invariant tensors, for instance,

$$(3.14)\qquad 1)\ {}^*L^i{}_j={}^*g^{ir}L_{rj},\qquad 2)\ {}^*L^h{}_{ik}={}^*g^{hr}L_{rik},\ \cdots,\quad \text{etc.}$$

Let $g(x, y)$ be the determinant consisting of components g_{ij} of the fundamental tensor. It is well-known ([15]) that the determinant $\bar{g}$ of $\bar{F}^n$, obtained from F^n by a Randers change, is given by

$$(3.15)\qquad \bar{g}=g(\bar{L}/L)^{n+1}.$$

Thus $g\neq 0$ implies $\bar{g}\neq 0$. Therefore,

$$(3.16)\qquad {}^*L=L/g^{1/n+1}$$

is an R-invariant relative scalar which was called a *relative fundamental function of weight* $-2/(n+1)$ *of* F^n by Matsumoto [15]. Differentiating (3.16) by y^i and putting

$$(3.17)\qquad K_i=\dot{\partial}_i(\log{}^*L),$$

this is written as

$$(3.17')\qquad K_i=l_i/L-2C_i/(n+1),$$

because of $\dot{\partial}_ig=2gC_i$ and $\dot{\partial}_iL=l_i$. Therefore this vector K_i is R-invariant; this fact can be also observed from (3.9′). Moreover we differentiate K_i by y^j and get

$$(3.18)\qquad K_{ij}:=\dot{\partial}_jK_i=(h_{ij}-l_il_j)/L^2-2(T_{ij}-l_iC_j-l_jC_i)/(n+1)L$$
$$-2C_mC_j{}^m{}_i/(n+1),$$

where we refer to $T_{ij}=LC_i|_j+C_il_j+C_jl_i$.

As it has been seen, we obtain two systems of R-invariant quantities of F^n:

(i) $L_{ij},\ L_{ijk},\ L_{hijk},\ \cdots\cdots\cdots\cdots\cdots\cdots,$

(ii) ${}^*L,\ K_i,\ K_{ij},\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots.$

Now, from these systems, we can construct many R-invariant tensors. First, putting

$$(3.19)\qquad H_{hij}:=(L_{hij}+L_{hi}K_j+L_{ij}K_h+L_{jh}K_i)/2,$$

(3.3) and (3.17′) lead to

$$(3.19')\qquad H_{hij}=[C_{hij}-(h_{ij}C_h+h_{jh}C_i+h_{hi}C_j)/(n+1)]/L,$$

and

$$(3.20)\qquad H_{hijk}:=\dot{\partial}_kH_{hij}=[T_{hijk}-(h_{ij}T_{hk}+h_{hj}T_{ik}+h_{hi}T_{jk})/(n+1)]/L^2$$
$$-2H_{ijh}l_k/L+\mathfrak{S}_{(ijh)}\{-H_{jkh}l_i/L+C_i{}^m{}_kH_{mhj}\}$$

is also R-invariant.

Secondly we contract H_{hij} in (3.19) by $*g^{hk}$ and get the R-invariant tensor

$$*H_i{}^k{}_j := H_{ihj}*g^{hk} = LH_i{}^k{}_j + 2Ly^k C_r H_i{}^r{}_j/(n+1)\,, \tag{3.21}$$

where we put $H_i{}^k{}_j = H_{isj}g^{sk}$. (3.19′) and (3.21) give the R-invariant tensor:

$$\begin{aligned} U_{hijk} &:= H_{hrk}*H_i{}^r{}_j - H_{hrj}*H_i{}^r{}_k \\ &= S_{hijk}/L - \mathfrak{A}_{(jk)}\{(C_{hrk}L_{ij} + C_{rij}L_{hk})C^r/(n+1) \\ &\qquad - (L_{ij}C_hC_k + L_{hk}C_iC_j + C^2L_{hk}h_{ij})/(n+1)^2\}\,. \end{aligned} \tag{3.22}$$

Thirdly from (3.3) and (3.19′) we obtain the following R-invariant tensors:

(3.23) (1) $M_{ijk} := -L_{ijk} + 2H_{ijk} = L_{ki}K_j + L_{ij}K_k + L_{jk}K_i\,,$

(2) $M_{hijk} := \dot{\partial}_k M_{hij} = \mathfrak{S}_{(ijh)}\{L_{hik}K_j + L_{hi}K_{jk}\}\,.$

Finally (3.13) and (3.17′) yield

$$K_i*g^{ij} = 0\,, \tag{3.24}$$

which will play an important role later on.

§4. Properties of the R-invariant tensors.

We shall treat C-reducible Finsler spaces. Matsumoto and Hōjō [17] proved a remarkable theorem: The metric functions of C-reducible Finsler spaces are confined solely to the Randers metric ($L=\alpha+\beta$) and the Kropina one ($L=\alpha^2/\beta$).

Definition 4.1. A Finsler space F^n ($n\geqq 3$) is called *C-reducible* if the $h(hv)$-torsion tensor C_{ijk} is written as

$$C_{hij} = (h_{hi}C_j + h_{ij}C_h + h_{jh}C_i)/(n+1)\,.$$

For H_{hij} of (3.19), from the above definition we get

Theorem 4.1. *The R-invariant tensor H_{hij} of* (3.19) *vanishes if and only if the Finsler space is C-reducible.*

Corollary 4.1. *A C-reducible Finsler space remains to be C-reducible by any Randers change.*

As $L_{ij} = \dot{\partial}_i\dot{\partial}_j L$ and $K_i = \dot{\partial}_i \log *L$, from (3.3) the second fundamental tensor L_{ijk} is written as $\partial_i\partial_j\partial_k L$. Thus from (3.19) and Theorem 4.1 we can state

Theorem 4.2. *A Finsler space F^n is C-reducible if and only if its fundamental function L satisfies a system of differential equations*

$$\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L = -\mathfrak{S}_{(ijk)}\{(\dot{\partial}_i\dot{\partial}_j L)(\dot{\partial}_k \log *L)\}\,,$$

where we put $*L = L/g^{1/n+1}$ and $g = \det(\dot{\partial}_j\dot{\partial}_i L^2/2)$.

