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Sobolev spaces of Wiener functionals and Malliavin's calculus

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Introduction.

The notion of Sobolev spaces of Wiener functionals was first introduced by D. W. Stroock [10] and I. Shigekawa [7] to formulate Malliavin's calculus rigorously and study it systematically. However it cannot be denied that their Sobolev spaces sometimes appeared very complicated. It is mainly because they should have dealt with both the derivative operator D and its dual D^*, or the Ornstein-Uhlenbeck operator $L(=-D^*D)$, not only on L^2 but also on all L^p-spaces over the Wiener space. But in 1982, P. A. Meyer pointed out the possibility to remove those apparent complications; that is, he proved the equivalence of the two norms defined in terms of L and D respectively. ([3], [4])

In the present paper, we first aim to develop Meyer's results and prove the equivalence among several Sobolev-type norms. In doing this, there are two useful tools; the Wiener chaos decomposition of L^2 and the hypercontractivity of the Ornstein-Uhlenbeck semigroup. Combining these two, we follow Shigekawa's idea to prove Theorem 1.1, which offers a sufficient condition for a linear operator to be bounded on L^p.

Next we construct the Sobolev spaces of Wiener functionals and discuss their properties. In particular, our definition allows of negative indices and such spaces contain what we call generalized Wiener functionals. In this context, we consider the composition of Schwartz's distributions and Wiener functionals, which was first studied by S. Watanabe [11]. This presents another approach to Malliavin's calculus.

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1. Basic notions.

Let (W, H, μ) be an *abstract Wiener space*. i.e., W is a separable Banach space, H is a separable Hilbert space densely and continuously imbedded in W, and μ is a Gaussian measure on W with mean 0 satisfying the condition,

$$\int_W (l, w)(l', w)\mu(dw)=\langle l, l'\rangle_H, \qquad l, l'\in W^*\subset H^*=H$$

where $(,)$ and $\langle , \rangle_H$ denote the pairing of W^* and W, and the inner product of H, respectively. Let E be a separable Hilbert space, with the norm $| \ |_E$ and the inner product $\langle , \rangle_E$. We call a mapping $f: W \to E$ $\bar{\mathscr{B}}^{\mu}(W)/\mathscr{B}(E)$-measurable an *E-valued Wiener functional,* where $\mathscr{B}(W)$ and $\mathscr{B}(E)$ are the topological σ-fields on W and E respectively and $\bar{\mathscr{B}}^{\mu}(W)$ is the completion of $\mathscr{B}(W)$ with respect to the measure μ. As usual, two Wiener functionals are identified whenever they coincide μ-almost everywhere. For $1 \leqq p < \infty$, if f is an E-valued Wiener functional and $|f(w)|_E^p$ is μ-integrable on W, we say f belongs to $L^p(E) = L^p(W: E)$. The norm of $f \in L^p(E)$ is defined by

$$\|f\|_{L^p(E)} \equiv \left(\int_W |f(w)|_E^p \mu(dw) \right)^{1/p}.$$

$L^p(\boldsymbol{R}^1)$ will be denoted simply by L^p.

Now, we introduce a useful family of Wiener functionals called polynomial functionals.

Definition 1.1. (i) An $\boldsymbol{R}^1$-valued Wiener functional f is said to be a *polynomial functional,* if $\exists n \in \boldsymbol{N}$, $\exists l_1, \cdots, l_n \in W^*$ and $\exists \tilde{f}: \boldsymbol{R}^n \to \boldsymbol{R}^1$, polynomial in n variables, such that

$$f(w) = \tilde{f}((l_1, w), \cdots, (l_n, w)), \qquad w \in W. \tag{1.1}$$

The totality of such functionals is denoted by $\boldsymbol{P}$.

(ii) An E-valued Wiener functional f is said to be an *E-valued polynomial functional,* if $\exists m \in \boldsymbol{N}$, $\exists f_1, \cdots, f_m \in \boldsymbol{P}$, and $\exists e_1, \cdots, e_m \in E$ such that

$$f(w) = \sum_{i=1}^{m} f_i(w) e_i, \qquad w \in W. \tag{1.2}$$

The totality of such functionals is denoted by $\boldsymbol{P}(E)$.

In the expression (1.1) for $f \in \boldsymbol{P}$, we can always assume, by Schmidt's orthogonalization method, that the system $\{l_i\}_{i=1}^n$ forms an orthonormal system (ONS) with respect to the inner product of H. Similarly, in (1.2), we always assume $\{e_i\}_{i=1}^m$ to be an ONS of E. As W^* is dense in $H^* = H$, $\boldsymbol{P}(E)$ is a dense linear subspace of $L^p(E)$, for every $1 \leqq p < \infty$.

Next we shall introduce some important operators acting on $\boldsymbol{P}$ or $\boldsymbol{P}(E)$, such as Fréchet derivative and Ornstein-Uhlenbeck operator. They are the analogues of gradient and Laplacian in the finite dimensional case.

Definition 1.2 (*the Fréchet derivative*). (i) For $f \in \boldsymbol{P}$, the *Fréchet derivative* $Df(w) \in W^*$ *at* $w \in W$ is defined by,

$$(Df(w), v) \equiv \frac{d}{dt} f(w + tv) \Big|_{t=0}, \qquad v \in W. \tag{1.3}$$

Since $W^* \subset H^* = H$, Df can be regarded as an element of $\boldsymbol{P}(H)$. We will often deal D as an operator mapping $\boldsymbol{P}$ into $\boldsymbol{P}(H)$.

