J. Math. Kyoto Univ. (JMKYAZ)
24-2 (1984) 305-321

On convergence of holomorphic abelian differentials on the Teichmüller spaces of arbitrary Riemann surfaces

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(Received January 6, 1983)

Introduction

On the Teichmüller space of an arbitrary Riemann surface, we can consider at least two kinds of convergence of holomorphic abelian differentials. The one of them is concerned with the distortion estimates with respect to the Dirichlet norm, and is, in a sense, the most natural kind of convergence, which we call the metrical convergence (, cf. Definition 2). The other is concerned with the geometrical structure of the squares of abelian differentials, which we call the geometrical convergence (, cf. Definition 3).

We have already investigated the relation between these two kinds of convergence in the case of compact surfaces ([11]). In this paper, we will treat with the case of general surfaces and show that the geometrical convergence implies the metrical one in the case of square integrable differentials (Theorem 2). We also give sufficient conditions under which the metrical convergence implies the geometrical one (Theorems 3 and 4).

§1 is preliminaries from the theory of Teichmüller spaces and quasi-conformal mappings. The definitions of two kinds of convergence and main theorems are stated and proved in §2. Finally, as applications of main theorems, we will show in §3 that several fundamental differentials converge both metrically and geometrically.

§1. Preliminaries on the theory of Teichmüller spaces

Let a Riemann surface R^* be arbitrarily given. Then consider all pairs (R, f) of a Riemann surface R and a quasiconformal mapping f from R^* onto R. We say that (R_1, f_1) and (R_2, f_2) are equivalent if $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping from R_1 onto R_2. *The Teichmüller space* $T(R^*)$ is, by definition, the set of all equivalence classes of pairs as above. The Teichmüller space $T(R^*)$ has the usual Teichmuller metric, cf. [1] Ch. VI, and we call the topology induced by this metric *the Teichmüller topology* on $T(R^*)$. A point of $T(R^*)$ is called a marked Riemann surface, and we denote simply by R the point corresponding to any (R, f) when the marking (, i.e. the mapping) f is clear from the context. Also in the sequel, excluding trivial cases, we assume that the universal covering surface of R^* is conformally equivalent to the open unit disk $U_1=\{|z|<1\}$ and $T(R^*)$ is non-trivial.

We can define the Teichmüller space $T(R^*)$ by using Fuchsian groups acting on U_1. Let G^* be a Fuchsian equivalent of R^* acting on U_1. Here we also assume that $1, \sqrt{-1}, -1$ are contained in the limit set $L(G^*)$ of G^*. Such a Fuchsian group is called *a normalized Fuchsian group*. Now fix a normalized Fuchsian equivalent G^* of R^*, and consider all quasiconformal self-mapping of U_1 which leave $1, \sqrt{-1}, -1$ fixed. Such a mapping F is called to be *compatible* with a normalized Fuchsian group G if $F \circ g \circ F^{-1}$ is conformal for every g in G. If F is compatible with G, then F induces an isomorphism F_* of G into another normalized Fuchsian group, which is called *a quasiconformal isomorphism* of G (induced by F). Returning to G^*, we call the set of images of G^* by all quasiconformal isomorphisms of G^* *the reduced Teichmüller space* $T^{\#}(G^*)$ of G^*. Then it is well-known ([3] Lemma 2) that $T(R^*)$ can be identified with $T^{\#}(G^*)$.

Let R and R' be points in $T(R^*)$ and let G and G' correspond to R and R', respectively, in $T^{\#}(G^*)$. Then for every marking-preserving quasiconformal self-mapping f from R onto R', there is the unique quasiconformal self-mapping F of U_1 which leaves $1, \sqrt{-1}, -1$ fixed and is compatible with G such that F_* is an isomorphism from G onto G' and the projection of F on R is identical with the given f. We call this F *the lift of* f on U_1 (with respect to G).

Next in general, for every Riemann surface R, every Borel subset E of R and every homeomorphism f from R into another surface which is quasiconformal on E, we denote by $K(f, E)$ the maximal dilatation of f on E, namely,

$$K(f, E)=\text{ess.sup}_{z\in E}(|f_z|+|f_{\bar z}|)/(|f_z|-|f_{\bar z}|),$$

where f_z and $f_{\bar z}$ are the generalized derivatives of f with respect to z and $\bar z$ with generic local parameter z. Then the convergence of a sequence $\{R_n\}_{n=1}^{\infty}$ to R_0 in $T(R^*)$ in the sense of the Teichmüller topology is equivalent to the condition that there is a sequence of marking-preserving quasiconformal mapping f_n from R_0 onto R_n such that $\lim_{n\to\infty} K(f_n, R_0)=1$. Relating to this characteirzation of the Teichmüller topology, we can consider the following topology on $T(R^*)$ which is apparently weaker than the Teichmüller topology.

Definition 1. Let P be a given set of punctures of R^*, and for every R in $T(R^*)$, denote by P_R the set of punctures of R corresponding to P. We say that a sequence $\{R_n\}_{n=1}^{\infty}$ in $T(R^*)$ converges to $R_0\in T(R^*)$ in the sence of *the* (P-)*weak topology* if there is a sequence of marking-preserving homeomorphisms f_n from R_0 onto R_n such that for every neighbourhood V of P_{R_0} (in $R_0\cup P_{R_0}$) we can find an N satisfying that f_n is quasiconformal on R_0-V for every $n\geq N$ and it holds that $\lim_{n\to\infty} K(f_n, R_0-V)=1$.

We call such a sequence $\{f_n\}_{n=1}^{\infty}$ as above *a weakly admissible sequence for* $\{R_n\}_{n=0}^{\infty}$ (with respect to P). If P is empty (, i.e. $\lim_{n\to\infty} K(f_n, R_0)=1$. then we call it simply *an admissible sequence for* $\{R_n\}_{n=0}^{\infty}$.

Here it is clear that the convergence in the sense of the Teichmüller topology implies that in the sense of any (P-)weak topology. If the given set P of punctures of R^* is a finite set, then we can show the converse, hence in such a case, the (P-)weak topology is equivalent to the Teichmüller topology. Namely, we can show the following.

Theorem 1. *Let a finite set P of punctures of R^* be given, and suppose that R_n converges to R_0 in $T(R^*)$ in the sense of the (P-)weak topology. Then R_n converges to R_0 also in the sense of the Teichmüller topology.*

To prove Theorem 1, first we note the following

Lemma 1. *Let R_n converge to R_0 in the sense of the (P-)weak topology with an abitrarily given (finite or infinite) set P of punctures of R^*.* Then we can *find a sequence of marking-preserving M-quasiconformal mappings f_n from R_0 onto R_n with some M $(<+\infty)$ and a sequence $\{V_k\}_{k=1}^{\infty}$ of neighbourhoods of P_{R_0} such that $\bigcap_k V_k=P_{R_0}$, $V_k\supset V_{k+1}$ and $\lim_{n\to\infty} K(f_n, R_0-V_k)=1$ for every k.*

Proof (cf. the proof of [10] I Lemma 2). Let $\{g_n\}_{n=1}^{\infty}$ be a weakly admissible sequence for $\{R_n\}_{n=0}^{\infty}$, and $P_{R_0}=\{p_j\}_{j=1}^{N}$ $(N\leqq+\infty)$. Then for every j, we can find a fundamental neighbourhood system $\{V_n^j\}_{n=1}^{\infty}$ of p_j (on $R_0\cup\{p_j\}$) such that $V_n^j-V_{n+1}^j$ is a doubly connected region with the smooth compact boundary curves whose whose modulus is not less than one for every n. Also we may assume that $\{V_1^j\}_{j=1}^{N}$ are mutually disjoint, and set $V_n=\bigcup_{j=1}^{N} V_n^j$ for every n. Then from the assumption, for every n, there is an N_n such that f_m is 2-quasi-comformal on $R_0-\overline{V_{n+2}}$ for every $m\geq N_n$. Here we may assume that N_{n+1} is greater than N_n for every n.