Next, we introduce

$$V_{hijk}:=\dot{\partial}_k H_{hij}-\mathfrak{S}_{(hij)}\{H_{hir}{}^*H_j{}^r{}_k-K_iH_{hjk}\}+2K_kH_{hij}. \tag{4.1}$$

By (3.17′) and (3.19′) it is written as

$$V_{hijk}=T_{hijk}/L^2-\mathfrak{S}_{(hij)}\{h_{ij}T_{hk}/L^2+H_{ihk}C_j-h_{jk}H_{hir}C^r\}/(n+1) \tag{4.1′}$$
$$-C_kH_{hij}/(n+1).$$

Thus we get the R-invariant tensor V_{hijk} in relation to the T-tensor T_{hijk}. Since $T_{hijk}y^k=0$ and $H_{ijk}y^k=0$, contraction of (4.1′) by $^*g^{ik}$ yields

$$V_{hj}:=V_{hijk}{}^*g^{ik}=((n-1)T_{hj}-Th_{hj})/(n+1)L \tag{4.2}$$
$$-(n-1)LH_{hjr}C^r/(n+1),$$

which is also R-invariant. Moreover, owing to these tensors we construct a new R-invariant tensor

$$4^*T_{hijk}=V_{hijk}+V_{khij}+V_{jkhi}+V_{ijkh} \tag{4.3}$$
$$+2\mathfrak{S}_{(hij)}\{L_{ik}V_{hj}+L_{ij}V_{hk}\}/(n+1),$$

which, in virtue of (4.1) and (4.2), is written in the form

$$^*T_{hijk}=[T_{hijk}-T(h_{ik}h_{hj}+h_{hk}h_{ij}+h_{jk}h_{hi})/(n^2-1)]/L^2 \tag{4.3′}$$
$$-H_{jik}C_h-H_{hjk}C_i-H_{ihk}C_j-H_{hij}C_k.$$

Hence we get

Theorem 4.3. *The T-tensor T_{hijk} of a Finsler space is written as*

$$T_{hijk}=T(h_{ik}h_{hj}+h_{hk}h_{ij}+h_{jk}h_{hi})/(n^2-1) \tag{4.4}$$
$$+L^2(C_iH_{hjk}+C_jH_{ihk}+C_hH_{jik}+C_kH_{hij}),$$

if and only if the R-invariant tensor $^*T_{hijk}$ of (4.3) *vanishes.*

Corollary 4.2. *If the T-tensor T_{hijk} of F^n is written as* (4.4), *the T-tensor $\bar{T}_{hijk}$ of $\bar{F}^n$, obtained from F^n by a Randers change, is also written in the same form as* (4.4).

If a Finsler space F^n is C-reducible, Theorem 4.1 shows that the tensor H_{hij} vanishes, and so R-invariant tensors V_{hijk} and $^*T_{hijk}$ also vanish. Therefore the T-tensor T_{hijk} of a C-reducible Finsler space ([16]) is written as

$$T_{hijk}=T(h_{ik}h_{hj}+h_{hk}h_{ij}+h_{jk}h_{hi})/(n^2-1).$$

Next, we shall deal with the R-invariant tensor U_{hijk} defined by (3.22). It is obvious that U_{hijk} vanishes if a Finsler space F^n is C-reducible. Contracting (3.22) by $^*g^{ik}$ and then $^*g^{hj}$, we get

$$U_{hj}:=U_{hijk}{}^*g^{ik}=S_{hj}+\{(n-3)(C_{hrj}C^r-C_hC_j/(n+1)) \tag{4.5}$$
$$+2C^2h_{hj}/(n+1)\}/(n+1),\qquad S_{hj}=S_{hijk}g^{ik},$$

$$U:=U_{hj}{}^*g^{hj}=L(S+(n-2)C^2/(n+1)),\qquad S=S_{hj}g^{hj}. \tag{4.6}$$

On the other hand, from the definition of (3.22), the above quantities can be respectively written as follows:

$$U_{hj}=L^2H_{hrk}H_i{}^r{}_jg^{ik}, \tag{4.5'}$$

$$U=L^3H_{hrk}H_{mij}g^{hm}g^{ri}g^{kj}, \tag{4.6'}$$

where $H_{irk}g^{ik}=0$ and $H_{ijk}y^i=0$ are used. We now assume that g_{ij} is positive definite. Then (4.6′) shows that $U=0$ is equivalent to $H_{hij}=0$, and (4.6) does $S+(n-2)C^2/(n+1)=0$. Consequently, from Theorem 4.1 we have

Theorem 4.4. *A Finsler space F^n $(n>2)$ is C-reducible if and only if the v-scalar curvature S is given by*

$$S=-(n-2)C^2/(n+1)\,,$$

provided that g_{ij} be positive definite.

Remark. Since $S=C_{abc}C^{abc}-C^2$, Corollary 4.2 resembles closely to the following fact: If the v-scalar curvature S is given by $S=-C^2$ and g_{ij} is positive definite, a Finsler space is Riemannian.

Corollary 4.3. *A Finsler space F^n is C-reducible if and only if the v-Ricci tensor S_{hj} is written in the form*

$$S_{hj}=\{(n-3)(C_hC_j/(n+1)-C_{hrj}C^r)-2C^2h_{hj}/(n+1)\}/(n+1)\,, \tag{4.7}$$

provided that g_{ij} be positive definite.

Moreover we shall give another R-invariant tensor in relation to the v-curvature tensor S_{hijk}. Since $U_{hj}=\bar{U}_{hj}$ and $U=\bar{U}$ in (4.5), (4.6), it holds

$$\begin{aligned}&(1)\quad C_{hrj}C^r/(n+1)=C_hC_j/(n+1)^2+\{\bar{C}_{hrj}\bar{C}^r-\bar{C}_h\bar{C}_j/(n+1)\}/(n+1)\\ &\qquad\qquad +(\bar{S}_{hj}-S_{hj})/(n-3)+2(\bar{C}^2\bar{h}_{hj}-C^2h_{hj})/(n+1)^2(n-3)\,,\\ &(2)\quad C^2/(n+1)=\bar{L}\{\bar{S}\bar{L}/(n-2)+\bar{C}^2\bar{L}/(n+1)\}/L-S/(n-2)\,.\end{aligned} \tag{4.8}$$

Substituting from (4.8) into (3.22) and putting

$${}^*S_{hijk}:=[S_{hijk}+\mathfrak{A}_{(jk)}\{h_{ij}S_{hk}+h_{hk}S_{ij}-Sh_{ij}h_{hk}/(n-2)\}/(n-3)]/L\,, \tag{4.9}$$

we get ${}^*S_{hijk}={}^*\bar{S}_{hijk}$, that is, ${}^*S_{hijk}$ is R-invariant.

Here we introduce the concept of *S4-like* Finsler space ([14]):

Definition 4.2. A non-Riemannian Finsler space F^n $(n>4)$ is called *S4-like*, if the v-curvature tensor S_{hijk} is written in the form

$$L^2S_{hijk}=h_{hj}M_{ik}+h_{ik}M_{hj}-h_{hk}M_{ij}-h_{ij}M_{hk}\,,$$

where M_{ij} is a symmetric and indicatory tensor.

Assume that F^n be S4-like. Then the tensor M_{ij} of the above definition is given by $\{S_{ij}-Sh_{ij}/2(n-2)\}/(n-3)$ and accordingly (4.9) gives ${}^*S_{hijk}=0$ immediately. Thus we have

Theorem 4.5. *A non-Riemannian space* F^n $(n>4)$ *is S4-like if and only if the R-invariant tensor* $^*S_{hijk}$ *of* (4.10) *vanishes.*

Corollary 4.4. *If a Finsler space* F^n $(n>4)$ *is S4-like, the Finsler space* $\bar{F}^n$, *obtained from* F^n *by a Randers change of the metric, is also S4-like.*

§ 5. Projective Randers change.