(ii) For $f \in \boldsymbol{P}(E)$ with an expression $f=\sum_{i=1}^{m} f_i e_i$, $(f_i \in \boldsymbol{P},\ e_i \in E)$, $Df \in \boldsymbol{P}(H \otimes E)$ is defined by

$$Df(w) \equiv \sum_{i=1}^{m} Df_i(w) \otimes e_i, \qquad w \in W. \tag{1.4}$$

Here $H \otimes E$ is the tensor product of Hilbert spaces H and E; the totality of all continuous bilinear forms on $H \times E$ with finite Hilbert-Schmidt norm which is endowed as the norm of $H \otimes E$. For $h \in H$ and $e \in E$, $h \otimes e \in H \otimes E$ is defined by $(h \otimes e)[\cdot, \cdot] \equiv \langle h, \cdot \rangle_H \langle e, \cdot \rangle_E$ and thus it holds that

$$\langle h \otimes e, h' \otimes e' \rangle_{H \otimes E} = \langle h, h' \rangle_H \langle e, e' \rangle_E, \qquad h' \in H \text{ and } e' \in E. \tag{1.5}$$

By definition (ii), we can iterate the operation D; $D^2 f \equiv D(Df), \cdots, D^k f \equiv D(D^{k-1} f)$ and $D^k f$ is an element of $\boldsymbol{P}(\underbrace{H \otimes \cdots \otimes H}_{k} \otimes E)$, for $f \in \boldsymbol{P}(E)$.

Definition 1.3 (*the Ornstein-Uhlenbeck semigroup and operator*). (i) We define an operator T_t: $L^1 \to L^1$, $t \geqq 0$, by

$$T_t f(w) \equiv \int_W f(e^{-t} w + \sqrt{1-e^{-2t}}\, v) \mu(dv), \qquad f \in L^1. \tag{1.6}$$

It is known that the family $\{T_t\}_{t \geq 0}$ forms a μ-symmetric contraction semigroup on L^p for every $1 \leqq p < \infty$. We call it the *Ornstein-Unhlenbeck semigroup*.

(ii) The infinitesimal generator of the semigroup $\{T_t\}_{t \geq 0}$ is called the *Ornstein-Uhlenbeck operator* and denoted by L. L has the explicit form on $\boldsymbol{P}$;

$$Lf(w) = \operatorname{trace} D^2 f(w) - (Df(w), w), \quad w \in W, \quad f \in \boldsymbol{P}. \tag{1.7}$$

(For $V \in H \otimes H$, trace V is defined by $\sum_{i=0}^{\infty} \langle V, h_i \otimes h_i \rangle_{H \otimes H}$, if the sum is absolutely convergent, where $\{h_i\}_{i=1}^{\infty}$ is complete orthonormal system (CONS) of H. The value of trace V, when it exists, is independent of the choice of $\{h_i\}_{i=1}^{\infty}$).

It is clear that T_t and L are operators mapping $\boldsymbol{P}$ into itself. But for later use, it is convenient to consider these operators on $\boldsymbol{P}(E)$. To do this in general, let S be a linear operator mapping $\boldsymbol{P}$ into itself. For $f \in \boldsymbol{P}(E)$ with an expression $f=\sum_{i=1}^{m} f_i e_i$, $f_i \in \boldsymbol{P}$, $e_i \in E$, we define $Sf \equiv \sum_{i=1}^{m} (Sf_i) e_i$. This definition, as well as (1.4), does not depend on the particular choice of the expression of f. In the present paper, any linear operator mapping $\boldsymbol{P}$ into $\boldsymbol{P}$ will always be considered as an operator mapping $\boldsymbol{P}(E)$ into $\boldsymbol{P}(E)$ in this manner.

Now we introduce the Wiener-chaos decomposition (or the Wiener-Itô decomposition) of L^2. First of all, we notice that the system $\{\sqrt{n!}\, H_n(x)\}_{n=0}^{\infty}$ of Hermite polynomials is a CONS of $L^2\left(\boldsymbol{R}^1, \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx\right)$, where

$$H_n(x) \equiv \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right), \quad n=0, 1, \cdots, \quad x \in \boldsymbol{R}^1.$$

We construct a CONS of $L^2=L^2(W:\boldsymbol{R}^1)$ using this system. Let $\Lambda\equiv\{a=(a_1, a_2, \cdots); a_i=0, 1, 2, \cdots, a_i=0$ except for finitely many i's$\}$ and $\Lambda_n\equiv\{a\in\Lambda; \sum_i a_i=n\}$. Fix a CONS $\{l_i\}_{i=1}^{\infty}$ of H such that $l_i\in W^*$ and set for $a\in\Lambda$, $H_a(w)\equiv\prod_{i=1}^{\infty}H_{a_i}((l_i, w))$, $w\in W$.

Proposition 1.1. *The system* $\{\sqrt{a!}\,H_a(w); a\in\Lambda\}$ *forms a CONS of* L^2, *where* $a!=a_1!\,a_2!\cdots$, *if* $a=(a_1, a_2, \cdots)\in\Lambda$ *and* $0!=1$.

The proof is standard and omitted.

Let Z_n be the closed subspace of L^2 spanned by the family $\{\sqrt{a!}\,H_a(w); a\in\Lambda_n\}$. Then, due to the above proposition, L^2 is decomposed into the orthogonal direct sum of $\{Z_n\}_{n=0}^{\infty}$; $L^2=Z_0\oplus Z_1\oplus\cdots$. This orthogonal decomposition is called the *Wiener chaos decomposition* or the *Wiener-Itô decomposition.* It is important to notice that this decomposition is independent of a particular choice of CONS $\{l_i\}_{i=1}^{\infty}$; indeed, it holds that $Z_n=\{f\in L^2; T_tf=e^{-nt}f$ for all $t>0\}$. Let J_n be the orthogonal projection to Z_n. Then $J_n(\boldsymbol{P})=Z_n\cap\boldsymbol{P}\subset\boldsymbol{P}$ and $J_n(\boldsymbol{P})$ is dense in Z_n. For every $f\in\boldsymbol{P}$, there exists $n\geqq 0$ such that $f\in Z_0\oplus Z_1\cdots\oplus Z_n$. And in particular, we have

$$T_t=\sum_{n=0}^{\infty}e^{-nt}J_n \quad \text{on } \boldsymbol{P} \text{ or on } L^2, \tag{1.8}$$

and

$$L=\sum_{n=0}^{\infty}(-n)J_n \quad \text{on } \boldsymbol{P} \text{ or on } \mathcal{D}(L)\equiv\left\{f\in L^2;\ \sum_{n=0}^{\infty}n^2\|J_nf\|_{L^2}^2<\infty\right\}. \tag{1.9}$$