Fix n and let m be any integer such that $N_n\leq m<N_{n+1}$. And take conformal mappings H_j and $\tilde{H}_j$ from $V_n^j-\{p_j\}$ and $g_m(V_n^j-\{P_j\})$ onto $U_1'=\{0<|z|<1\}$, respectively, for every j. Here because the modulus of $V_n^j-V_{n+1}^j$ and $g_m(V_n^j-V_{n+1}^j)$ are not less than 1 and 1/2, respectively, we can find an absolute constant r less than one (depending on neither n nor j) such that each of the regions $H_j(V_n^j-\overline{V_{n+1}^j})$ and $\tilde{H}_j(g_m(V_n^j-\overline{V_{n+1}^j}))$ contains the annulus $\{r<|z|<1\}$ for every j. Now by [8] Theorem II–8–1, there are M_1-quasiconformal self-mapping $\{h_j\}_{j=1}^{N}$ of U_1 with an M_1 depending only on r such that $h_j\equiv\tilde{H}_j\circ g_m\circ H_j^{-1}$ on the annulus $\{r\leq|z|<1\}$ for every j. Next because $h_j(0)$ is contained in $\{|z|<r\}$, the mapping $T_j(z)=(z-h_j(0))/(1-h_j(0)\cdot\bar{z})$ is an $M_2\left(=\frac{1+r}{1-r}\right)$-quasiconformal self-mapping of U_1 such that $T_j(h_j(0))=0$ and $T_j(z)\equiv z$ on the circle $\{|z|=1\}$ for every j.

Thus set $f_m\equiv g_m$ on R_0-V_n and $f_m\equiv(\tilde{H}_j)^{-1}\circ T_j\circ h_j\circ H_j$ for every j, then from above we can see that f_m is $M_1\cdot M_2$-quasiconformal mapping from R_0 onto R_m for every m. For the whole sequence $\{f_m\}_{m=1}^{\infty}$, it holds that $\lim_{m\to\infty} K(f_m, R_0-V_k)=$

$\lim_{m\to\infty} K(g_m, R_0 - V_k)$ for every fixed k, which implies that $\{f_m\}_{m=1}^\infty$ and $\{V_k\}_{k=1}^\infty$ are desired sequences with $M = M_1 \cdot M_2$. q.e.d.

Next the following fact is essentially well-known.

Lemma 2. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of M-quasiconformal self-mappings of U_1 which leave 1, $\sqrt{-1}$, -1 fixed and are compatible with a given normalized Fuchsian group G. Then there is a subsequence $\{F_{n'}\}_{n'=1}^\infty$ whcih converges locally uniformly to an M-quasiconformal self-mapping of U_1 which also leaves 1, $\sqrt{-1}$, -1 fixed and is compatible with G.*

Proof. First by the reflection principle, we may consider that every F_n is also an M-quasiconformal self-mapping of the Riemann sphere $\hat{C}$. Restrict every F_n on $D = \hat{C} - \{1, \sqrt{-1}, -1\}$, then by [8] Theorems II–5–1 and II–5–5 there is a subsequence $\{F_{n'}\}_{n'=1}^\infty$ which converges locally uniformly on D to a function F. Here F is an M-quasiconformal self-mapping of D or a constant 1, $\sqrt{-1}$ or -1. But, for example, if $F \equiv 1$, then since $F_n(I) = I$ for every n, where I is the circular arc between $\sqrt{-1}$ and -1 not containing $z=1$ on $\{|z|=1\}$, we should have a contradiction. Hence F is an M-quasiconformal self-mapping of D. And it is clear that F leave U_1, 1, $\sqrt{-1}$, -1 fixed and is compatible with G. q.e.d.

Also we note the following, in a sense well-known

Lemma 3. *Let $\{R_n\}_{n=0}^\infty$ and P be as in Lemma 1 and $\{f_n\}_{n=1}^\infty$ be a weakly admissible sequence for $\{R_n\}_{n=0}^\infty$. Let G_n be the point in $T^\sharp(G^*)$ corresponding to R_n for every n and F_n be the lift of f_n on U_1 with respect to G_0. Then F_n converges locally uniformly to the identical mapping on U_1, and hence G_n converges to G_0 elementwise (, i. e. algebraically).*

Proof. As in the proof of Lemma 1, we can construct a sequence of M-quasiconformal mappings $\tilde{f}_n$ from R_0 onto R_n (with a suitblae M) such that for every V_k obtained in Lemma 1 there is an N such that $\tilde{f}_n \equiv f_n$ on $R_0 - V_k$ for every $n \geq N$. Let $\tilde{F}_n$ be the lift of $\tilde{f}_n$ on U_1 with respect to G_0, then by Lemma 2 every subsequence of $\{\tilde{F}_n\}_{n=1}^\infty$ contains a subsequence which converges locally uniformly to an M-quasiconformal self-mapping F. Also because it holds that $\lim_{n\to\infty} K(\tilde{f}_n, R_0 - V_k) = \lim_{n\to\infty} K(f_n, R_0 - V_k) = 1$ for every V_k from above and $\tilde{F}_n$ leaves 1, $\sqrt{-1}$, -1 fixed for every n, we see by [8] Theorem IV–5–2 that F is conformal and leave 1, $\sqrt{-1}$, -1 fixed, hence is the identical mapping. Thus we can conclude the assertions. q.e.d.

Proof of Theorem 1. First let $P_{R_0} = \{p_j\}_{j=1}^N$ with a finite N, and $G_n \in T^\sharp(G^*)$ correspond to R_n for every n. Also let g_j be a parabolic element of G_0 corresponding to a loop freely homotopic to p_j on R_0 for every j. Then we can take a so-called cusped region, say H_j for g_j, namely, H_j is an open disk in U_1 such that $g_j(H_j) = H_j$ and $g(H_j) \cap H_j$ is empty for every g in $G_0 - \{g_j^n$: n is any integer$\}$ (, cf. for example, [4] IV 9.10 Theorem). Here we may also assume that $\{G_0(H_j)\}_{j=1}^N$ are mutually disjoint, where $G_0(H_j)=\bigcup_{g\in G_0} g(H_j)$. Then $V_j=(G_0(H_j)/G_0)\cup\{p_j\}$ is a neighbourhood of p_j for every j, and set $V=\bigcup_{j=1}^{N} V_j$.

Next from the assumption and Lemma 1, we can find a weakly admissible sequence of M-quasiconformal mappings f_n from R_0 onto R_n. And take another neighbourhood V' of P corresponding also to cusped regions such that V' is contained in V, then we can decompose every f_n as $f_n=f_{n,1}\circ f_{n,2}$ so that $f_{n,1}$ and $f_{n,2}$ are M-quasiconformal mappings from some $R_n'=f_{n,2}(R_0)$ in $T(R^*)$ onto R_n and from R_0 onto R_n', respectively, which are conformal on $f_{n,2}(V')$ and R_0-V', respectively. (Such a decomposition can be obtained from the decomposition of the complex dilatation of f_n.) And we have that

$$\lim_{n\to\infty} K(f_{n,1},\ R_n')=\lim_{n\to\infty} K(f_{n,1},\ R_n'-f_{n,2}(V'))=\lim_{n\to\infty} K(f_n,\ R_0-V')=1.$$

Hence if we can construct an admissible sequence $\{h_n\}_{n=1}^{\infty}$ for $\{R_n'\}_{n=0}^{\infty}$ (with $R_0'=R_0$) then we can conclude that $\{f_{n,1}\circ h_n\}_{n=1}^{\infty}$ is an admissible sequence for $\{R_n\}_{n=0}^{\infty}$, that is, R_n converges to R_0 in the sense of the Teichmüller topology.