In this section, we shall treat a special class of Randers satisfying

$$\partial_i b_j - \partial_j b_i = 0\,, \quad \text{i.e.,} \qquad F_{ij}=0\,, \tag{5.1}$$

that is, the covariant vector $b_i(x)$ is gradient. Such a change is called a *projective Randers change.*

Hashiguchi and Ichijyō [7] have shown an interesting result: A Randers change is projective, that is, any geodesic remains to be a geodesic by the change, if and only if $b_i(x)$ is gradient.

Throughout the present section, we restrict ourselves to projective Randers changes and from now on, we shall call a quantity which is invariant under a projective Randers change, a *projective R-invariance.*

From (1.11), (1.11) 1) and (5.1) we get

$$D^i := \bar{G}^i - G^i = \alpha y^i, \qquad \alpha = E_{00}/2\bar{L}\,. \tag{5.2}$$

Since $N^i{}_j = \dot{\partial}_j G^i$, the above gives

$$D^i{}_j = \bar{N}^i{}_j - N^i{}_j = \alpha_j y^i + \alpha \delta^i{}_j\,, \qquad \alpha_j = \dot{\partial}_j \alpha\,. \tag{5.3}$$

We consider the R-invariant vector K_i of (3.17′). Differentiating covariantly by x^j, we get

$$\bar{K}_{i\top 0} = K_{i|0} - 2K_{ir}D^r - K_r D^r{}_i\,, \tag{5.4}$$

where the symbol $(\top)$ denotes the h-covariant differentiation with respect to $C\bar{\Gamma}$.

On the other hand, from $l_{i|j} = L_{|j} = 0$, (3.17′) yields

$$K_{i|0} = -2C_{i|0}/(n+1) = -2P_i/(n+1)\,, \qquad (P_i = P_i{}^r{}_r)\,.$$

In the same manner, we get $\bar{K}_{i\top 0} = -2\bar{P}_i/(n+1)$. Therefore, (5.4) is written as

$$\bar{P}_i = P_i - (n+1)\alpha K_i/2 + (n+1)\alpha_i/2\,. \tag{5.5}$$

Differentiating this by y^j, we get

$$\bar{P}_{ij} = P_{ij} - (n+1)(\alpha_j K_i + \alpha K_{ij})/2 + (n+1)\alpha_{ij}/2\,, \tag{5.6}$$

where we put $P_{ij} = \dot{\partial}_j P_i$ and $\alpha_{ij} = \dot{\partial}_j \alpha_i$.

Next, (5.3) gives

$$\bar{G}_j{}^i{}_k = G_j{}^i{}_k + \alpha_{jk} y^i + \alpha_j \delta^i{}_k + \alpha_k \delta^i{}_j\,. \tag{5.7}$$

Further partial differentiation by y^h yields

$$\bar{G}_j{}^i{}_{kh}=G_j{}^i{}_{kh}+\alpha_{jkh}y^i+\alpha_{jk}\delta^i{}_h+\alpha_{jh}\delta^i{}_k+\alpha_{kh}\delta^i{}_j\,. \tag{5.8}$$

Summing this with respect to $i=h$, from the homogeneity of α_{jk}, we get

$$\alpha_{jk}=(\bar{G}_{jk}-G_{jk})/(n+1)\,,\qquad G_i{}^r{}_{jr}=G_{ij}\,. \tag{5.9}$$

Substituting from (5.9) into (5.6), we get

$$\bar{P}_{ij}=P_{ij}-(n+1)(\alpha_jK_i+\alpha K_{ij})/2+(\bar{G}_{ij}-G_{ij})/2\,. \tag{5.6$'$}$$

Contracting this by the R-invariant tensor $*g^{ij}$ given by (3.13) and referring to $G_{ij}y^i=0$ and (3.24), we have

$$\bar{P}=P-(n+1)\alpha{}^*K/2+\bar{G}-G\,, \tag{5.10}$$

where we put $P=P_{ij}{}^*g^{ij}$, $*K=K_{ij}{}^*g^{ij}$ and $G=G_{ij}{}^*g^{ij}/2$. If we put

$$\Phi=-2(P-G)/(n+1){}^*K\,, \tag{5.10) 1)}$$

(5.10) is written as

$$\alpha=\bar{\Phi}-\Phi\,. \tag{5.10$'$}$$

For α_i in (5.5), we have

$$\alpha_i=\dot{\partial}_i(\bar{\Phi}-\Phi)=2(\bar{P}_i-P_i)/(n+1)+(\bar{\Phi}-\Phi)K_i\,. \tag{5.11}$$

Further, by substituting from (5.11) into (5.8), we have

$$\begin{aligned}\bar{G}_{jk}-G_{jk}&=\dot{\partial}_k\{2(\bar{P}_j-P_j)+(n+1)(\bar{\Phi}-\Phi)K_j\}\\&=2(\bar{P}_{jk}-P_{jk})+(n+1)\{\dot{\partial}_k(\bar{\Phi}-\Phi)K_j+(\bar{\Phi}-\Phi)K_{jk}\}\,.\end{aligned} \tag{5.12}$$

Consequently D^i in (5.2) and $D^i{}_j$ in (5.3) are respectively rewritten in the form

$$D^i=(\bar{\Phi}-\Phi)y^i, \tag{5.2$'$}$$

$$D^i{}_j=\{2(\bar{P}_j-P_j)/(n+1)+(\bar{\Phi}-\Phi)K_j\}\,y^i+(\bar{\Phi}-\Phi)\delta^i{}_j\,, \tag{5.3$'$}$$

and $\bar{G}_j{}^i{}_k$ in (5.7) are written as

$$\begin{aligned}\bar{G}_j{}^i{}_k=G_j{}^i{}_k&+(\bar{G}_{jk}-G_{jk})y^i/(n+1)+\{2(\bar{P}_j-P_j)/(n+1)+(\bar{\Phi}-\Phi)K_j\}\,\delta^i{}_k\\&+\{2(\bar{P}_k-P_k)/(n+1)+(\bar{\Phi}-\Phi)K_k\}\,\delta^i{}_j\,.\end{aligned} \tag{5.7$'$}$$

Thus from (5.3), (5.3′), (5.7) and (5.7′) we obtain projectively R-invariant connection coefficients as follows:

$$\begin{aligned}&(1)\quad {}^*N^i{}_j:=N^i{}_j+I^i{}_j\,,\\&(2)\quad {}^*G_j{}^i{}_k=G_j{}^i{}_k+I_j{}^i{}_k\,,\end{aligned} \tag{5.13}$$

where tensors $I^i{}_j$ and $I_j{}^i{}_k$ are respectively given by

$$\begin{aligned}&(1)\quad I^i{}_j=-\{2P_j/(n+1)+\Phi K_j\}\,y^i-\Phi\delta^i{}_j\,,\\&(2)\quad I_j{}^i{}_k=-[G_{jk}y^i+(2P_j+(n+1)\Phi K_j)\delta^i{}_k\\&\qquad\qquad +(2P_k+(n+1)\Phi K_k)\delta^i{}_j]/(n+1)\,.\end{aligned} \tag{5.14}$$