Finally, noting the property $\frac{d}{dx}H_n(x)=H_{n-1}(x)$ of Hermite polynomials, we can easily verify the following relation on $\boldsymbol{P}$ and $\boldsymbol{P}(E)$;

$$DJ_n=J_{n-1}D,\ n\geqq 1 \quad\text{and}\quad DJ_0=0. \tag{1.10}$$

More precisely, we have for $a=(a_1, a_2, \cdots)\in\Lambda$,

$$\langle DH_a, l_i\rangle_H\begin{cases} =H_{a(i)} & \text{if } a_i>0 \text{ where } a(i)=(a_1, \cdots, a_{i-1}, a_i-1, a_{i+1}, \cdots) \\ =0 & \text{if } a_i=0 \end{cases} \tag{1.11}$$

Our aim in the remainder of this section is to prove the following theorem which was first obtained by Meyer [4], and whose proof was simplified by Shigekawa [8]. We will follow Shigekawa's idea.

Theorem 1.1. *Let* $T_\varphi: \boldsymbol{P}\to\boldsymbol{P}$ *be given by* $T_\varphi\equiv\sum_{n=0}^{\infty}\varphi(n)J_n$, *where* $\{\varphi(n)\}_{n=0}^{\infty}$ *is a real sequence. If there exist a function* $h(x)$ *analytic on some neighborhood of the origin and a positive constant* α *such that* $\varphi(n)=h(n^{-\alpha})$, *then* T_φ *can uniquely extend to a bounded linear operator on* L^p *for each* $1<p<\infty$.

For the proof, we require the following well-known result by Nelson [5] (cf. also Neveu [6]).

Proposition 1.2 (*the hypercontractivity of* T_t). *For* $1<p<\infty$, *put* $q(t)=e^{2t}(p-1)+1$ (*note that* $q(t)\geqq p$). *Then we have* $\|T_t f\|_{L^{q(t)}}\leqq\|f\|_{L^p}$, *for every* $f\in L^p$.

As a consequence, we have

Lemma 1.1. *For every* $1<p<\infty$, *and* $n=0, 1, \cdots$, *there exists a positive constant* $c_{p,n}$ *such that*

$$\|J_n f\|_{L^p}\leqq c_{p,n}\|f\|_{L^p}, \qquad \text{for all} \quad f\in \boldsymbol{P}.$$

Therefore J_n *can be considered as a bounded linear operator on* L^p.

Proof. If $p=2$, the assertion is clear. In the case $p>2$, take $t>0$ such that $p=e^{2t}+1$. Then by Proposition 1.2 and (1.8),

$$\|e^{-nt}J_n f\|_{L^p}=\|T_t J_n f\|_{L^p}\leqq\|J_n f\|_{L^2}\leqq\|f\|_{L^2}\leqq\|f\|_{L^p}.$$

Hence $\|J_n f\|_{L^p}\leqq e^{nt}\|f\|_{L^p}$, for $f\in\boldsymbol{P}$.

In the case $1<p<2$, the dual operator J_n^* is a bounded operator on L^p, because J_n is bounded in L^q with $q>2$. But since $J_n^*=J_n$ on $\boldsymbol{P}$, the assertion is obvious. q.e.d.

In order to prove Theorem 1.1, we need one more lemma.

Lemma 1.2. *For* $1<p<\infty$ *and* $n\in\boldsymbol{N}$, *there exists a positive constant* $c_{p,n}$ *such that*

(i) $\|T_t(I-J_0-\cdots-J_{n-1})f\|_{L^p}\leqq c_{p,n}e^{-nt}\|f\|_{L^p}$, *for all* $f\in L^p$,

(ii) $\|\{R(I-J_0-\cdots-J_{n-1})\}^j f\|_{L^p}\leqq c_{p,n}n^{-j}\|f\|_{L^p}$, *for all* $j\in\boldsymbol{N}$ *and* $f\in L^p$,

where I *denotes the identity operator and* $R\equiv\int_0^\infty(T_t-J_0)dt$ *is the potential operator for* L.

Proof. If $p=2$, the assertion is clear. In the case $p>2$, take $t_0>0$ such that $p=\exp(2t_0)+1$. Then Proposition 1.2 implies that

$$\begin{aligned}\|T_{t+t_0}(I-J_0-\cdots-J_{n-1})f\|_{L^p}&=\|T_{t_0}T_t(I-J_0-\cdots-J_{n-1})f\|_{L^p}\\&\leqq\|T_t(I-J_0-\cdots-J_{n-1})f\|_{L^2}.\end{aligned}$$

By (1.8), we have

$$\begin{aligned}\|T_t(I-J_0-\cdots-J_{n-1})f\|_{L^2}&=\sqrt{\sum_{j=n}^\infty e^{-2jt}\|J_j f\|_{L^2}^2}\\&\leqq e^{-nt}\sqrt{\sum_{j=n}^\infty\|J_j f\|_{L^2}^2}\\&\leqq e^{-nt}\|f\|_{L^2}\leqq e^{-nt}\|f\|_{L^p},\end{aligned}$$

and hence $\|T_{t_0+t}(I-J_0-\cdots J_{n-1})f\|_{L^p}\leqq e^{-nt}\|f\|_{L^p}$. Therefore we have that

$$\|T_t(I-J_0-\cdots-J_{n-1})f\|_{L^p}\leqq e^{-n(t-t_0)}\|f\|_{L^p}, \qquad \text{for} \quad t\geqq t_0.$$

But if $0\leqq t<t_0$,

$$\|T_t(I-J_0-\cdots-J_{n-1})f\|_{L^p}\leqq\|(I-J_0-\cdots-J_{n-1})f\|_{L^p}$$

by the contractivity of T_t. Consequently,

$$\|T_t(I-J_0-\cdots-J_{n-1})f\|_{L^p}\leqq\exp(nt_0)[1\vee\|I-J_0-\cdots-J_{n-1}\|_{L^p}]e^{-nt}\|f\|_{L^p}$$

In the case $1<p<2$, using the duality, the proof is the same as in Lemma 1.1. Thus (i) is proved.