Now to construct such a sequence, first fix a conformal mapping $S_{j,n}$ from U_1 onto $H=\{\mathrm{Im}\, z>0\}$ which maps the fixed point of $g_{j,n}=(F_n)_*(g_j)$ to the infinity, where F_n is the lift of $f_{n,2}$ on U_1 with respect to G_0 and $F_0(z)\equiv z$. Here we also assume that $S_{j,n}^{-1}\circ S_{j,0}(0)=0$ for every n and j. Then note that because the fixed point of $g_{j,n}$ converges to $g_{j,0}$ by Lemma 3, we see that $S_{j,n}$ converges to $S_{j,0}$ locally uniformly. Set $S_{j,0}(H_j)=H_j^*=\{\mathrm{Im}\ z>c_j\}$ with some positive c_j and let V' correspond to $H_j'=\{\mathrm{Im}\, z>c_j'\}$ by $S_{j,0}$. Also set $L_j=\{\mathrm{Im}\, z=c_j''\}$ with $c_j''=(c_j+c_j')/2$ and $F_{n,j}=S_{j,n}\circ F_n\circ S_{j,0}^{-1}$, then $F_{n,j}$ is conformal, hence holomorphic, in a neighbourhood of L_j for every n and j. Let $F_{n,j}(x+\sqrt{-1}\cdot c_j'')=u_{n,j}(x)+\sqrt{-1}\cdot v_{n,j}(x)$, then $(u_{n,j})'(x)$ and $(v_{n,j})'(x)$ (and $v_{n,j}(x)$ itself) are periodic from the construction (whose periods are multiples of the real number e_j, where $S_{j,0}\circ g_j\circ S_{j,0}^{-1}(z)=z+e_j$, hence independent of n), and converge to 1 and 0, respectively, uniformly on L_j, for F_n converges locally uniformly to the identical mapping on L_j by Lemma 3, and $F_{n,j}$ is holomorphic in a neighoburbood of L_j independent of n.

In particular, if n is sufficiently large, then $u_{n,j}(x)$ is strictly monotonously increasing, so if we set

$$\tilde{F}_{n,j}(x+\sqrt{-1}\cdot y)=u_{n,j}(x)+\sqrt{-1}\cdot((y-c_j'')+v_{n,j}(x))$$

on $H_j''=\{\mathrm{Im}\, z>c_j''\}$, then $\tilde{F}_{n,j}$ is a homeomorphism from H_j'' onto $F_{n,j}(H_j'')$ such that $\tilde{F}_{n,j}\equiv F_{n,j}$ on L_j. Moreover, because it holds that

$$2(\tilde{F}_{n,j})_z=((u_{n,j})'(x)+1+\sqrt{-1}\cdot(v_{n,j})'(x)),\quad \text{and}$$

$$2(\tilde{F}_{n,j})_{\bar{z}}=((u_{n,j})'(x)-1+\sqrt{-1}\cdot(v_{n,j})'(x))\quad \text{on}\quad H_j''$$

$K(F_{n,j},\ H_j'')$ converges to 1 as n tends to $+\infty$ for every j. Thus defining

$$\tilde{F}_n=F_n\quad \text{on}\quad U_1-\bigcup_{j=1}^{N} G_0(S_{j,0}^{-1}(H_j'')),\quad \text{and}$$

$$\tilde{F}_n=(F_n)_*(g)\circ(S_{j,n})^{1-}\circ\tilde{F}_{n,j}\circ S_{j,0}\circ g^{-1}\quad \text{on}\quad g(S_{j,0}^{-1}(H_j''))$$

for every $g\in G_0$ and every j, we have a sequence of quasiconformal self-mapping $\tilde{F}_n$ of U_1 compatible with G_0 such that $\lim_{n\to\infty} K(F_n, U_1)=\lim_{n\to\infty} K(F_n, \bigcup_{j=1}^{N} G_0(S_{j,0}^{-1}(H_j''))) = 1$. Hence projecting F_n on R_0, we have a desired admissible sequence for $\{R_n\}_{n=0}^{\infty}$. q.e.d.

Finally we also note the following lemma which can be obtained by a slight modification of the above proof of Theorem 1.

Lemma 4. *Let R_n converge to R_0 in $T(R^*)$ and a finite set P of puctures of R^* be given. Then there is an admissible sequence $\{f_n\}_{n=1}^{\infty}$ for $\{R_n\}_{n=0}^{\infty}$ such that f_n is conformal on some neighbourhood V of P_{R_0} (on $R_0\cup P_{R_0}$) for every large n.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be any admissible sequence for $\{R_n\}_{n=0}^{\infty}$, and we use the same notations as in the proof of Theorem 1. Here we also take two sufficiently large d_j and d_j' (other than c_j'') such that $d_j'>d_j>\max\{c_j'', \max_{L_j} v_{n,j}(x)\}$ for every j. And first on $\{c_j''<y<d_j\}$ we define

$$\tilde{\tilde{F}}_{n,j}(x+\sqrt{-1}\cdot y)=u_{n,j}(x)+\sqrt{-1}\cdot\left(\frac{y-c_j''}{d_j-c_j''}(d_j-v_{n,j}(x))+v_{n,j}(x)\right).$$

Next letting $S_{j,n}\circ g_n\circ S_{j,n}^{-1}(z)=z+e_{j,n}$(, or equivalently, $e_{j,n}=u_{n,j}(x+e_j)-u_{n,j}(x)$), set $e(j, n)=e_{j,n}/e_j$(which is positive and converges to 1 as n tends to $+\infty$ by Lemma 3) and define on $\{d_j<y<d_j'\}$

$$\tilde{\tilde{F}}_{n,j}(x+\sqrt{-1}\cdot y)=\frac{y-d_j}{d_j'-d_j}\cdot e(j, n)\cdot x+\frac{d_j'-y}{d_j'-d_j}\cdot u_{n,j}(x)+\sqrt{-1}\cdot y.$$

Finally on $\{d_j'<y\}$ we define

$$\tilde{\tilde{F}}_{n,j}(z)=e(n, j)\cdot z-\sqrt{-1}\cdot(e(n, j)-1)\cdot d_j'.$$

And using $\tilde{\tilde{F}}_{n,j}(z)$ instead of $\tilde{F}_{n,j}(z)$ in the proof of Theorem 1, we can construct an admissible sequence $\{h_n\}_{n=1}^{\infty}$ for $\{R_n'\}_{n=0}^{\infty}$ such that every h_n is conformal on V'' which corresponds to $\bigcup_{j=1}^{N} G_0(S_{j,0}^{-1}(\mathrm{Im}\, z>d_j'))$. Here taking d_j' sufficiently large, we may assume that $h_n^{-1}\circ f_{n,2}(V')$ contains V'' for every n. (This is possible, because $h_n^{-1}\circ f_{n,2}$ is weakly admissible for $\{S_n\}_{n=0}^{\infty}$ with $S_n=R_0$ for every n, hence by Lemma 3 $h_n^{-1}\circ f_{n,2}(V')$ tends to V'.) Thus we conclude that $\{f_{n,1}\circ h_n\}_{n=1}^{\infty}$ is an admissible sequence for $\{R_n\}_{n=0}^{\infty}$ such that every $f_{n,1}\circ h_n$ is conformal on V''. q.e.d.

Remark. It is very likely that any (P-) weak topology is equivalent to the Teichmüller topology, though the author has no proof at present. Anyway, we shall use essentially not the weak topology, but the Teichmüller topology, and we need only Lemmas 3 and 4 in the sequel.

§2. Continuity of holomorphic abelian differentials

Let R^* be an arbitrary Riemann surface and for every R in $T(R^*)$ set

$$A_1(R)=\{\theta: \theta \text{ is a holomorphic abelian differential on } R\}.$$

Then we can consider the following basic kind of continuity of elements in $A_1(R)$ on $T(R^*)$, which is in the sequel always equipped with the Teichmüller topology.