Moreover we can derive projectively R-invariant quantities from P_i and G_{jk}. First substituting from (5.11) into (5.5), it is clear that the vector

$$*P_i := 2P_i/(n+1) - \Phi_i + \Phi K_i\,, \qquad \Phi_i = \dot{\partial}_i \Phi \tag{5.15}$$

satisfies $*P_i = *\bar{P}_i$. Therefore we immediately get a projectively R-invariant tensor

$$*P_{ij} = \dot{\partial}_j P_i = 2P_{ij}/(n+1) - \dot{\partial}_j(\Phi K_i - \dot{\partial}_i \Phi)\,. \tag{5.16}$$

Secondly from (5.12) and (5.16) we get another projective R-invariance

$$*U_{jk} := G_{jk} - \dot{\partial}_k \{2P_j + (n+1)\Phi K_j\}\,. \tag{5.17}$$

Hence we obtain

Proposition 5.1. *The covariant vector* $*P_i$ *in* (5.15) *and the tensor* $*U_{ij}$ *in* (5.17) *are projectively R-invariant.*

Next, in terms of connection coefficients $(G_j{}^i{}_k,\ N^i{}_j)$, the $(v)h$-torsion tensor $R_j{}^i{}_k$ and the hv-curvature tensor $G_i{}^h{}_{jk}$ are respectively written as

$$R_j{}^i{}_k = \mathfrak{A}_{(jk)} \{\partial_k N^i{}_j - G_j{}^i{}_r N^r{}_k\}\,, \qquad G_i{}^h{}_{jk} = \dot{\partial}_i G_j{}^h{}_k\,.$$

From $*N^i{}_j$ and $*G_j{}^i{}_k$ in (5.13), we can introduce projectively R-invariant tensors

$$*R_j{}^i{}_k := \mathfrak{A}_{(jk)} \{\partial_k {*N^i{}_j} - {*G_j{}^i{}_r} {*N^r{}_k}\}\,, \qquad *G_i{}^h{}_{jk} = \dot{\partial}_i {*G_j{}^h{}_k}\,.$$

These tensors are rewritten in the form

$$*R_j{}^i{}_k = R_j{}^i{}_k + \mathfrak{A}_{(jk)} \{I^i{}_{j;\,k} + I_j{}^i{}_r I^r{}_k\}\,, \tag{5.18}$$

$$*G_i{}^h{}_{jk} = G_i{}^h{}_{jk} + I_i{}^h{}_{jk}\,, \qquad I_i{}^h{}_{jk} = \dot{\partial}_k I_i{}^h{}_j\,, \tag{5.19}$$

where the symbol (;) denote the h-covariant differentiation with respect to $B\Gamma$.

We are concerned with the tensor $*R_j{}^i{}_k$. If F^n is a Berwald space, that is, $G_j{}^i{}_k = G_j{}^i{}_k(x)$, then $P_j{}^i{}_k = 0$ and $G_j{}^i{}_{kh} = 0$, therefore the scalar Φ in (5.10) 1) vanishes. Consequently from (5.14) and (5.18.) we can state

Proposition 5.2. *If a space* F^n *is a Berwald space, the* $(v)h$*-torsion tensor* $R_j{}^i{}_k$ *is equal to the projective R-invariant tensor* $*R_j{}^i{}_k$.

Next, contracting (5.18) by y^j, we have

$$*R_0{}^i{}_k = R_0{}^i{}_k + 2I^i{}_{;\,k} - I^i{}_{k;\,0} + I^i{}_r I^r{}_k - 2I_k{}^i{}_r I^r, \tag{5.20}$$

where $2I^i = I^i{}_0$. Further, contracting this by L_{hi}, and then substituting from (5.14) it is obvious that the tensor

$$*R_{h0k} := L_{hi} {*R_0{}^i{}_k} = (R_{h0k} + \Psi h_{hk})/L \tag{5.21}$$

satisfies $*\bar{R}_{h0k} = *R_{h0k}$, where Ψ is given by

$$\Psi := \Phi_{;\,0} - \Phi^2. \tag{5.21) 1}$$

Further $*\bar{R}_{h0k}*\bar{g}^{hk}=*R_{h0k}*g^{hk}$ gives

$$\Psi=-(R_{00}-\bar{R}_{00})/(n-1)+\bar{\Psi}\,,\qquad R_{00}=R_0{}^r{}_{0r}\,. \tag{5.22}$$

We substitute from (5.22) into (5.21), and get a projectively R-invariant tensor:

$$W_{hk}=\{R_{h0k}-R_{00}h_{hk}/(n-1)\}/L\,. \tag{5.23}$$

We treat a Finsler space F^n $(n>2)$ of scalar curvature $K(x, y)$. Such a space is characterized by the following equation:

$$R_{h0k}=L^2Kh_{hk}\,,\qquad K=R_{00}/(n-1)L^2. \tag{5.24}$$

Thus from (5.23) and (5.24) we can state

Theorem 5.1. *A Finsler space is of scalar curvature, if and only if the projectively R-invariant tensor W_{hk} in* (5.23) *vanishes identically.*

Corollary 5.1. *If a Finsler space F^n is of scalar curvature, then $\bar{F}^n$, obtained from F^n by a projectively Randers change, is also of scalar curvature.*

Finally we consider the second fundamental tensor L_{ijk} of the Randers change. Differentiating covariantly this tensor, we get

$$\bar{L}_{ijk\top 0}=L_{ijk|0}-2L_{ijkr}D^r-L_{rjk}D^r{}_i-L_{irk}D^r{}_j-L_{ijr}D^r{}_k\,, \tag{5.25}$$

where we used the relation (5.3). From $L_{|j}=0$ and $L_{ij|k}=0$, (3.3) leads to

$$\bar{L}_{ijk\top 0}=2\bar{P}_{ijk}/\bar{L}\,,\qquad L_{ijk|0}=2P_{ijk}/L\,. \tag{5.26}$$

By means of (5.2′) and (5.3′), it is clear that the tensor

$$\begin{aligned}*P_{ijk}:&=P_{ijk}/L-(L_{jk}P_i+L_{ki}P_j+L_{ij}P_k)/(n+1)\\&\quad-\Phi(L_{ijk}+L_{jk}K_i+L_{ki}K_j+L_{ij}K_k)/2\end{aligned} \tag{5.27}$$

satisfies $*P_{ijk}=*\bar{P}_{ijk}$. Because (3.19′) gives

$$2H_{ijk|0}=\{P_{ijk}-(h_{ij}P_k+h_{jk}P_i+h_{ki}P_j)/(n+1)\}/L\,, \tag{5.28}$$

the projective R-invariant tensor $*P_{ijk}$ is rewritten as

$$*P_{ijk}=2H_{ijk|0}-\Phi H_{ijk}\,. \tag{5.27′}$$

We shall introduce the concept of P-reducibility which is defined as follows:

Definition 5.1. A Finsler space is called *P-reducible* if the $v(hv)$-torsion tensor P_{ijk} is written as

$$P_{ijk}=(h_{ij}P_k+h_{jk}P_i+h_{kh}P_j)/(n+1)\,.$$

It is obvious that a C-reducible Finsler space is P-reducible. Therefore Theorem 4.1 leads to

Theorem 5.2. *A P-reducible Finsler space F^n is C-reducible if and only if* *the projectively R-invariant tensor* $^*P_{ijk}$ *vanishes, provided* $\Phi\neq 0$.