As for (ii), note that $R(I-J_0-\cdots-J_{n-1})=\int_0^\infty T_t(I-J_0-\cdots-J_{n-1})dt$. By (i), we have

$$\begin{aligned}
&\|\{R(I-J_0-\cdots-J_{n-1})\}^2\|_{L^p}\\
&=\left\|\int_0^\infty\int_0^\infty T_t(I-J_0-\cdots-J_{n-1})T_s(I-J_0-\cdots J_{n-1})dtds\right\|_{L^p}\\
&=\left\|\int_0^\infty\int_0^\infty T_{t+s}(I-J_0-\cdots-J_{n-1})dtds\right\|_{L^p}\\
&\leqq\int_0^\infty\int_0^\infty\|T_{t+s}(I-J_0-\cdots-J_{n-1})\|_{L^p}dtds\\
&\leqq\int_0^\infty\int_0^\infty c_{p,n}e^{-n(t+s)}dtds\\
&=c_{p,n}n^{-2}.
\end{aligned}$$

Similarly, we can estimate $\|\{R(I-J_0-\cdots-J_{n-1})\}^j\|_{L^p}$ and obtain (ii). q.e.d.

Now we can proceed to the proof of Theorem 1.1. First we will give the proof in the case $\alpha=1$. Take $k\in\boldsymbol{N}$ such that $h(x)=\sum_{j=0}^\infty a_jx^j$ is absolutely convergent for $|x|\leqq\frac{1}{k}$. Next we devide T_φ into two parts;

$$T_\varphi=T_\varphi^{(1)}+T_\varphi^{(2)},\qquad\text{where}\quad T_\varphi^{(1)}=\sum_{n=0}^{k-1}\varphi(n)J_n\quad\text{and}\quad T_\varphi^{(2)}=\sum_{n=k}^\infty\varphi(n)J_n.$$

Then $T_\varphi^{(1)}$ is bounded on L^p by Lemma 1.1. On the other hand, the following equality holds.

$$T_\varphi^{(2)}=\sum_{j=0}^\infty a_j\{R(I-J_0-\cdots-J_{k-1})\}^j\tag{1.12}$$

Indeed, for $f_n\in Z_n(n\geqq k)$, since $R(I-J_0-\cdots-J_{k-1})f_n=(1/n)f_n$, we have

$$\begin{aligned}
\sum_{j=0}^\infty a_j\{R(I-J_0-\cdots-J_{k-1})\}^jf_n&=\sum_{j=0}^\infty a_j\left(\frac{1}{n}\right)^jf_n\\
&=h\left(\frac{1}{n}\right)f_n=\varphi(n)f_n.
\end{aligned}$$

But by Lemma 1.2 (ii), the right hand side of (1.12) is convergent in the L^p-operator norm. Thus $T_\varphi^{(2)}$ is bounded on L^p.

Next we will prove the case of general $\alpha>0$. We may restrict ourselves to the case $0<\alpha<1$. Put $\tilde{T}_t\equiv\int_0^\infty T_s\lambda_t(ds)$, where λ_t is a probability measure on $\boldsymbol{R}^1$ determined by $\int_0^\infty e^{-us}\lambda_t(ds)=e^{-u^\alpha t}$, i.e., λ_t is a one-sided stable distribution of order α and $\tilde{T}_t$ is the α-subordination of T_t. Then by Lemma 1.2,

$$\|\tilde{T}_t(I-J_0-\cdots-J_{n-1})\|_{L^p}\leqq\int_0^\infty\|T_s(I-J_0-\cdots-J_{n-1})\|_{L^p}\lambda_t(ds)\leqq c_{p,n}e^{-n^\alpha t}.$$

and similarly,

$$\|\{\tilde{R}(I-J_0-\cdots-J_{n-1})\}^j\|_{L^p}\leqq c_{p,n}n^{-\alpha j},\quad\text{where}\quad \tilde{R}\equiv\int_0^\infty(\tilde{T}_t-J_0)dt.$$

Now the proof runs in the same way as above. q.e.d.

2. Equivalence of norms on $\boldsymbol{P}(E)$.

The operator $L=\sum_{n=0}^\infty(-n)J_n$ with domain $\mathcal{D}(L)=\{f\in L^2;\ \sum_{n=0}^\infty n^2\|J_nf\|_{L^2}^2<\infty\}$ is a non-positive definite self-adjoint operator on L^2. Therefore we can define $C=-\sqrt{-L}$, which we call the *Cauchy operator*. We note that C also maps $\boldsymbol{P}$ into itself; $C=\sum_{n=0}^\infty(-\sqrt{n})J_n$. According to our convention, C is also considered as an operator mapping $\boldsymbol{P}(E)$ into itself.

First we shall state the following theorem which was obtained, in the case of $E=\boldsymbol{R}^1$, by Meyer [3] as an application of the Littlewood-Paley-Stein inequalities.

Theorem 2.1. *For $1<p<\infty$, there exist positive constants c_p and c_p' such that*

$$c_p\|Cf\|_{L^p(E)}\leqq\|Df\|_{L^p(H\otimes E)}\leqq c_p'\|Cf\|_{L^p(E)},\qquad \textit{for}\quad f\in\boldsymbol{P}(E). \tag{2.1}$$

Note that the equality $\|Cf\|_{L^2(E)}=\|Df\|_{L^2(H\otimes E)}$ holds on $\boldsymbol{P}(E)$.

To make notations brief, we shall introduce the following;

(i) The norm of $L^p(E)$ will be denoted simply by $\|\ \|_p$ whatever E may be.