Definition 2. (, cf. [11] Definition 1). Let R_n converge to R_0 in $T(R^*)$ and θ_n in $A_1(R_n)$ be given for every n. We say that θ_n *converges to* θ_0 *metrically* if there is an admissible sequence, or more generally, a weakly admissible sequence (with respect to a given set P of punctures of R^*) $\{f_n\}_{n=1}^{\infty}$ for $\{R_n\}_{n=0}^{\infty}$ such that

$$(*) \qquad \lim_{n\to\infty} \|\theta_n \circ f_n - \theta_0\|_E = 0$$

for every compact set E in R_0, where and in the sequel, $\theta\circ f$ implies the pull-back of θ by f and $\|\theta\|_X$ implies the Dirichlet norm of θ on a Borel set X on the surface where θ is defined.

Also we say that θ_n *converges to* θ_0 *strongly metrically*, if the set P is contained in the set P_0 of all such punctures of R_0 that are poles of θ_0 and for every neighbourhood V of P_0, it holds that

$$(*') \qquad \lim_{n\to\infty} \|\theta_n \circ f_n - \theta_0\|_{R_0 - V} = 0.$$

Next we set

$$CA_1(R)=\{\theta\in A_1(R): \theta^2 \text{ has closed trajectories (cf. [12] §1)}\}.$$

And for every θ in $CA_1(R)$ and every simple closed curve c on R, we call the doubly connected region in R swept out by all compact regular trajectories of θ^2 freely homotopic to c on R *the characteristic ring domain of* θ *for* c on R, whenever the region is non-empty. A characteristic ring domain of θ is called *degenerate* if it is conformally equivalent to a punctured disk (, and then θ has a simple pole at the puncture). Each characteristic ring domain of θ determines the free homotopy class of a simple closed curve separating two boundary components and oriented so that the period of θ along it is positive. We denote by $L(\theta)$ the set of all free homotopy classes determined by characteristic ring domains of θ, and by $L'(\theta)$ the subset of $L(\theta)$ consisting of all elements corresponding to degenerate characteristic ring domains. Also for every c in $L(\theta)$, we denote by $W_{c,\theta}$ the characteristic ring domain of θ for c, and by $m_{c,\theta}$ and by $a_{c,\theta}$ the modulus of $W_{c,\theta}$ and the period $\int_c\theta$ of θ along c (, which is positive from above), respectively. Here recall that $m_{c,\theta}=+\infty$ if and only if $W_{c,\theta}$ is degenerate and that $\|\theta\|_R^2=2\cdot\sum_{c\in L(\theta)} a_{c,\theta}^2\cdot m_{c,\theta}$ (, which may be infinite). Also we set $m_{c,\theta}=0$ if $c\notin L(\theta)$.

Now we can define another kind of continuity on $T(R^*)$ for elements in the above special subset $CA_1(R)$ of $A_1(R)$.

Definition 3. (, cf. [11] Definition 2). Let R_n converge to R_0 in $T(R^*)$ and θ_n in $CA_1(R_n)$ be given for every n. We say that θ_n *converges to* θ_0 *geometrically* if the following conditions holds.

1) For every c in $L(\theta_0)$, there is an N_c such that c belongs also to $L(\theta_n)$ for

every $n \geq N_c$.

2) It holds that $\lim_{n\to\infty} a_{c,\theta_n} = a_{c,\theta_0}$ and $\lim_{n\to\infty} m_{c,\theta_n} = m_{c,\theta_0}$ for every c in $L(\theta_0)$.

3) It holds that $\lim_{n\to\infty}(\sum_{c\in L''} a^2_{c,\theta_n} \cdot m_{c,\theta_n}) = \sum_{c\in L''} a^2_{c,\theta_0} \cdot m_{c,\theta_0} < +\infty$, where $L'' = L(\theta_0) - L'(\theta_0)$.

4) It holds that $\lim_{n\to\infty} (\|\theta_n\|^2_{X_n} - 2 \cdot \sum_{c\in L''} a^2_{c,\theta_n} \cdot m_{c,\theta_n}) = 0$, where $X_n = R_n - \bigcup_{c\in L'(\theta_0)} W_{c,\theta_n}$ for every n.

Here note that if $L(\theta_0)$ is a finite set, which is the case at least if R^* is of finite type, then the above condition 3) follows from the condition 2). Also note that if θ_0 is square integrable, then $L'' = L(\theta_0)$ and the conditions 3) and 4) implies that $\lim_{n\to\infty} \|\theta_n\|_{R_n}$ exists and is equal to $\|\theta_0\|_{R_0}$.

Now as relations between (strongly) metrical convergence and the geometrical one, we can show the following theorems, which are the main results of this paper and generalize [11] Corollary 1.

Theorem 2. *Let R_n converge to R_0 in $T(R^*)$ and θ_n in $CA_1(R_n)$ be given for every n. If θ_0 is square integrable and θ_n converges to θ_0 geometrically, then θ_n converges to θ_0 metrically.*

Theorem 3. *Let $\{R_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ be as in Theorem 2. Suppose that*

1) *the set $L'(\theta_0)$ is a finite set,*

2) *letting P be the (finite) set of punctures of R_0 corresponding to $L'(\theta_0)$ (, which are poles of θ_0), it holds that $\|\theta_0\|_{R_0 - V}$ is finite for every neighbourhood V of P (on $R_0 \cup P$), and*

3) *for every c in $L(\theta_0)$ there is an N_c such that the period $\int_c \theta_n$ is real for every $n \geq N_c$.*

Now if θ_n converges to θ_0 strongly metrically, then θ_n converges to θ_0 geometrically.

Corollary 1. *Let $\{R_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ be as in Theorem 2. If θ_0 is square integrable, the condition 3) in Theorem 3 holds, and θ_n converges to θ_0 strongly metrically, then θ_n converges to θ_0 geometrically.*

Theorem 4. *Let $\{R_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ be as in Theorem 2. Suppose that the condition 3) in Theorem 3 holds and that*

4) $$\limsup_{n\to\infty} \|\theta_n\|_{X_n} \leq \|\theta_0\|_{X_0} < +\infty,$$

where X_n are as in Definition 3. If θ_n converges to θ_0 metrically, then θ_n converges to θ_0 geometrically.

Corollary 2. *Let $\{R_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ be as in Theorem 2. Suppose that θ_0 is square integrable, the condition 3) in Theorem 3 holds, and*

4′) $$\limsup_{n\to\infty} \|\theta_n\|_{R_n} \leq \|\theta_0\|_{R_0} \quad (< +\infty).$$

Now if θ_n converges to θ_0 metrically, then θ_n converges to θ_0 geometrically.

To prove theorems, first we note the following

Proposition 1. *Suppose that R_n converges to R_0 in $T(R^*)$ in the sense of the (P-)weak topology with any given P, and θ_n in $A_1(R_n)$ is given for every n. Then the following conditions are mutually equivalent.*

1) *θ_n converges to θ_0 metrically.*

2) *For every weakly admissible sequence $\{f_n\}_{n=1}^{\infty}$ for $\{R_n\}_{n=0}^{\infty}$, the condition $(*)$ in Definition 2 holds for every compact set E in R_0.*

3) *For every n, let $a_n(z)dz$ be the lift of θ_n on U_1 with respect to G_n in $T^{\#}(G^*)$ corresponding to R_n (, i.e. the G_n-invariant form on U_1 corresponding to θ_n), then $a_n(z)$ converges to $a_0(z)$ locally uniformly on U_1.*

The proof of Proposition 1 is essentially the same as that of [10] II Proposition. But for the sake of completeness, we include the proof.