Theorem 5.3. *A P-reducible Finsler space* F^n *remains to be P-reducible by a projective Randers change if and only if the quantity* Φ *in* (5.10) 1) *is projective R-invariant, provided that* F^n *be non-C-reducible.*

Proof. Assume that both of two Finsler spaces F^n and $\bar{F}^n$ are P-reducible. Then (5.27′) implies $(\bar{\Phi}-\Phi)H_{ijk}=0$, and so $\Phi=\bar{\Phi}$ holds, because of $H_{ijk}\neq 0$.

Conversely if $\Phi=\bar{\Phi}$ holds, (5.27′) gives $H_{ijk|0}=\bar{H}_{ijk\top 0}$, so that the P-reducibility of F^n implies that of $\bar{F}^n$.

§ 6. Douglas tensor.

In the theory of projective changes in Finsler spaces, we have two essential projective invariants, one is the projective h-curvature tensor $W_i{}^h{}_{jk}$, which is related to a Finsler space of scalar curvature and the other is the projective hv-curvature tensor $D_i{}^h{}_{jk}$. These tensors $W_i{}^h{}_{jk}$ and $D_i{}^h{}_{jk}$ are called the *Weyl curvature tensor* and the *Douglas tensor* respectively.

In this section we shall deal with the projectively R-invariant tensor $^*G_i{}^h{}_{jk}$ of (5.19). By means of (5.14) 2) $^*G_i{}^h{}_{jk}$ is written as

$$(6.1)\qquad \begin{aligned} {}^*G_i{}^h{}_{jk}=&G_i{}^h{}_{jk}-(G_{ij\cdot k}y^h+G_{ij}\delta^h{}_k)/(n+1)\\ &-\dot{\partial}_k\{2P_j+(n+1)\Phi K_j\}\delta^h{}_i/(n+1)\\ &-\dot{\partial}_k\{2P_i+(n+1)\Phi K_i\}\delta^h{}_j/(n+1)\,. \end{aligned}$$

From $^*G_i{}^h{}_{jk}$ and $^*U_{ij}$ in (5.17), we can introduce the projectively R-invariant tensor

$$D_i{}^h{}_{jk} := {}^*G_i{}^h{}_{jk}-({}^*U_{jk}\delta^h{}_i+{}^*G_{ik}\delta^h{}_j)/(n+1)\,,$$

which, in virtue of (5.17) and (6.1), is written as

$$(6.2)\qquad D_i{}^h{}_{jk}=G_i{}^h{}_{jk}-[G_{ij\cdot k}y^h+\mathfrak{S}_{(ijk)}\{G_{ij}\delta^h{}_k\}]/(n+1)\,.$$

This tensor is nothing but the well-known Douglas tensor. We have attempted to derive this from projective Randers change.

We now assume that the tensor $^*G_i{}^h{}_{jk}$ vanishes. Then (6.1) gives

$$(6.3)\qquad \begin{aligned} G_i{}^h{}_{jk}=&[G_{ij.k}y^h+G_{ij}\delta^h{}_k+\dot{\partial}_k\{2P_i+(n+1)\Phi K_i\}\delta^h{}_j\\ &+\dot{\partial}_k\{2P_j+(n+1)K_j\}\delta^h{}_i]/(n+1)\,. \end{aligned}$$

Summing (6.3) with respect to $h=k$, we get

$$G_{ij}=\dot{\partial}_j\{2P_i+(n+1)\Phi K_i\}\,,$$

so that $G_i{}^h{}_{jk}$ is rewritten in the form

$$(6.3')\qquad G_i{}^h{}_{jk}=(G_{ij.k}y^h+G_{ij}\delta^h{}_k+G_{ki}\delta^h{}_j+G_{kj}\delta^h{}_i)/(n+1)\,.$$

Therefore, (6.2) and (6.3′) imply $D_i{}^h{}_{jk}=0$. Hence we can state

Proposition 6.1. *If the projectively R-invariant tensor* $*G_i{}^h{}_{jk}$ *vanishes, then the Douglas tensor* $D_i{}^h{}_{jk}$ *vanishes.*

Next, we consider a relation between the *hv*-curvature tensor $P_i{}^h{}_{jk}$ with respect to $C\Gamma$ and the projective *hv*-curvature tensor $D_i{}^h{}_{jk}$. Contracting (6.2) by y_h, we have

$$G_{ij.k}=-(n+1)D_i{}^r{}_{jk}y_r/L^2-\mathfrak{S}_{(ijk)}\{G_{ij}l_k\}/L\,, \tag{6.4}$$

so that $D_i{}^h{}_{jk}$ is rewritten in the form

$$D_i{}^h{}_{jk}=D_i{}^r{}_{jk}y_r y^h/L^2+G_i{}^h{}_{jk}-\mathfrak{S}_{(ijk)}\{G_{ij}h^h{}_k\}/(n+1)\,. \tag{6.2$'$}$$

Since $\dot\partial_k F_i{}^h{}_j=P_i{}^h{}_{jk}+C_i{}^h{}_{k|j}-C_i{}^h{}_s P_j{}^s{}_k$, the equation $G_i{}^h{}_j=F_i{}^h{}_j+P_i{}^h{}_j$, which shows the relation between the Cartan connection $C\Gamma$ and the Berwald connection $B\Gamma$, gives

$$G_i{}^h{}_{jk}=P_i{}^h{}_{jk}+C_i{}^h{}_{k|j}-C_i{}^h{}_s P_j{}^s{}_k+\dot\partial_k P_i{}^h{}_j\,, \tag{6.5}$$

$$G_{ij}=P_i{}^r{}_{jr}+C_{i|j}-C_i{}^r{}_s P_j{}^s{}_r+\dot\partial_r P_i{}^r{}_j\,. \tag{6.6}$$

Substituting from (6.5) and (6.6) into (6.2′), we get

$$\begin{aligned} h^h{}_r D_i{}^r{}_{jk}=&P_i{}^h{}_{jk}+C_i{}^h{}_{k|j}-C_i{}^h{}_s P_j{}^s{}_k+2P_{ijk}y^h+\dot\partial_k P_i{}^h{}_j\\ &-\mathfrak{S}_{(ijk)}\{(P_j{}^r{}_{kr}+C_{j|k}-C_j{}^r{}_s P_k{}^s{}_r+\dot\partial_r P_j{}^r{}_k)h^h{}_i\}/(n+1)\,. \end{aligned} \tag{6.7}$$

