J. Math. Kyoto Univ. (JMKYAZ)
24-2 (1984) 281-284

A note on the local solvability of the Cauchy problem

Dedicated to Professor Sigeru MIZOHATA on his sixtieth birthday

By

Tatsuo NISHITANI

(Received December 10, 1982)

1. Introduction and results

In this note, we improve some results in the previous paper [3]. Let $p(x, D)$ be a differential operator of order m with coefficients in $\gamma^{(s)}(V)$, where V is a neighborhood of the origin in $\boldsymbol{R}^{n+1}$,

$$x=(x_0, x_1,\dots, x_n),\quad D=(D_0, D_1,\dots, D_n),\quad D_j=\frac{1}{i}\frac{\partial}{\partial x_j},$$

and $\gamma^{(s)}(V)$ denotes the set of all functions $f(x)\in C^\infty(V)$ such that for any compact set K in V, there are constants C, A with

$$|D^\alpha f(x)|\leqq CA^{|\alpha|}(|\alpha|!)^s,\ x\in K,$$

for all multi-indexes $\alpha\in N^{n+1}$.

By the definition, $\gamma^{(1)}(V)$ coincides with the set of real analytic functions in V. For convenience sake, we set $\gamma^{(\infty)}(V)=C^\infty(V)$. We denote by $p_m(x, \xi)$ the principal symbol of $p(x, D)$, and suppose that the hyperplan $\{x_0=0\}$ is non-characteristic for $p(x, D)$. Heareafter it will be assumed that $p_m(x, 1, 0,\dots, 0)=1$. Let us consider the following problem.

$$(p,\ \phi(x'))_\mu;\begin{cases} p(x, D)u=0 \\ D_0^j u(0, x')=0,\ 0\leqq j\leqq \mu-1, \\ D_0^\mu u(0, x')=\phi(x'), \end{cases}$$

where $x'=(x_1,\dots, x_n)$, $0\leqq\mu\leqq m-1$. Then we have

Theorem 1.1. *Let $s=\infty$. Suppose that the characteristic equation $p_m(0, \xi_0, \hat{\xi}')=0$, $\hat{\xi}'=(1, 0,\dots, 0)$ has μ real and ν non-real roots ($\mu+\nu=m$, $\nu\geqq 1$). Then there is a sequence of positive number $\{C_n\}$ with the following property: let $g(x_1)$ be any C^0-function defined near the origin for which $(p, g(x_1))_\mu$ has a local C^m-solution near the origin. Then $g(x_1)$ is C^∞ in a neighborhood of the origin and we have*

$$\limsup_{n\to\infty} (|g^{(n)}(0)|/C_n)^{1/n} \leqq 1,$$

where $g^{(n)}(0) = \left(\frac{d}{dx_1}\right)^n g(0)$.

Theorem 1.2. *Let* $1 < s < \infty$. *Under the same hypothesis that of theorem* 1.1, *there is a positive constant* A *having the following property: let* $g(x_1)$ *be a* C^0-*function as in theorem* 1.1, *then* $g(x_1)$ *belongs to* $\gamma^{(s)}$ *near the origin, and moreover we have*

$$\limsup_{n\to\infty} (|g^{(n)}(0)|/(n!)^s)^{1/n} \leqq A.$$

Theorem 1.3. ([3]). *Let* $s=1$. *Assume the same assumption in theorem* 1.1, *then one can find a positive constant* A *so that: if* $g(x_1)$ *is a* C^0-*function near the origin for which* $(p, g(x_1))_\mu$ *has a local* C^m-*solution defined in* $B_r = \{x; |x| < r\}$, *then* $g(x_1)$ *is analytic at the origin and has the following estimate,*

$$\limsup_{n\to\infty} (|g^{(n)}(0)|/n!)^{1/n} \leqq A/r.$$

Corollary 1.1. *Let* $1 < s \leqq \infty$. *If* $(p, \phi(x'))_\mu$ *has a local* C^m-*solution in a neighborhood of the origin for any* $\phi(x') \in \gamma^{(s)}(\boldsymbol{R}^n)$, *then the characteristic equation* $p_m(0, \xi_0, \xi') = 0$ *must have more than* $\mu+1$ *real roots for every* $\xi' \in \boldsymbol{R}^n \backslash \{0\}$.

Remark 1.1. In the case when $(p, g(x_1))_\mu$ has a local C^m-solution in a semi-neighborhood, we can obtain the corresponding results (cf. theorem 2.1 in [3]).