Proof. First suppose that θ_n converges to θ_0 metrically with a weakly admissible sequence $\{f_n\}_{n=1}^{\infty}$ for $\{R_n\}_{n=0}^{\infty}$. Let F_n be the lift of f_n on U_1 with respect to G_0, then by Lemma 3, $F_n(z)$ converges to $F_0(z)\equiv z$ locally uniformly on U_1 and G_n converges to G_0 elementwise. Hence for every $z_0 \in U_1$, we can find a positive $r=r(z_0)$ such that $D(z_0, r)=\{|z-z_0|<r(z_0)\}$ is projected univalently into R_n for every sufficiently large n. Here we can also assume that $r(z_0)\leq \frac{1}{2}(1-|z_0|)$ for every $z_0 \in U_1$ and $r(z_0)^{-1}$ is locally bounded on U_1. Now take a compact set E_1 in U_1 arbitrarily, then $E_1'=\overline{\bigcup_{z\in E_1} D(z, r(z))}$ is also compact from above, and because $F_n^{-1}(z)$ also converges to $F_0^{-1}(z)\equiv z$ uniformly on E_1' there is a compact set E_2 in U_1 which contains $F_n^{-1}(E_1')$ for every n. So we can take a compact set E in R_0 which contains the projection of E_2 into R_0. And then for every sufficiently large n and every z_0 in E_1 it holds that

$$|a_n(z_0)|\leq \left(\frac{1}{\pi r(z_0)^2}\cdot \iint_{D(z_0,r(z_0))} |a_n(z)|^2 dxdy\right)^{1/2}$$

$$\leq \frac{1}{\sqrt{\pi}r(z_0)}\cdot\left(\iint_{F_n^{-1}(D(z_0,r(z_0)))} |a_n(F_n(z))|^2\cdot(|(F_n)_z|^2-|(F_n)_{\bar z}|^2)dxdy\right)^{1/2}$$

$$\leq \frac{1}{\sqrt{2\pi}\cdot r(z_0)}\cdot\|\theta_n\circ f_n\|_E \leq \frac{1}{\sqrt{2\pi}\cdot r(z_0)}\cdot(\|\theta_n\circ f_n-\theta_0\|_E+\|\theta_0\|_E).$$

So from the assumption, we can see that $\{a_n(z)\}_{n=1}$ are locally uniformly bounded on U_1, hence makes a normal family. Also, similarly, we have that for every $z_0 \in E_1$,

$$|a_n(z_0)-a_0(z_0)| \leq \frac{1}{\sqrt{\pi}\cdot r(z_0)}\|a_n-a_0\|_{D(z_0,r)}$$

$$\leq \frac{1}{\sqrt{\pi}\cdot r(z_0)}(\|a_n-a_n\circ F_n\|_{D(z_0,r)}+\|a_n\circ F_n\cdot((F_n)_z-1)\|_{D(z_0,r)}$$

$$+\|a_n\circ F_n\cdot(F_n)_z-a_0\|_{D(z_0,r)}).$$

Here the first and the second terms in the right hand side converge to 0 by Lemma 3 and [8] Theorem V–5–3. And the last term is majorized by $\|\theta_n\circ f_n-\theta_0\|_E$. hence also converges to 0 from the assumption. Thus we conclude that the condition 3) holds.

Next suppose that the condition 3) holds, and let $\{f_n\}_{n=1}^{\infty}$ be any weakly admissible sequence for $\{R_n\}_{n=0}^{\infty}$. Then for every compact set E of R_0 we have that

$$\|\theta_n\circ f_n-\theta_0\|_E$$
$$\leq\|a_n\circ F_n\cdot((F_n)_z-1)\|_D+\|a_n\circ F_n-a_0\|_D+\|a_n\circ F_n\cdot(F_n)_{\bar z}\|_D,$$

where D is a compact set in U_1 whose projection on R_0 covers E. Hence by Lemma 3, [8] Theorem V–5–3 and the locally uniform convergence of $a_n(z)$ to $a_0(z)$, we can conclude that the condition 2) holds. And since it is trivial that 2) implies 1), we have the equivalence. q.e.d.

In particular, the metrical convergence does not depend on the choice of a weakly admissible sequence. Also the metrical convergence implies the following weak convergence.

Corollary 3 (*cf.* [10] II Corollary 1). *Let $\{R_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ be as in Proposition* 1, *and suppose that θ_n converges to θ_0 metrically. Then for every* 1-*cycle d on R^* (, hence on every R in $T(R^*)$), it holds that*

$$\lim_{n\to\infty}\int_d\theta_n=\int_d\theta_0.$$

Proof. As before, let $G_n\in T^*(G^*)$ correspond to R_n, and $a_n(z)dz$ and $F_n(z)$ be the lifts of θ_n with respect to G_n and of f_n with respect to G_0, respectively, for every n, We need to show the assertion only in case that d is a simple closed curve. And let $d_0\in G_0$ be an element corresponding to d and set $d_n=(F_n)_*(d_0)$, then by Lemma 3, it holds that

$$\lim_{n\to\infty}F_n(z)=z \quad\text{and}\quad \lim_{n\to\infty}d_n\circ F_n(z)=d_0(z) \quad\text{for every}\quad z\in U_1.$$

Now fix z_0 in U_1. Then since $a_n(z)$ converges locally uniformly to $a_0(z)$ by Proposition 1, we have that

$$\lim_{n\to\infty}\int_d\theta_n=\lim_{n\to\infty}\int_{F_n(z_0)}^{d_n\circ F_n(z_0)}a_n(z)dz=\int_{z_0}^{d_0(z_0)}a_0(z)dz=\int_d\theta_0.$$

Thus we have the assertion. q.e.d.

Next the following lemma is crusial for the proof of Theorems.

Lemma 5 (*cf.* [11] Lemma 3). *Let $\{R_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ be as in Proposition* 1 *abd $\{f_n\}_{n=1}^{\infty}$ be any weakly admissible sequence for $\{R_n\}_{n=0}^{\infty}$. Suppose that θ_n converges to θ_0 metrically, θ_0^2 has a compact regular trajectory c_0 and $\int_{c_0}\theta_n$ is real for every n. Then for every neighbourhood V of c_0 in R_0, there is an N such that θ_n^2 has a compact trajectory c_n freely homotopic to $f_n(c_0)$ on R_n whose preimage $f_n^{-1}(c_n)$ on R_0 is contained in V for every $n\geq N$.*

Proof. Let G_n correspond to R_n in $T(G^*)$ and F_n be the lift of f_n on U_1 with respect to G_0 for every n. Take a point p on c_0 and fix a lift z_0 of p with respect to G_0 on U_1. And let U and C_0 be the connected components of the lifts of V and c_0 with respect to G_0 on U_1 which contain z_0, and consider the function $u_n(z)=\operatorname{Im}\int_{F_n(z_0)}^{z} a_n(t)dt$ on U for every n. Then since $u_0\equiv 0$ on C_0, there is a (suitably small) positive ε such that the component, say U_ε, of $\{z\colon |u_0(z)|<\varepsilon\}$ containing C_0 is contained in U and the boundary of U_ε in U_1 consists of two analytic curves (whose projections into R_0 are compact regular trajectories in V freely homotopic to c_0). Because $F_n(z)$ and $u_n(z)$ converge to $F_0(z)\equiv z$ and $u_0(z)$, respectively, locally uniformly on U_1 (by Lemma 3 and the metrical convergence of $\{\theta_n\}_{n=1}^{\infty}$ to θ_0), $u_n(F_n(z))$ also converges to $u_0(z)$ locally uniformly on U_1. Hence in particular, there is an N such that we can find a suitably long compact arc from $F_n(z_0)$ contained in $\{z\colon u_n(F_n(z))=0\}\cap U_\varepsilon$ whose projection on R_0 covers a simple closed curve, say c_n', in V freely homotopic to c_0 for every $n\geq N$. Then from the construction, we can conclude that $c_n=f_n(c_n')$ is a compact regular trajectory of θ_n^2 on R_n for every $n\geq N$, which show the assertion. q.e.d.

Lemma 6. *Let* $\{R_n\}_{n=0}^{\infty}$ *be as in Proposition* 1, $\theta_n\in CA_1(R_n)$ *converge to* $\theta_0\in CA_1(R_0)$ *metrically, and* $\{f_n\}_{n=1}^{\infty}$ *be a weakly admissible sequence for* $\{R_n\}_{n=0}^{\infty}$. *Then the set* $\bigcup_{n=1}^{\infty}(\operatorname{Int}\bigcap_{m=n}^{\infty} f_m^{-1}(W_{c,\theta_m}))$ (, *which is called the Carathéodory kernel of* $\{f_n^{-1}(W_{c,\theta_n})\}_{n=1}^{\infty}$,) *is coincident with* W_{c,θ_0} *for every* c *in* $L(\theta_0)$.