We shall be concerned with a Landsberg space F^n, which is defined by $P_i{}^h{}_{jk}=0$. If the Douglas tensor $D_i{}^h{}_{jk}$ of this F^n vanishes, then (6.7) implies

$$C_{hik|j}=(h_{hi}C_{j|k}+h_{hj}C_{k|i}+h_{hk}C_{i|j})/(n+1)\,, \tag{6.8}$$

which immediately gives

$$C_{h|j}=C^r{}_{|r}h_{hj}/(n-1)\,. \tag{6.9}$$

Further (6.6) and (6.9) yield $G_{ij}=C^r{}_{|r}h_{ij}/(n-1)$. Differentiating this by y^k and referring to $\dot\partial_k h_{ij}=2C_{ijk}-(h_{ik}l_j+h_{jk}l_i)/L$, we get

$$\dot\partial_k G_{ij}=\{(\dot\partial_k C^r{}_{|r})h_{ij}+C^r{}_{|r}(2C_{ijk}-h_{ik}l_j/L-h_{jk}l_i/L)\}/(n-1)\,. \tag{6.10}$$

Owing to $\dot\partial_k G_{ij}=\dot\partial_j G_{ik}$, (6.10) gives $\dot\partial_j C^r{}_{|r}=-C^r{}_{|r}l_j/L$, so that (6.10) is rewritten in the form

$$\dot\partial_k G_{ij}=C^r{}_{|r}\{2C_{ijk}-(h_{ik}l_j+h_{jk}l_i+h_{ij}l_k)/L\}/(n-1)\,. \tag{6.10$'$}$$

On the other hand, from (6.4) and $G_{ij}=C^r{}_{|r}h_{ij}/(n-1)$, we have

$$\dot\partial_k G_{ij}=-C^r{}_{|r}(h_{ij}l_k+h_{jk}l_i+h_{ki}l_j)/(n-1)L\,. \tag{6.11}$$

Comparing (6.10′) with (6.11), we obtain $C^r{}_{|r}=0$, so that (6.8) and (6.9) lead to $C_{ijk|h}=0$, that is, the Finsler space is a Berwald space.

Conversely, if a Finsler space F^n is a Berwald space, then the Douglas tensor $D_i{}^h{}_{jk}$ vanishes obviously. Summarizing up all the above, we can state

Theorem 6.1. *Assume that a Finsler space F^n is a Landsberg space. The Douglas tensor $D_i{}^h{}_{jk}$ of F^n vanishes if and only if the Finsler space F^n is a Berwald space.*

§ 7. Decomposable tensors.

The tangent vector space F_x^n with the origin removed at any point x of F^n is regarded as a Riemannian space with the fundamental quadratic form $ds^2=g_{ij}(x, y)dy^i dy^j$. The indicatrix I_x at x is a hypersurface of the Riemannian space F_x^n which is defined by the equation $L(x, y)=1$ (x is fixed) ([13]).

Definition 7.1. A Finsler tensor U_{ij} is called *indicatory* if U_{ij} satisfies equations $U_{ij}l^i=U_{ij}l^j=0$.

From an arbitrary tensor U_{ij} we get an indicatory tensor $'U_{ij}$ which is given by

$$'U_{ij}=U_{lm}h^l{}_i h^m{}_j .$$

It is clear that the tensors h_{ij}, C_{ijk}, P_{ijk}, S_{hijk} and T_{hijk} are indicatory. From (3.2), (3.19′), (4.4′), (4.10) and (5.23) we have

Proposition 7.1. *Tensors L_{ij}, H_{hij}, $*T_{hijk}$, $*S_{hijk}$, W_{hk} and $*P_{ijk}$ are indicatory.*

Here we treat R-invariant tensors in § 3. First, for the covariant vector K_i in (3.17′), by applying the above indicatorization we immediately get $'K_i=-2C_i/(n+1)$. For the second fundamental tensor L_{ijk} of the Randers change in (3.3), from the homogeneity we get $L_{i00}=0$, and hence by applying the above indicatorization it follows that $'L_{ijk}=2C_{ijk}/L$. Moreover, for M_{ijk} and M_{hijk} in (3.21), we have

$$'M_{ijk}=-2(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1)L\,, \tag{7.1}$$

$$'M_{hijk}=\mathfrak{S}_{(ijh)}\{-4C_{hik}C_j/(n+1)L+h_{ij}(h_{hk}/L-2T_{hk}/(n+1))/L^2\}. \tag{7.2}$$

Now, for the $h(hv)$-torsion tensor C_{ijk}, it is obvious that F^n is Riemannian if the tensor C_{ijk} vanishes, and Deicke's theorem ([12]) shows that F^n is also Riemannian even if the torsion vector C_i vanishes. Therefore for above indicatory tensors we can state

Proposition 7.2. (1) *One of indicatorized tensors $'K_i$ and $'M_{ijk}$ vanishes if and only if the Finsler space F^n is a Riemannian space.*

(2) *The indicatorized tensor $'L_{ijk}$ vanishes if and only if the Finsler space is a Riemannian space.*

Proof. If $'K_i$ vanishes, from $'K_i=-2C_i/(n+1)$ we immediately $C_i=0$. Next, if $'M_{ijk}$ vanishes, in virtue of $h_{ij}C^i=C_j$, (7.1) yields $C_j=0$. In any case, owing to Deicke's theorem asserts that the space F^n is Riemannian. For $'L_{ijk}$ of (2), we observe that $'L_{ijk}=0$ is equivalent to $C_{ijk}=0$, so F^n is Riemannian. It is clear that the converses of those facts are true.

Next, we will treat another hypersurface of the Riemannian space F^n_x called a *g-hypercone* which is defined by the equation $g(x, y)=c$ (c=const., $\neq 0$) ([24]). The concept of g-hyperconic tensor is defined as follows:

Definition 7.2. Let V_{ijk} be a Finsler tensor, for instance, of (0, 3)-type. If V_{ijk} satisfies $V_{ijk}N^i=V_{ijk}N^j=V_{ijk}N^k=0$, then V_{ijk} is called a *g-hyperconic tensor*, where N^i is the unit normal of g-hypercone, i.e., $N^i=C^i/C$, $C^2=C_iC^i$.

A method to derive the g-hyperconic tensor $''V_{ij}$ from an arbitrary Finsler tensor V_{ij} is the following, similar to the case of indicatory tensor:

$$''V_{ij}=V_{mn}N^m{}_iN^n{}_j\,,$$

where $N_{ij}=g_{ij}-N_iN_j$, $N^m{}_i=g^{mr}N_{ri}$, $N_i=g_{ij}N^j$.