(ii) For two norms $\|\ \|$ and $\|\ \|'$ on a linear space K, we write $\|f\|\lesssim\|f\|'$ or $\|f\|'\gtrsim\|f\|$, if there exists a positive constant c such that $\|f\|\leqq c\|f\|'$ for all $f\in K$. If both $\|f\|\lesssim\|f\|'$ and $\|f\|\gtrsim\|f\|'$ hold, we write $\|f\|\sim\|f\|'$, and say that the norms $\|\ \|$ and $\|\ \|'$ are *equivalent* to each other. Under these notations, (2.1) is rewritten as $\|Cf\|_p\sim\|Df\|_p$.

We shall deduce Theorm 2.1 from Meyer's result in the case $E=\boldsymbol{R}^1$. For this, we need the following lemma.

Lemma 2.1 (*Khintchin's inequalities*). *Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $\{r_i(\omega)\}_{i=1}^\infty$, $\omega\in\Omega$, be a sequence of i.i.d. random variables on Ω with $P(r_i=1)=P(r_i=-1)=1/2$ (Rademacher's system of random variables). Then,*

(i) *For $0<p<\infty$, we have*

$$\Big(\sum_{i=1}^\infty|a_i|^2\Big)^{p/2}\sim\boldsymbol{E}\Big(\Big|\sum_{i=1}^\infty r_ia_i\Big|^p\Big),\qquad \textit{for all}\quad\{a_i\}_{i=1}^\infty\in l^2.$$

Here $\boldsymbol{E}$ *denotes the integration under the probability measure* P.

(ii) *Let* G *be a separable Hilbert space and* $1<p<\infty$. *Then we have*

$$\left(\sum_{i=1}^{\infty}|a_i|_G^2\right)^{p/2}\sim\boldsymbol{E}\left(\left|\sum_{i=1}^{\infty}r_ia_i\right|_G^p\right),$$

for all G*-valued sequences* $\{a_i\}_{i=1}^{\infty}$ *such that* $\sum_{i=1}^{\infty}|a_i|_G^2<\infty$.

(iii) *Let* $1<p<\infty$. *Then we have*

$$\left(\sum_{i=1}^{\infty}a_{ii}\right)^p\sim\boldsymbol{E}\left(\left|\sum_{i=1,j=1}^{\infty\ \ \infty}a_{ij}r_ir_j\right|^p\right)$$

for all non-negative definite matrices (a_{ij}), $i, j\in\boldsymbol{N}$, *such that* $\sum_{i=1,j=1}^{\infty\ \ \infty}|a_{ij}|^2<\infty$.

Proof. The proof for (i) is found in Stein [9] and its Hilbert version (ii) is in Burkholder [1]. The assertion (iii) is easily derived from (ii). Indeed, if (a_{ij}) is a finite matrix, we can find a matrix $(\tilde{a}_{ij})$ such that $a_{ij}=\sum_k\tilde{a}_{ik}\tilde{a}_{kj}$. Applying (ii) for $\tilde{a}_i\equiv\{\tilde{a}_{ij}\}_j\in l^2$, we obtain (iii). For an infinite matrix, an approximation by finite matrices will complete the proof. q.e.d.

Now, we will proceed to the proof of Theorem 2.1 for $\boldsymbol{P}(E)$. Let f be an element of $\boldsymbol{P}(E)$ and have an expression $f(w)=\sum_{i=1}^{m}f_i(w)e_i$, $w\in W$, where $f_i\in\boldsymbol{P}$ and $\{e_i\}_{i=1}^{m}$ is an ONS of E. Take a Rademacher's system $\{r_i\}_{i=1}^{\infty}$ on a probability space $(\Omega, \mathcal{F}, P)$, and put $X(\omega, w)=\sum_{i=1}^{m}r_i(\omega)f_i(w)$, $\omega\in\Omega$, $w\in W$. Then $X(\omega, \cdot)\in\boldsymbol{P}$ for all ω. First, Lemma 2.1 (i) implies that

$$\boldsymbol{E}(|CX(\cdot, w)|^p)=\boldsymbol{E}\left(\left|\sum_{i=1}^{m}r_iCf_i(w)\right|^p\right)\sim\left(\sum_{i=1}^{m}|Cf_i(w)|^2\right)^{p/2}=|Cf(w)|_E^p$$

for all $w\in W$. Integrating both hand sides with μ, we get

$$\int_W\boldsymbol{E}(|CX|^p)d\mu=\boldsymbol{E}(\|CX\|_p^p)\sim\|Cf\|_p^p.$$

On the other hand, since the matrix $(\langle Df_i(w), Df_j(w)\rangle_H)$ is non-negative definite, by Lemma 2.1 (iii), we have

$$\boldsymbol{E}(|DX(\cdot, w)|_H^p)=\boldsymbol{E}(|\sum_{i,j}r_ir_j\langle Df_i(w), Df_j(w)\rangle_H|^p)\sim(\sum_i|Df_i(w)|_H^2)^{p/2}$$

for all $w\in W$. Similarly, the integration with μ leads us to

$$\int_W\boldsymbol{E}(|DX|_H^p)d\mu=\boldsymbol{E}(\|DX\|_p^p)\sim\|Df\|_p^p.$$

But Meyer's result for $\boldsymbol{R}^1$-valued polynomials implies that $\|CX(\omega, \cdot)\|_p\sim\|DX(\omega, \cdot)\|_p$ for all ω, which completes the proof. q.e.d.

Here is another useful consequence of Lemma 2.1.

Lemma 2.2. *Let* $1<p<\infty$. *If* $S: \boldsymbol{P}\to\boldsymbol{P}$ *is a bounded linear operator in the*

L^p-norm, it is also bounded as an operator $S: \boldsymbol{P}(E)\rightarrow\boldsymbol{P}(E)$ in the $L^p(E)$-norm.