Proof. First by Lemma 5, we can easily see that for every compact subset E of W_{c,θ_0} there is an M such that $f_n^{-1}(W_{c,\theta_n})$ contains E for every $n\geq M$. So $\operatorname{Int}E$ is contained in $N=\bigcup_{n=1}^{\infty}(\operatorname{Int}(\bigcap_{m=n}^{\infty} f_m^{-1}(W_{c,\theta_m})))$. And since E is arbitrary, we conclude that W_{c,θ_0} is contained in N.

Next for every p and every compact neighbourhood V_p of P in $R_0-\overline{W_{c,\theta_0}}$, there is a compact regular trajectory of θ_0^2 intersecting with int V_p, for θ_0 belongs to $CA_1(R_0)$, hence again by Lemma 5, there is an M such that $V_p-f_n^{-1}(W_{c,\theta_n})$ is non-empty for every $n\geq M$. So we see that V_p-N is non-empty, and since V_p is arbitrary and N is open, we conclude that $p\notin N$. Thus we have shown that N is contained in $\overline{W_{c,\theta_0}}$.

Finally if $N-W_{c,\theta_0}$ were not empty, then there would be a non-trivial simple closed curve c' in N which is not freely homotopic to c. This is a contradiction, for $f_n(c')$ should be contained in W_{c,θ_n} for every sufficiently large n (from the definition of N), hence freely homotopic to c. Thus we have that $N=W_{c,\theta_0}$. q.e.d.

Lemma 7. *Under the same assumption as in Lemma* 6, *we further assume that every* f_n *is* M*-quasiconformal with some* M, *then it holds that*

$$\lim_{n\to\infty} m_{c,\theta_n}=m_{c,\theta_0} \quad \text{for every} \quad c\in L(\theta_0).$$

Moreover for every $c\in L(\theta_0)$, *setting* $H_{c,n}(p)=b_n\cdot\exp\left(\frac{2\pi\sqrt{-1}}{a_{c,\theta_n}}\cdot\int_{p_n}^{p}\theta_n\right)$

on W_{c,θ_n} *with a* p_n *in* W_{c,θ_n} *and a real* b_n *such that* $H_{c,n}(W_{c,\theta_n})=W_n=\{1<|z|<\exp\cdot(2\pi m_{c,\theta_n})\}$ *for every* n, *the mapping* $(H_{c,n}\circ f_n)^{-1}$ *converges locally uniformly to* $(H_{c,0})^{-1}$ *on* W_0 *up to a rotation of* W_0.

Proof. First by Lemma 5, we can see that $\liminf_{n\to\infty} m_{c,\theta_n}\geq m_{c,\theta_0}$, for $f_n(E)$ is contained in W_{c,θ_n} for every compact set E in W_{c,θ_0} and every n larger than an integer depending on E, and $\lim_{n\to\infty} K(f_n, E)=1$ from the assumption. And taking a subsequence if necessary, we may assume that $\lim_{n\to\infty} m_{c,\theta_n}=m$ (which is not less than m_{c,θ_0}). Then by Lemma 6, we can see (, cf. [8] Theorem II–5–4) that $(H_{c,n}\circ f_n)^{-1}$ converges locally uniformly on $W=\{1<|z|<\exp(2\pi m)\}$ to a conformal mapping, say H_c, from W onto W_{c,θ_0}. And since $H_{c,0}\circ H_c$ is a conformal mapping from W onto W_0, we conclude that $m=m_{c,\theta_0}$. Hence taking a subsequence is unnecessary, and $(H_{c,n}\circ f_n)^{-1}$ converges locally uniformly on W_0 to $(H_{c,0})^{-1}$ up to a rotation of W_0.
q. e. d.

Here we state the following result which follows at once from Corollary 3 and Lemmas 5 and 7.

Proposition 2. *Let* R_n *converge to* R_0 *in* $T(R^*)$ *and* $\theta_n\in CA_1(R_n)$ *be given for every* n. *Suppose that the condition* 3) *in Theorem* 3 *holds for* $\{\theta_n\}_{n=0}^{\infty}$ *and that* θ_n *converges to* θ_0 *metrically, then the conditions* 1) *and* 2) *of Definition* 3 *hold.*

Finally we recall tha following extremal property of every θ in $CA_1(R)$, which is essentially due to J. A. Jenkins.

Proposition 3 ([5] Theorem *B*). *Let* R *be an arbitrary Riemann surface,* $\theta\in CA_1(R)$ *with a finite Dirichlet norm be given, and* $b(z)|dz|$ *be any non-negative measurable conformal density on* R *such that*

$$\int_{\tilde{c}} b(z)|dz|\geq a_{c,\theta}$$

for every $c\in L(\theta)$ *and every curve* $\tilde{c}$ *on* R *freely homotopic to* c. *Then it holds that*

$$\iint_R b(z)^2dxdy\geq \sum_{c\in L(\theta)} a_{c,\theta}^2\cdot m_{c,\theta}\left(=\frac{1}{2}\|\theta\|_R^2\right),$$

where $z=x+\sqrt{-1}\cdot y$ *is a generic local parameter. And the equation holds if and only if* $b(z)|dz|\equiv|\theta|$.

Proof. Using the Cauchy-Schwarz' inequality, it is routine to show that

$$a_{c,\theta}^2\cdot m_{c,\theta}\leq \iint_{W_{c,\theta}} b(z)^2dxdy$$

for every c in $L(\theta)$ and that the equality holds if and only if $b(z)|dz|\equiv|\theta|$. And taking the sum for all c in $L(\theta)$, we have the assertion. q. e. d.

Proof of Theorem 2. Suppose that θ_0 is square integrable and θ_n converges to θ_0 geometrically. Then as noted before, $\lim_{n\to\infty}\|\theta_n\|_{R_n}=\|\theta_0\|_{R_0}$, hence we may assume that $\{\|\theta_n\|_{R_n}\}_{n=0}^{\infty}$ is a bounded sequence.

Let $a_n(z)dz$ be the lift of θ_n on U_1 with respect the group G_n in $T^{\#}(G^*)$ corresponding to R_n for every n. Then similarly as the first part of the proof of Proposition 1, we can see that $\{a_n(z)\}_{n=0}^{\infty}$ are locally uniformly bounded on U_1. Hence we can find a subsequence, which is also denoted by $\{a_n(z)\}_{n=0}^{\infty}$, converging to a holomorphic function, say $a(z)$, locally uniformly on U_1. It is clear that $a(z)dz$ is G_0-invariant, so let $\theta \in A_1(R_0)$ correspond to $a(z)dz$, then θ converges to θ metrically by Proposition 1. Hence by Corollary 3 and the condition 2) of Definition 3, it holds that

$$\int_c \theta = \lim_{n\to\infty} a_{c,\theta_n} = a_{c,\theta_0} = \int_c \theta_0$$

for every c in $L(\theta_0)$.