2. Proofs of theorems 1.1 and 1.2

Let us suppose that $p_m(0, \xi_0, \hat{\xi}') = 0$ has μ real roots and ν non-real roots ($\mu+\nu=m$, $\nu \geqq 1$). Then from lemma 3.3 in [3], there are a neighborhood W of 0 in $\boldsymbol{R}^{n+1}$, a conic neighborhood Γ of $\hat{\xi}'$ in $\boldsymbol{R}^n \backslash \{0\}$ and symbols q, r on $W \times (R \times \Gamma)$ which satisfy the followings;

$$(-1)^m p^t(x, \xi_0, \xi') \circ q(x, \xi_0, \xi') = r(x, \xi_0, \xi'), \tag{2.1}$$

with

$$r(x, \xi_0, \xi') = \xi_0^\mu + \sum_{j=0}^{\mu-1} a^j(x, \xi') \xi_0^j, \quad q(x, \xi) = \sum_{k=0}^{\infty} q_k(x, \xi), \tag{2.2}$$

where $a^j(x, \xi')$ is a symbol independent of ξ_0 of class s with order $(j, 0)$ on $W \times (R \times \Gamma)$ and $q(x, \xi)$ is a symbol of class s with order $(0, -\nu)$ satisfying the following estimate,

$$|q_{k(\beta)}^{(\alpha)}(x, \xi)| \leqq C A^{k+|\alpha+\beta|} |\xi|^{-\nu-1} |\xi'|^{1-k-|\alpha|} (k+|\beta|)!^s \alpha!, \tag{2.3}$$

for $k+|\alpha| \geqq 1$, $(x, \xi) \in W \times (R \times \Gamma)$. Here p^t denotes the transposed operator of p.

Remark 2.1. In the case when $s=\infty$, the constant $CA^{k+|\alpha+\beta|}(k+|\beta|)!^s \alpha!$ in (2.3) should be replaced by a constant $C_{k,\alpha,\beta}$ which depends on k, α, and β.

From lemma 2.2 in [2], one can take $v_{N,r}(x)\in C_0^\infty(\boldsymbol{R}^{n+1})$ so that $v_{N,r}(x)=1$ on B_r, vanish for $|x|\geqq 2r$ and satisfy

$$|D^\alpha v_{N,r}(x)|\leqq C_0(NA_0/r)^{|\alpha|}, \tag{2.4}$$

when $|\alpha|\leqq N$, where C_0, A_0 is independent of N ($N=1, 2,\dots,$). Now assume that $(p, g(x_1))_\mu$ has a solution $u(x)\in C^m(B_{3r})$. Then applying the same reasoning as [3], we have that

$$\int e^{ix\xi}\Big(\sum_{j=0}^{N-m} R_j v_{N,r}\Big)u\,dx=-\int e^{ix\xi}\Big(\sum_{N\geqq k+l\geqq N-m,N-m\geqq k,m\geqq l} P_l Q_k v_{N,r}\Big)u\,dx. \tag{2.5}$$

Now estimate the right hand side of (2.5).

Proposition 2.1. *Let $1<s<\infty$, $1\leqq N-2m\leqq k\leqq N-m$. Then we have*

$$|D^\gamma(Q_k v_{N,r})|\leqq CA^N N^{sN}|\xi|^{-\nu-1}|\xi'|^{1+2m-N}(A_1 r^{-1})^{|\gamma|}|\gamma|!^s,$$

when $(x, \xi)\in W\times(R\times\Gamma)$, $|\xi'|\geqq 1$, $N\geqq(2r^{-1})^{1/(s-1)}$, where C, A, A_1 do not depend on r, N. If $s=\infty$ and $1\leqq N-2m\leqq k\leqq N-m$, the following estimate holds

$$|D^\gamma(Q_k v_{N,r})|\leqq C_1 C_{N,|\gamma|}r^{-N-|\gamma|}|\xi|^{-\nu-1}|\xi'|^{1+2m-N},$$

for $(x, \xi)\in W\times(R\times\Gamma)$, $|\xi'|\geqq 1$, where C_1, $C_{N,|\gamma|}$ are independent of r.