Proof. Let $f\in\boldsymbol{P}(E)$ be expressed as $f(w)=\sum_{i=1}^{m}f_i(w)e_i$, $w\in W$, where $f_i\in\boldsymbol{P}$, and $\{e_i\}_{i=1}^{m}$ is an ONS of E. It is sufficient to show that

$$\left\|\sqrt{\sum_{i=1}^{m}|Sf_i|^2}\right\|_p\lesssim\left\|\sqrt{\sum_{i=1}^{m}|f_i|^2}\right\|_p.$$

To show this, take a Rademacher's system $\{r_i(\omega)\}_{i=1}^{\infty}$, $\omega\in\Omega$, and set $X(\omega, w)=\sum_{i=1}^{m}r_i(\omega)f_i(w)$. By Lemma 2.1 (i), we have that $\boldsymbol{E}(|X(\cdot, w)|^p)\sim\left(\sum_{i=1}^{m}|f_i(w)|^2\right)^{p/2}$ for all w. Hence, $(\boldsymbol{E}\|X\|_p^p)^{1/p}\sim\left\|\sqrt{\sum_{i=1}^{m}|f_i|^2}\right\|_p$. Noting that $SX(\omega, \cdot)=\sum_{i=1}^{m}r_i(\omega)Sf_i(\cdot)$, similarly we have $(\boldsymbol{E}\|SX\|_p^p)^{1/p}\sim\left\|\sqrt{\sum_{i=1}^{m}|Sf_i|^2}\right\|_p$. On the other hand, since $X(\omega, \cdot)\in\boldsymbol{P}$, $\|SX(\omega, \cdot)\|_p\lesssim\|X(\omega, \cdot)\|_p$ by the hypothesis. Consequently, it holds that $\boldsymbol{E}\|SX\|_p^p\lesssim\boldsymbol{E}\|X\|_p^p$ and the proof is complete. q.e.d.

The first assertion of the following theorem was established by Meyer [4] in the case $E=\boldsymbol{R}^1$.

Theorem 2.2. *For any $k\in\boldsymbol{N}$ and $1<p<\infty$,*

(i) $$\|D^kf\|_p\lesssim\|C^kf\|_p, \qquad f\in\boldsymbol{P}(E),$$

(ii) $$\|D^kf\|_p\gtrsim\|C^kf\|_p, \qquad f\in\boldsymbol{P}_{k+1}(E),$$

where $\boldsymbol{P}_{k+1}(E)\equiv\{f\in\boldsymbol{P}(E); (J_0+\cdots+J_k)f=0\}$.

For the proof we show the following lemma from which it is immediately obtained by induction.

Lemma 2.3. (i) $\|CD^kf\|_p\lesssim\|D^kCf\|_p$, $\quad f\in\boldsymbol{P}(E)$.

(ii) $$\|CD^kf\|_p\gtrsim\|D^kCf\|_p, \qquad f\in\boldsymbol{P}_{k+1}(E).$$

Proof. First we shall prove (i). Let $T_k: \boldsymbol{P}\rightarrow\boldsymbol{P}$ be defined by $T_k=\sum_{n=k+1}^{\infty}\sqrt{1-\frac{k}{n}}J_n$. Then we have $T_kD^kC=CD^k$. Indeed, since $C=\sum_{n=1}^{\infty}-\sqrt{n}J_n$, $T_kD^kC=T_kD^k\sum_{n=1}^{\infty}-\sqrt{n}J_n=T_k\sum_{n=1}^{\infty}-\sqrt{n}D^kJ_n$. By applying (1.10) k-times, we see that

$$\begin{aligned}T_kD^kC&=T_k\sum_{n=k+1}^{\infty}-\sqrt{n}J_{n-k}D^k=\sum_{n=k+1}^{\infty}\sqrt{1-\frac{k}{n}}(-\sqrt{n})J_{n-k}D^k\\&=\sum_{n=k+1}^{\infty}-\sqrt{n-k}\,J_{n-k}D^k\\&=CD^k.\end{aligned}$$

By Theorem 1.1 and Lemma 2.2, T_k extends boundedly on $L^p(E)$. Therefore, $\|CD^kf\|_p=\|T_kD^kCf\|_p\leqq\|T_k\|_p\|D^kCf\|_p$ for $f\in\boldsymbol{P}(E)$, implying (i).

Next we shall prove (ii). Let $T'_k: \boldsymbol{P} \to \boldsymbol{P}$ be defined by $T'_k = \sum_{n=k+1}^{\infty} \left(1-\frac{k}{n}\right)^{-1/2} J_n$. Then since $T_k T'_k = T'_k T_k = I - J_0 - \cdots - J_k$, we have $D^k Cf = T'_k C D^k f$ for $f \in \boldsymbol{P}_{k+1}(E)$. Now the proof proceeds as in (i). q.e.d.

Theorem 2.3. *For any $1<p, q<\infty$ and $k \in \boldsymbol{N}$, we have the following relations on $\boldsymbol{P}(E)$.*

(i) $$\|f\|_p \lesssim \|f\|_1 + \|C^k f\|_p .$$

(ii) $$\|(J_0 + \cdots + J_k) f\|_p \sim \|(J_0 + \cdots + J_k) f\|_q .$$

(iii) $$\|C^k f\|_p \lesssim \|f\|_q + \|D^k f\|_p .$$

Proof. (i) Let $f_0 = J_0 f$ and $f_1 = f - f_0$. Then the equality $C^k f = C^k f_1$ holds, and hence we have $f_1 = V^k C^k f$ where $V = \sum_{n=1}^{\infty} -n^{-1/2} J_n$. As V is bounded in $L^p(E)$ by Theorem 1.1 and Lemma 2.2, we see that $\|f_1\|_p \lesssim \|C^k f\|_p$. On the other hand, as $f_0 = \int_W f(w)\mu(dw)$, we have $\|f_0\|_p = |f_0|_E \leqq \|f\|_1$, and consequently, $\|f\|_p \lesssim \|f_1\|_p + \|f_0\|_p \lesssim \|f\|_1 + \|C^k f\|_p$.