Now we are concerned with a Finsler tensor X_{ij}. The g-hyperconic tensor and the indicatory tensor derived from X_{ij} are respectively given by

$$\text{(7.3)}\qquad \begin{aligned} &(1)\quad ''X_{ij}=X_{ij}-(N_iX_{gj}+N_jX_{ig})+X_{gg}N_iN_j\,,\\ &(2)\quad 'X_{ij}=X_{ij}-(l_iX_{0j}+l_jX_{i0})/L+X_{00}l_il_j/L^2, \end{aligned}$$

where the suffix g stands for the contraction by N^i. Contracting (7.3) 2) by N^i and then N^j successively we have

$$'X_{gj}=X_{gj}-X_{g0}l_j/L\,,\qquad 'X_{gg}=X_{gg}\,,$$

by means of $N^il_i=0$. Thus (7.3) leads to

$$\text{(7.4)}\qquad ''('X_{ij})='X_{ij}-X_{g0}(N_il_j+N_jl_i)/L+X_{gg}N_iN_j-(N_iX_{gj}+N_jX_{ig})\,.$$

Consequently, from (7.3) and (7.4) we get

Proposition 7.3. *Let X_{ij} be a Finsler tensor, for instance, of* (0, 2)*-type. Then X_{ij} can be written as*

$$\text{(7.5)}\qquad X_{ij}='X_{ij}+''X_{ij}-''('X_{ij})+{}^*X_{ij}\,,$$

where we put ${}^*X_{ij}=(X_{g0}N_il_j+X_{0g}N_jl_i)/L$.

Remark. A (0, r)-type, for instance, Finsler tensor $X_{ij\cdots k}$ can be similarly written as (7.5), i.e.,

$$X_{ij\cdots k}='X_{ij\cdots k}+''X_{ij\cdots k}-''('X_{ij\cdots k})+{}^*X_{ij\cdots k}\,.$$

For tensors $'X_{ij}$, $''X_{ij}$ and ${}^*X_{ij}$ in (7.5), from (7.3) and (7.4) we can state

Proposition 7.4. *An arbitrary Finsler tensor X_{ij} satisfies*

$$\text{(7.6)}\qquad (1)\quad '({}^*X_{ij})=''({}^*X_{ij})=0\,,\qquad (2)\quad '(''X_{ij})=''('X_{ij})\,.$$

Proposition 7.5. *A Finsler tensor X_{ij} is indicatory (resp. g-hyperconic) if and only if X_{ij} satisfies*

$$*X_{g0}=0 \quad \text{and} \quad ''X_{ij}=''('X_{ij}) \quad (\text{resp. } 'X_{ij}='(''X_{ij}))\,.$$

Hence we are led to the following definition:

Definition 7.3. Let X_{ij} be a Finsler tensor, for instance, of (0, 2)-type. If X_{ij} is written as

$$X_{ij}='X_{ij}+''X_{ij}+{}^*X_{ij}\,, \tag{7.7}$$

then X_{ij} is called a *decomposable tensor.*

From the above and (7.5) we have

Theorem 7.1. *A* (0, 2)*-type, for instance, Finsler tensor X_{ij} is decomposable if and only if the tensor $''('X_{ij})$ vanishes.*

Remark. It is noted that Propositions 7.3, 7.4, 7.5 and Theorem 7.1 are easily generalized for a (0, r)-type Finsler tensor.

Now, we consider Finsler spaces with the torsion tensor C_{ijk} of which satisfies some tensor equations ([18], [19], [20]).

Definition 7.4. A non-Riemannian Finsler space is called *quasi-C-reducible* (resp. *semi-C-reducible,* resp. *C2-like*), if the $h(hv)$-torsion tensor C_{ijk} is written in the form (1) (resp. (2), resp. (3))

$$\begin{aligned} &(1)\quad C_{ijk}=P_{ij}C_k+P_{jk}C_i+P_{ki}C_j\,,\\ &(2)\quad C_{ijk}=p(h_{ij}C_k+h_{jk}C_i+h_{ki}C_j)/(n+1)+qC_iC_jC_k/C^2, \qquad p+q=1\,,\\ &(3)\quad C_{ijk}=C_iC_jC_k/C^2, \quad (C^2\neq 0). \end{aligned} \tag{7.8}$$

Further, we consider, Finsler spaces in which C_{ijk} satisfies the tensor equation

$$\begin{gathered} C_{ijk}=(C_{ij}+C_{jk}C_i+C_{ki}C_j)/C^2+2QC_iC_jC_k/C^3,\\ (C_{ij}=C_{irj}C^r, \quad Q=-(1/C^3)C^iC^jC^kC_{ijk})\,. \end{gathered} \tag{7.9}$$

It is noted that the torsion tensor C_{ijk} of a semi-C-reducible or $C2$-like Finsler space are written in the form (7.9).

Here we deal with decomposability of some R-invariant tensors. First for K_i in (3.17′), we have $''K_i=l_i/L$ and $'K_i=-2C_i/(n+1)$, so that the R-invariant vecotr K_i is written as

$$K_i='K_i+''K_i\,. \tag{7.10}$$

Secondly, for M_{ijk} in (3.21), (7.1) and (7.2) give $''('M_{ijk})=0$, therefore this tensor is also written in the form

(7.11) $$M_{ijk}={}'M_{ijk}+{}''M_{ijk}+{}^*M_{ijk}\,.$$

Finally, for the second fundamental tensor L_{ijk} of the Randers change in (3.3), by applying the method of (7.3) we have

$$''('L_{ijk})=C_{ijk}-(C_{ijg}N_k+C_{gjk}N_i+C_{igk}N_j)+2C_{ggg}N_iN_jN_k\,,$$

therefore, which leads to $''('L_{ijk})=0$ if C_{ijk} of F^n is written as (7.9). Paying attention to $N^i=C^i/C$, the converse is true also. Consequently from Theorem 7.1, (7.10), (7.11) and (7.12) we can state

Theorem 7.2. (1) *The R-invariant vector K_i of* (3.17) *and the R-invariant tensor M_{ijk} of* (3.21) *are decomposable.*

(2) *The R-invariant tensor L_{ijk} of* (3.3) *is decomposable if and only if the $h(hv)$-torsion tensor C_{ijk} of F^n is written in the form* (7.9).

We consider a condition for a Finsler space to be semi-C-reducible. If F^n is semi-C-reducible, then the tensor L_{ijk} is the decomposable tensor, because of Theorem 7.2. Thus we obtain

Theorem 7.3. *A Finsler space is semi-C-reducible if and only if the $\bar{R}$-invariant tensor L_{ijk} is the decomposable tensor and the $h(hv)$-torsion tensor C_{ijk} satisfies*

$$C_{ijr}C^r=pC^2h_{ij}/(n+1)+\{1-(n-1)p/(n+1)\}\,C_iC_j\,,$$

where p is a scalar.

§8. Relation between projective change and β-change.