From this proposition and the same procedure in [3], it follows that

$$\Big|\int d\xi_0\int e^{ix\xi}\Big(\sum_{N\geqq k+l>N-m,N-m\geqq k,m\geqq l} P_l Q_k v_{N,r}\Big)u\,dx\Big|\leqq \tag{2.6}$$

$$\leqq Cr^{-m}\sup_{|x|\leqq 2r,j\leqq m}|D_0^j u|A^N N^{sN}|\xi'|^{1+3m-N},$$

when $|\xi'|\geqq 1$, $\xi'\in\Gamma$, $N\geqq(2r^{-1})^{1/(s-1)}$, $1<s<\infty$, where C, A is independent of r and N. If $s=\infty$, this term is estimated by

$$C_N r^{-N-m}\sup_{|x|\leqq 2r,j\leqq m}|D_0^j u|\,|\xi'|^{1+3m-N}, \tag{2.7}$$

for $\xi'\in\Gamma$, $|\xi'|\geqq 1$, where C_N does not depend on r.

Using these estimates, we have

Lemma 2.1. *Let $1<s<\infty$. Suppose that $(p, g(x_1))_\mu$ has a solution $u(x)\in C^m(B_{3r})$, then there are constants C, A independent of r, N such that*

$$|\widehat{\tilde v_{N,r}g}(\xi')|\leqq CA^N N^{sN}(1+|\xi'|)^{-N}\{\sup_{|x|\leqq 2r,j\leqq m}|D_0^j u|+\sup_{|x|\leqq 2r}|g|\},$$

for $\xi'\in\boldsymbol{R}^n$, $N=1, 2,\dots$, $N\geqq(2r^{-1})^{1/(s-1)}$. If $s=\infty$, we have

$$|\widehat{\tilde v_{N,r}g}(\xi')|\leqq(C_N^2+N^{2N})(1+|\xi'|)^{-N}\{\sup_{|x|\leqq 2r,j\leqq m}|D_0^j u|+\sup_{|x|\leqq 2r}|g|\},$$

for $\xi'\in\boldsymbol{R}^n$, $N=1, 2,\dots$, $N\geqq r^{-1}$, where $\tilde v_{N,r}(x')=v_{N,r}(0, x')$ and $\widehat{\tilde v_{N,r}g}(\xi')$ denotes the Fourier transform of $\tilde v_{N,r}(x')g(x_1)$ with respect to x'.

Proof. If we integrate (2.5) by ξ_0, then the Fourier inversion formula gives that

$$|\int e^{ix'\xi'}\tilde{v}_{N,r}(x')D_0^\mu u(0, x')dx'| \leqq \text{the right hand side of (2.6)}. \tag{2.8}$$

Since $p_m(0, \xi_0, -\hat{\xi}')=0$ has also μ real roots and ν non-real roots, contarcting Γ if necessary, we may assume that the estimate (2.8) holds for $\xi' \in \Gamma \cup (-\Gamma)$, with $-\Gamma=\{\xi'; -\xi' \in \Gamma\}$.

On the other hand, in the complement of $\Gamma \cup (-\Gamma)$, one can easily get the following estimate,

$$|\widehat{\tilde{v}_{N,r}g}(\xi')| \leqq C_2 A_2^N N^{sN} |\xi'|^{-N} \sup_{|x|\leqq 2r} |g|, \tag{2.9}$$

when $N \geqq (r^{-1})^{1/(s-1)}$, where C_2, A_2 is independent of N and r. Hence the estimates (2.8) and (2.9) show this lemma when $1<s<\infty$. In the case $s=\infty$, it suffice to remark the inequalities,

$$|\widehat{\tilde{v}_{N,r}g}(\xi')| \leqq C_N r^{-N} |\xi'|^{-N} \sup_{|x|\leqq 2r} |g|, \tag{2.10}$$

in the complement of $\Gamma \cup (-\Gamma)$ with a constant C_N (possibly different from that of (2.7)) independent of r, and

$$r^{-N} C_N \leqq C_N^2 + N^{2N} \quad \text{if} \quad N \geqq r^{-1}. \tag{2.11}$$

Proofs of theorems. In view of the identity

$$\left(\frac{d}{dx_1}\right)^n (\tilde{v}_{N,r}g)(0) = \left(\frac{d}{dx_1}\right)^n g(0),$$

the theorems follow from lemma 2.1 and the inverse Fourier transformation.

Department of Mathematics
Kyoto University

References

[1] L. Boutet de Monvel and P. Krée, Pseudo-differential operators and Gevrey classes, Ann. Inst. Fourier Grenoble, **17** (1967), 295–323.

[2] L. Hörmander, Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, Comm. Pure Appl. Math., **24** (1971), 671–704.

[3] T. Nishitani, On the Lax-Mizohata theorem in the analytic and Gevrey classes, Jour. Math. Kyoto Univ., **18** (1978), 509–521.