(ii) Let $q>p>1$. It is sufficient to show the following;

$$\|(J_0 + \cdots + J_k) f\|_p \gtrsim \|(J_0 + \cdots + J_k) f\|_q .$$

To this end, take $t>0$ such that $q = e^{2t}(p-1)+1$. If $E = \boldsymbol{R}^1$ i.e., $f \in \boldsymbol{P}$, then by Proposition 1.2, we have $\|(J_0 + \cdots + J_k) f\|_p \geqq \|T_t (J_0 + \cdots + J_k) f\|_q$. In general, $\|(J_0 + \cdots + J_k) f\|_p \gtrsim \|T_t (J_0 + \cdots + J_k) f\|_q$ holds, by the same argument as in Lemma 2.2. But Lemma 1.1 implies that $T_t^{-1}(J_0 + \cdots + J_k)$ is bounded on $L^q(E)$, hence we have $\|T_t (J_0 + \cdots + J_k) f\|_q \gtrsim \|(J_0 + \cdots + J_k) f\|_q$. This concludes the proof.

(iii) Put $f_0 = (J_0 + \cdots + J_k) f$ and $f_1 = f - f_0$. Then Theorem 2.2 (ii) implies that $\|C^k f_1\|_p \lesssim \|D^k f_1\|_p$. On the other hand, we have

$$\|D^k f_1\|_p \leqq \|D^k f\|_p + \|D^k f_0\|_p \lesssim \|D^k f\|_p + \|C^k f_0\|_p$$

and also,

$$\|C^k f_0\|_p = \|C^k (J_0 + \cdots + J_k) f_0\|_p \lesssim \|f_0\|_p \sim \|f_0\|_q \lesssim \|f\|_q ,$$

hence $\|C^k f_1\|_p \lesssim \|D^k f_1\|_p \lesssim \|D^k f\|_p + \|f\|_q$. Finally we see that

$$\|C^k f\|_p \leqq \|C^k f_0\|_p + \|C^k f_1\|_p \lesssim \|D^k f\|_p + \|f\|_q .$$ q.e.d.

Now, we proceed to the main theorem of this section, which claims the equivalence of several Sobolev norms on $\boldsymbol{P}(E)$.

Theorem 2.4. *For $1<p<\infty$ and $k \in \boldsymbol{N}$, the following five norms on $\boldsymbol{P}(E)$ are equivalent to each other. Here, we put $C^0 = D^0 = I$.*

$$\|f\|_{p,k} = \|(I-C)^k f\|_p, \qquad \|f\|_{p,k}^{(1)} = \sum_{i=0}^{k} \|C^i f\|_p, \qquad \|f\|_{p,k}^{(2)} = \sum_{i=0}^{k} \|D^i f\|_p,$$

$$\|f\|_{p,k}^{(3)} = \|f\|_p + \|C^k f\|_p, \qquad \|f\|_{p,k}^{(4)} = \|f\|_p + \|D^k f\|_p$$

(ii) *The following two norms are also equivalent to the above norms;*

(a) $\|f\|_q+\|C^k f\|_p, \quad 1\leqq q<p.$ (b) $\|f\|_q+\|D^k f\|_p, \quad 1<q<p.$

(iii) *For* $1<p_0, \cdots, p_k<\infty$, *the two norms* $\|f\|_{p_0,\cdots,p_k}\equiv\sum_{i=0}^{k}\|D^i f\|_{p_i}$ *and* $\sum_{i=0}^{k}\|C^i f\|_{p_i}$ *are equivalent.*

(iv) *If* k *is even, we have* $\|f\|_p+\|L^{k/2}f\|_p\sim\sum_{i=0}^{k/2}\|L^i f\|_p\sim\|f\|_{p,k}$. *For an arbitrary* $r\in\boldsymbol{R}^1$, $\|f\|_{p,r}=\|(I-C)^r f\|_p\sim\|(I-L)^{r/2}f\|_p$ *holds.*

Remark. In general, for $T_\varphi=\sum_{n=0}^{\infty}\varphi(n)J_n$ with $\varphi(n)>0$ and $r\in\boldsymbol{R}^1$, we define $(T_\varphi)^r$ by $\sum_{n=0}^{\infty}(\varphi(n))^r J_n$. Since $I-C=\sum_{n=0}^{\infty}(1+\sqrt{n})J_n$ and $I-L=\sum_{n=0}^{\infty}(1+n)J_n$, we can define $(I-C)^r$ and $(I-L)^r$ in this manner.

Proof. (i) First we shall prove that $\|\ \|_{p,k}^{(3)}\sim\|\ \|_{p,k}^{(4)}$. Indeed, $\|\ \|_{p,k}^{(3)}\gtrsim\|\ \|_{p,k}^{(4)}$ is clear from Theorem 2.2 (i), and the converse relation is from Theorem 2.3 (iii). Similarly we can show $\|\ \|_{p,k}^{(1)}\sim\|\ \|_{p,k}^{(2)}$. Next we show that $\|\ \|_{p,k}^{(1)}\sim\|\ \|_{p,k}^{(3)}$. It is sufficient to prove that $\|C^i f\|_p\lesssim\|C^k f\|_p$, for $i=0, \cdots, k-1$. As the proof of Theorem 2.3 (i), we note that $C^i f=V^{k-i}C^k f$, where $V=\sum_{n=1}^{\infty}-n^{-1/2}J_n$ and that V is bounded on $L^p(E)$. Consequently, it holds that $\|C^i f\|_p\lesssim\|C^k f\|_p$. For the proof of $\|\ \|_{p,k}\sim\|\ \|_{p,k}^{(1)}$, we first note that $\|f\|_{p,k}\leqq\sum_{i=0}^{k}\binom{k}{i}\|C^i f\|_p\sim\|f\|_{p,k}^{(1)}$, and secondly that $\|f\|_{p,k}^{(3)}=\|f\|_p+\|\{I-(I-C)\}^k f\|_p\leqq\|f\|_p+\sum_{i=0}^{k}\binom{k}{i}\|(I-C)^i f\|_p$. But since $(I-C)^{-1}$ is a contraction on $L^p(E)$, we have $\|(I-C)^i f\|_p\leqq\|(I-C)^k f\|_p$, for $i=0, \cdots, k-1$, implying that $\|\ \|_{p,k}\gtrsim\|\ \|_{p,k}^{(1)}$.