Now we can show that $\|\theta\|_{R_0} \le \|\theta_0\|_{R_0}$ $(<+\infty)$. In fact, fix c in $L(\theta_0)$ and let E be any compact set in $W_{c,\theta_0} \subset R_0$. Then, since θ_n converges to θ metrically and $f_n^{-1}(W_{c,\theta_n})$ contains E for every sufficiently large n by Lemma 5, where $\{f_n\}_{n=1}^{\infty}$ is arbitrarily fixed admissible sequence for $\{R_n\}_{n=0}^{\infty}$, we have from the condition 2) of Definition 3 that

$$\begin{aligned}\|\theta\|_E^2 &= \lim_{n\to\infty} \|\theta_n \circ f_n\|_E^2 \le \lim_{n\to\infty} K(f_n, E)\cdot \|\theta_n\|_{f_n(E)}^2 \\ &\le \lim_{n\to\infty} \|\theta_n\|_{W_{c,\theta_n}}^2 = \lim_{n\to\infty} 2\cdot a_{c,\theta_n}^2 \cdot m_{c,\theta_n} \\ &= 2\cdot a_{c,\theta_0}^2 \cdot m_{c,\theta_0} = \|\theta_0\|_{W_{c,\theta_0}}^2.\end{aligned}$$

Because E is arbitrarily, we have that $\|\theta\|_{W_{c,\theta_0}} \le \|\theta_0\|_{W_{c,\theta_0}}$. And taking the sum for all c in $L(\theta_0)$, we have the desired inequality.

Finally consider the density $b(z)|dz| \equiv |\theta|$, then we have from above that $\int_{\tilde c} b(z)|dz| \ge |\int_{\tilde c} \theta| = a_{c,\theta_0}$ for every c in $L(\theta_0)$ and every $\tilde c$ freely homotopic to c on R_0. Hence by Proposition 3, we have that $b(z)|dz| \equiv |\theta_0|$, which implies that $\theta \equiv \theta_0$, for $\int_c \theta = \int_c \theta_0$ for every c in $L(\theta_0)$. Thus taking a subsequence is unnecessary, and we conclude that θ_n converges to θ_0 metrically. q.e.d.

Proof of Theorems 3 *and* 4. We have already shown (Proposition 2) that if θ_n converges to θ_0 metrically and $\{\theta_n\}_{n=0}^{\infty}$ satisfies the condition 3) in Theorem 3, then the conditions 1) and 2) of Definition 3 are satisfied.

Now to prove Theorem 3, further suppose that θ_n converges to θ_0 strongly metrically and θ_0 satisfies the conditions 1) and 2) in Theorem 3. Here note that P in the condition 2) is the set of all punctures where θ_0 has a pole, which follows from the condition 2). And take as a V in the condition 2) such a neighbourhood of P that $R_0 - V$ contains X_0 and every component of the relative boundary of V is a trajectory of θ_0^2 (contained in $R_0 - X_0$). Also let $\{f_n\}_{n=1}^{\infty}$ be a weakly admissible sequence with respect to P such that $\lim_{n\to\infty} \|\theta_n \circ f_n - \theta_0\|_{R_0 - V} = 0$ for every such a V as above, whose existence is assured by the strongly metrical convergence of θ_n to θ_0. Then by Lemma 5 and the condition 1), we can see that, for arbitrarily fixed V such as bove, there is an N such that $f_n^{-1}(X_n)$ is contained in $R_0 - V$ for every $n \ge N$. Hence we have that

$$\|\theta_n\|_{X_n} \leq \|\theta_n\|_{f_n(R_0-V)} \leq \|\theta_n \circ f_n\|_{R_0-V}.$$

for every $n \geq N$. So we have that

$$\limsup_{n\to\infty} \|\theta_n\|_{X_n} \leq \|\theta_0\|_{R_0-V},$$

which is finite by the condition 2. And since we can take V arbitrarily close to the set $R_0 - X_0$, we conclude that

$$\limsup_{n\to\infty} \|\theta_n\|_{X_n} \leq \|\theta_0\|_{X_0} \quad (<+\infty),$$

that is, the condition 4) in Theorem 4 holds. Thus the proof of Theorem 3 is reduced to that of Theorem 4.

Hence, turning to the proof of Theorem 4, suppose that the condition 4) in Theorem 4 holds. Then we have from the first paragraph of this proof and the Fatou's lemma that

$$\|\theta_0\|_{X_0} \geq \limsup_{n\to\infty} \|\theta_n\|_{X_n} \geq 2\cdot \limsup_{n\to\infty} \sum_{c\in L''} a^2_{c,\theta_n}\cdot m_{c,\theta_n}$$

$$\geq 2\cdot \liminf_{n\to\infty} \sum_{c\in L''} a^2_{c,\theta_n}\cdot m_{c,\theta_n} \geq 2\cdot \sum_{c\in L''} \liminf_{n\to\infty} a^2_{c,\theta_n}\cdot m_{c,\theta_n}$$

$$= 2\cdot \sum_{c\in L''} a^2_{c,\theta_0}\cdot \mathrm{m}_{c,\theta_0} = \|\theta_0\|_{X_0}.$$

Hence we have that

$$\lim_{n\to\infty} \sum_{c\in L''} a^2_{c,\theta_n}\cdot m_{c,\theta_n} = \sum_{c\in L''} a^2_{c,\theta_0}\cdot m_{c,\theta_0}, \quad \text{and}$$

$$\lim_{n\to\infty} (\|\theta_n\|_{X_n} - 2\cdot \sum_{c\in L''} a^2_{c,\theta_n}\cdot m_{c,\theta_n}) = 0,$$

that is, the conditions 3) and 4) of Definition 3 are satisfied. Thus we have shown Theorems 3 and 4. q. e. d.

§3. Applications of main theorems

First let a 1-cycle c on R^* (, which can be considered as a 1-cycle on R for every R in $T(R^*)$,) be given and let $\sigma_{c,R}$ be *the period reproducing differential for c on R* in the space $\Gamma_h(R)$ of all square integrable real harmonic differentials on R for every R (, see for example, [2] Ch. V §12, where $-{}^*\sigma_{c,R}$ is called the reproducing differential for c on R). And set $\theta_{c,R} = \sigma_{c,R} + \sqrt{-1}\,{}^*\sigma_{c,R}$, then we know the following

Proposition 4 ([7] Theorem 5′). *If R_n converges to R_0 in $T(R^*)$, then θ_{c,R_n} converges to θ_{c,R_0} strongly metrically for every given c.*

Proof. We will give an outline of another proof than that given in [7] (cf. [6] 13. Proposition). First let $\{f_n\}_{n=1}^\infty$ be an admissible sequence for $\{R_n\}_{n=0}^\infty$. Then from the well-known construction (, see [12] §2,) of ${}^*\sigma_{c,R}$ and the Minda's theorem ([9] Theorem 4) we can see that $({}^*\sigma_{c,R_n})\circ f_n - {}^*\sigma_{c,R_0}$ has $\{0\}$-behavior and is exact, hence belongs to $\Gamma_{e0}(R_0)$ for every n. So we conclude that for every n the inner product

$$\begin{aligned}&(\theta_{c,R_n}\circ f_n-\theta_{c,R_0},\ {}^*(\theta_{c,R_n}\circ f_n-\theta_{c,R_0}))_{R_0}\\&=-\iint_{R_0}(\theta_{c,R_n}\circ f_n-\theta_{c,R_0})\wedge\overline{(\theta_{c,R_n}\circ f_n-\theta_{c,R_n})}\\&=-2\sqrt{-1}(\sigma_{c,R_n}\circ f_n-\sigma_{c,R_0},\ {}^*(({}^*\sigma_{c,R_n})\circ f_n-{}^*\sigma_{c,R_0}))_{R_0}=0.\end{aligned}$$

Thus by the standard argument (,cf. the proof of [7] Theorem 1 or [6] Theorem 3) we can show that for every n

$$\|\theta_{c,R_n}\circ f_n-\theta_{c,R_0}\|_{R_0}\leq(2k_n/(1-k_n))\cdot\|\theta_{c,R_0}\|_{R_0},$$

where $k_n=(1-K(f_n,\ R_0))/(1+K(f_n,\ R_0))$. And since $\|\theta_{c,R_0}\|_{R_0}$ is finite, we have the assertion. q.e.d.

Remark. More refined results on the distortion estimates and the variation formulas for various harmonic and holomorphic differentials under quasiconformal deformations are investigated by Y. Kusunoki, F. Maitani and H. Shiga.