We shall return to a β-change of the metric. For two Finsler spaces $F^n=(M^n, L)$ and $\bar{F}^n=(M^n, \bar{L})$, if any geodesic on F^n is also a geodesic on $\bar{F}^n$ and the inverse is true, the change $L\to\bar{L}$ of the mctric is called *projective.* A geodesic on F^n is given by a system of dfferential equations

(8.1) $$dy^i/dt+2G^i(x, y)=\tau y^i, \qquad y^i=dx^i/dt\,,$$

where $\tau=(d^2s/dt)/(ds/dt)$. $G^i(x, y)$ are (2) p-homogeneous functions in y^i. We are now in a position to find a condition for a β-change to be projective. For this purpose we deal with the Euler-Lagrange differential equations $B_i=0$, where B_i is defined by

$$B_i=\partial_iL-d(\dot{\partial}_iL)/dt\,.$$

From (1.1) and (1.2), we have $Ldf_1/dt+\beta df_2/dt=0$, so that the Euler-Lagrange differential equations $\bar{B}_i=0$ for $\bar{F}^n$ are given by

$$\bar{B}_i=f_1B_i+2f_2F_{0i}-m_idf_2/dt=0\,.$$

In virtue of (1.2) and (1.4) 1), $\bar{B}_i$ are written as

(8.2) $$f\bar{B}_i=pLB_i-q_0B_rm^rm_i+A_i\,,$$

where the covariant vector A_i is defined by

(8.3) $$A_i=2qF_{0i}-q_0E_{00}m_i\,.$$

From (8.2) we get

Proposition 8.1. *A β-change $L\to\bar{L}=f(L,\beta)$ of the metric is projective if and only if the covariant vector A_i in* (8.3) *vanishes identically.*

Proof. Since $B_i=0$ (resp. $\bar{B}_i=0$) are equations of a geodesic on F^n (resp. $\bar{F}^n$), we immediately obtain $A_i=0$ if a β-change is projective.

Conversely if $A_i=0$ holds, then (8.2) shows that $B_i=0$ lead to $\bar{B}_i=0$. On the other hand, we observe from (8.2) and $A_i=0$ that $\bar{B}_i=0$ give $pLB_i-q_0B_rM^rm_i=0$. Contracting this by m^i and referring to $m^2=\nu$, $pL-\nu q_0\neq 0$, we get $B_rm^r=0$, so that $B_i=0$ hold. Consequently any geodesic remains to be a geodesic by a β-change.

We shall continue the discussion for the condition in Proposition 8.1, i.e., $A_i=0$, which, by means of $q=ff_2$ and $q_0=ff_{22}$, is written as

(8.4) $$2f_2F_{i0}+f_{22}m_iE_{00}=0\,.$$

On the other hand, (1.1) and (1.10) give

$$Lf_{122}+\beta f_{222}=-f_{22}\,,\qquad E_{00}=b_{i|j}y^iy^j.$$

Then, differentiating (8.4) by y^j, we have

(8.5) $$2f_2F_{ij}+\{2(m_iE_{j0}+m_jF_{i0})+E_{00}(\beta y_iy_j/L^2-m_iy_j-m_jy_i-\beta g_{ij})/L^2\}f_{22}+f_{222}E_{00}m_im_j=0\,.$$

Because F_{ij} in the above is skew-symmetric in indices, (8.5) leads to

(8.6) $$f_{22}(m_iE_{j0}+m_jE_{i0})+(f_{222}-f_{22}^2/f_2)E_{00}m_im_j+E_{00}f_{22}(\beta y_iy_j/L^2-m_iy_j-m_jy_i-\beta g_{ij})/L^2=0\,.$$

Contraction of this by b^ib^j does

(8.7) $$2f_{22}E_{\beta 0}+(f_{222}-f_{22}^2/f_2)\nu E_{00}-3\beta f_{22}E_{00}/L^2=0\,,$$

where we used $\nu=m_ib^i=b^2-\beta^2/L^2$ $(\neq 0)$. Moreover we contract (8.6) by g^{ij} and get

(8.8) $$2f_{22}E_{\beta 0}+(f_{222}-f_{22}^2/f_2)\nu E_{00}-(n+1)\beta f_{22}E_{00}/L^2=0\,.$$

Comparing (8.7) with (8.8), we can conclude

(8.9) $$(n-2)\beta f_{22}E_{00}=0\,,$$

therefore we are led discussions of two cases given by

(8.10) $$\text{(A)}\quad f_{22}=0\,,\qquad \text{(B)}\quad f_{22}\neq 0\,,$$

provided $n>2$.

First, we consider the case (A). Since f is positively homogeneous in two valuables L and β of degree one, we have $f=L+\beta$, that is, the β-change is Randers, and from (8.4) we obtain $F_{i0}=0$, which shows that the covariant vector field $b_i(x)$ is gradient.

Next, we shall treat the case (B). From (8.4) and (8.9) we immediately have $F_{ij}=0$ and $E_{00}=0$, thus we get $b_{i|j}y^j=0$, which implies that $(\partial_j b_i - b_m F_i{}^m{}_j)y^i=0$. Differentiating this by y^k and referring to $(\dot{\partial}_k F_i{}^m{}_j)y^i=P_j{}^m{}_k$, we get

$$b_{k|j}=b_m P_j{}^m{}_k\,. \tag{8.11}$$

Conversely, if $b_k(x)$ satisfies the equation (8.11), the symmetry property of $P_j{}^m{}_k$ and $P_j{}^m{}_0=0$ imply that $b_k(x)$ is a gradient vector, and $E_{j0}=0$. Therefore the covariant vector A_i in (8.3) vanishes identically. Summarizing up the above, from Proposition 8.1, we have

Theorem 8.1. *A β-change $L\to\bar{L}$ of the metric is projective if and only if one of the following facts holds good:*

(1) *It is a Randers change and the covariant vector field $b_i(x)$ is gradient.*

(2) *The covariant vector field $b_i(x)$ satisfies the equation $b_{i|j}=b_m P_i{}^m{}_j$, provided $n>2$.*

Now the concept of a projective change is closely related to Finsler spaces of scalar curvature. In fact, Szabó [25] showed the following:

Theorem A. *Let F^n be a Finsler space of scalar curvature. Then the Finsler space $\bar{F}^n$, obtained from F^n by a projective change of the metric, is also of scalar curvature.*

In case of a Riemannian space, "of scalar curvature" means "of constant curvature". Moreover, the $v(hv)$-torsion tensor $P_i{}^h{}_j$ of a Riemannian space always vanishes identically, and $b_{i|j}=0$ means that $b_i(x)$ is parallel with respect to the Riemannian connection constructed from the Riemannian metric α. Owing to Theorem A and Theorem 8.1, we can conclude

Corollary 8.1. *Assume that the covariant vector field $b_i(x)$ is parallel with respect to the Riemannian connection constructed from a Riemannian metric α. Then the Finsler space $\bar{F}^n=(M^n, (\alpha, \beta))$ with the (α, β)-metric is of scalar curvature if and only if a Riemannian space $F^n=(M^n, \alpha)$ is of constant curvature.*

Corollary 8.2. ([26]). *Assume that β is a closed differential one-form. Then a Randers space $\bar{F}^n=(M^n, L+\beta)$, where L is Riemannian, is of scalar curvature, if and only if a Riemannian space $F^n=(M^n, L)$ is of constant curvature.*

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