Here recall that for every $R\in T(R^*)$ and every 1-cycle c, $\theta_{c,R}$ belongs to $CA_1(R)$ whenever $\theta_{c,R}\not\equiv 0$ ([12] Proposition 2). And we can show the following

Theorem 5 (*cf.* [11] Proposition). *Let R_n converge to R_0 in $T(R^*)$ and a 1-cycle c such that $\theta_{c,R_0}\not\equiv 0$ be given. Then θ_{c,R_n} converges to θ_{c,R_0} geometrically.*

Proof. First recall that for every $R\in T(R^*)$ and every 1-cycle c, the period $\int_d {}^*\sigma_{c,R}$ is equal to the algebraic intersection number $c\times d$ for every given 1-cycle d. So if a simple closed curve d belongs to $L(\theta_{c,R_0})$, then it holds that

$$\int_d {}^*\sigma_{c,R_n}=c\times d=\int_d {}^*\sigma_{c,R_0}=0,$$

hence the condition 3) in Theorem 3 holds. And because $\|\theta_{c,R_n}\|^2_{R_n}=2\cdot\int_c\sigma_{c,R_n}$ converegs to $\|\theta_{c,R_0}\|^2_{R_0}=2\cdot\int_c\sigma_{c,R_0}$ by Proposition 4 and Corollary 3, the assertion follows from Proposition 4 and Corollary 1 or 2. q.e.d.

Next suppose that R^* admits the Green's functions (, i.e. $R^*\notin O_G$). And for every $R\in T(R^*)$ and every puncture p of R, set

$$\phi_{p,R}=\frac{\sqrt{-1}}{2\pi}\cdot(dg(\cdot,\ p)+\sqrt{-1}\ \cdot{}^*dg(\cdot,\ p)),$$

where $g(\cdot,\ p)$ is the Green's function on $R\cup\{p\}$ with the pole p, and call $\phi_{p,R}$ *the fundamental differential for p on R.*

Then it is clear that $\phi_{p,R}$ belongs to the class $A_1S_0(R\cup\{p\})$ defined in [12] §2, and since Im $\phi_{p,R}$ is exact, we see from [12] Proposition 1 that $\phi_{p,R}$ belongs to $CA_1(R)$. Also we can show the following

Proposition 5. *Let $R^* \notin O_G$ and fix a puncture p^* of R^*. Also let R_n converge to R_0 in $T(R^*)$ and p_n be the puncture of R_n corresponding to p^* for every n. Then ϕ_{p_n,R_n} converges to ϕ_{p_0,R_0} strongly metrically.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ and V be as in Lemma 4 with $P=\{p^*\}$, then for every large n, $\phi_{p_n,R_n}\circ f_n-\phi_{p_0,R_0}$ is holomorphic on $V\cup\{p_0\}$, hence square integrable closed differential on $R_0\cup\{p_0\}$. Also from the definition it is easily seen that $\mathrm{Im}\,(\phi_{p_n,R_n}\circ f_n-\phi_{p_0,R_0})$ has $\{0\}$-behavior and is exact, hence belongs to $\Gamma_{e0}(R_0\cup\{p_0\})$. Thus we can show as in the proof of Proposition 4 that

$$\|\phi_{p_n,R_n}\circ f_n-\phi_{p_0,R_0}\|_{R_0}\leq(2k_n/(1-k_n))\cdot\|\phi_{p_0,R_0}\|_{R_0-V}$$

(, cf. [6] Theorem 5), from which the assertion follows. q.e.d.

Theorem 6. *Under the same assumption as in Proposition 5, ϕ_{p_n,R_n} converges to ϕ_{p_0,R_0} geometrically.*

Proof. The conditions 1) and 2) in Theorem 3 are clearly satisfied from the definition, and the condition 3) in Theorem 3 also holds, for $\mathrm{Im}\,\phi_{p_n,R_n}$ is exact for every n. Hence the assertion follows from Proposition 5 and Theorem 3. q.e.d.

Finally suppose that R^* belongs to the class O_G. Then for every $R\in T(R^*)$ and every pair $\{p_1, p_2\}$ of punctures of R, there is a harmonic function $g(p;\, p_1, p_2)$ on R uniquely determined up to constants by the following conditions;

(a) $g(p;\, p_1, p_2)$ is bounded outside any neighbourhood of $\{p_1, p_2\}$ (on $R\cup\{p_1, p_2\}$), and

(b) $g(z_j;\, p_1, p_2)-(-1)^j\cdot\log|z_j|$ is harmonic in a neighbourhood of $z_j=0$, where $z_j=z_j(p)$ is a local parameter near p_j with $z_j(p_j)=0$.

These functions $g(p;\, p_1, p_2)$ are sometimes called the Green's function on the parabolic surface R with the pair of poles p_1 and p_2. Now we set

$$\phi_{p_1,p_2,R}=\frac{\sqrt{-1}}{2\pi}\cdot(dg(\cdot\,;\, p_1, p_2)+\sqrt{-1}{}^*dg(\cdot\,;\, p_1, p_2))$$

and call $\phi_{p_1p_2,R}$ also *the fundamental differential for the pair $\{p_1, p_2\}$ on R.* Recall that $\phi_{p_1,p_2,R}$ belongs to $CA_1(R)$ again by [12] Proposition 1. And by the same argument as above, we show the following

Proposition 6. *Let $R^*\in O_G$, and fix a pair $\{p_1^*, p_2^*\}$ of punctures of R^*. Also let R_n converge to R_0 in $T(R^*)$ and $p_{j,n}$ be the puncture of R_n corresponding to p_j^* for every n and each j. Then $\phi_{p_{1,n},p_{2,n},R_n}$ converges to $\phi_{p_{1,0},p_{2,0},R_0}$ strongly metrically.*

Theorem 7. *Under the same assumption as in Proposition 6, $\phi_{p_{1,n},p_{2,n},R_n}$ converges to $\phi_{p_{1,0},p_{2,0},R_0}$ geometrically.*

Remark. We can show again by the same argument as above that such results as Proposition 5 (or 6) and Theorem 6 (or 7) for every fixed linear combination of a finite number of the fundamental differentials for given punctures (or pairs of two punctures) of R^* with real coefficients.

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References

[1] L. Ahlfors, Lectures on quasiconformal mappings, Van Nostrand, (1966), 146 pp.

[2] L. Ahlfors and L. Sario, Riemann surfaces, Princeton Univ. Press, (1960), 382 pp.

[3] C. Earle, Reduced Teichmüller spaces, Trans. A. M. S., **126** (1967), 54–63.

[4] H. M. Farkas and I. Kra, Riemann surfaces, Springer, (1980), 337 pp.

[5] F. Gardiner, On Jenkins-Strebel differentials for open Riemann surfaces, Proc. Royal Soc. of Edinburgh, **86A** (1980), 315–325.

[6] Y. Kusunoki and F. Maitani, Variations of abelian differentials under quasiconformal deformations, Math. Z., **181** (1982), 435–450.

[7] Y. Kusunoki and M. Taniguchi, A continuity property of holomorphic differentials under quasiconformal deformations, Ann. Acad. Sci. Fenn., Ser. A. I. Math., **5** (1980), 207–226.

[8] O. Lehto and K. I. Virtanen, Quasiconformal mappings in the plane, Springer, (1973), 258 pp.

[9] C. D. Minda, Square integrable differentials on Riemann surfaces and quasiconformal mappings, Trans. A. M. S., **195** (1974), 365–381.

[10] M. Taniguchi, Remarks on topologies associated with squeezing a non-dividing loop on compact Riemann surfaces, J. Math. Kyoto Univ., **19** (1979); II, ibid., **21** (1981), 273–281.

[11] M. Taniguchi, Certain kinds of convergence of holomorphic abelian differentials on the augmented Teichmüller spaces, ibid., **22** (1982), 293–305.

[12] M. Taniguchi, Dirichlet finite harmonic differentials with integral periods on arbitrary Riemann surfaces, ibid., **23** (1983), 357–367.