(ii) This is an easy consequence of Theorem 2.3.

(iii) This is a consequence of Theorem 2.2 and Theorem 2.3.

(iv) It is easy to see that $\|f\|_p+\|L^{k/2}f\|_p\sim\sum_{i=0}^{k/2}\|L^i f\|_p\sim\|f\|_{p,k}^{(1)}$. Let $T=\sum_{n=0}^{\infty}\left(\frac{1+\sqrt{n}}{\sqrt{1+n}}\right)^r J_n$. Then both T and T^{-1} are bounded on $L^p(E)$ and $T(I-L)^{r/2}=(I-C)^r$ holds. From this, $\|f\|_{p,r}\sim\|(I-L)^{r/2}f\|_p$ follows. q.e.d.

Remark. By virture of the increasing property of L^p-norm in p, and the boundedness of the operator V, it is easy to verify that

$$\|f\|_{p_0,\cdots,p_k}\sim\sum_{i=0}^{k}\|C^i f\|_{p_i'}.$$

where $p_0'=p_0$, $p_k'=p_k$ and $p_i'=p_{i+1}\vee p_i$, $i=1, \cdots, k-1$. Thus we may always assume that $1<p_k\leqq\cdots\leqq p_1<\infty$ when we consider the norm $\|\ \|_{p_0,\cdots,p_k}$.

3. Sobolev spaces of Winier functionals.

In this section, we define Sobolev spaces of Wiener functionals and discuss the differential calculus on them. We adopt the norm $\|\ \|_{p,r}$ to define these spaces, since r can be any real number.

Definition 3.1. Let $\boldsymbol{D}_{p,r}(E)$ be the completion of $\boldsymbol{P}(E)$ by the norm $\| \ \|_{p,r}$, $r\in\boldsymbol{R}^1$, $1<p<\infty$. $\boldsymbol{D}_{p,r}(\boldsymbol{R}^1)$ will be denoted by $\boldsymbol{D}_{p,r}$.

The system of norms $\{\| \ \|_{p,r}\}_{1<p<\infty, r\in\boldsymbol{R}^1}$ is compatible on $\boldsymbol{P}(E)$ in the sense that if $\{f_i\}_{i=1}^{\infty}$, $f_i\in\boldsymbol{P}(E)$, is a Cauchy sequence in $\| \ \|_{p,r}$ converging to 0 in another norm $\| \ \|_{q,s}$, then it also converges to 0 in $\| \ \|_{p,r}$. It is because the operators $(I-C)^r$, $r\in\boldsymbol{R}^1$, are closable in each $L^p(E)$; this is easily shown by their symmetry on $\boldsymbol{P}(E)$ in $L^2(E)$. Since $(I-C)^r$, $r\leqq 0$, are contraction operators on $L^p(E)$, we have the following inclusion relation;

$$(3.1)\qquad 1<p\leqq q<\infty \ \text{ and } \ r\leqq s \longrightarrow \boldsymbol{D}_{q,s}(E)\underset{\rightarrow}{\subset}\boldsymbol{D}_{p,r}(E),$$

where "$\underset{\rightarrow}{\subset}$" stands for the continuous imbedding. Clearly $\boldsymbol{D}_{p,0}(E)=L^p(E)$ and thus we have the following diagram; $1<p\leqq q<\infty$, $0\leqq r\leqq s<\infty$

$$\begin{array}{ccccccccc} \boldsymbol{D}_{p,s}(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{p,r}(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{p,0}(E)=L^p(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{p,-r}(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{p,-s}(E) \\ \cup\uparrow & & \cup\uparrow & & \cup\uparrow & & \cup\uparrow & & \cup\uparrow \\ \boldsymbol{D}_{q,s}(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{q,r}(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{q,0}(E)=L^q(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{q,-r}(E) & \underset{\rightarrow}{\subset} & \boldsymbol{D}_{q,-s}(E) \end{array}$$

Similarly, if we set $\boldsymbol{D}_{[p_0,\dots,p_k]}(E)$ to be the completion of $\boldsymbol{P}(E)$ by the norm $\| \ \|_{p_0,\dots,p_k}$, we see that

$$\boldsymbol{D}_{p_1\vee p_0,k}(E)\underset{\rightarrow}{\subset}\boldsymbol{D}_{[p_0,\dots,p_k]}(E)\underset{\rightarrow}{\subset}\boldsymbol{D}_{p_k,k}(E)$$

where $1<p_0<\infty$ and $1<p_k\leqq\cdots\leqq p_1<\infty$.

We remark also that an element of $\boldsymbol{D}_{p,r}(E)$, $r<0$, is not necessarily an E-valued Wiener functional.

Theorem 3.1. (i) *For $1<p$, $q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $r\in\boldsymbol{R}^1$, we have*

$$(\boldsymbol{D}_{p,r}(E))^*=\boldsymbol{D}_{q,-r}(E),$$

under the standard identification of $(L^2(E))^*=L^2(E)$.

(ii) *Let E_1 and E_2 be two separable Hilbert spacses and let $1<p_1$, p_2, $q<\infty$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q}$ and k be a non-negative integer. Then for every $f\in\boldsymbol{D}_{p_1,k}(E_1)$ and $g\in\boldsymbol{D}_{p_2,k}(E)$, we have $f\otimes g\in\boldsymbol{D}_{q,k}(E_1\otimes E_2)$ and furthermore, the following estimate holds.*

$$\|f\otimes g\|_{q,k}\lesssim\|f\|_{p_1,k}\|g\|_{p_2,k}.$$

Proof. (i) $\boldsymbol{P}(E)$ being a dense subspace of both $\boldsymbol{D}_{p,r}(E)$ and $L^p(E)$, the identities $\|f\|_{p,r}=\|(I-C)^r f\|_p$ and $\|(I-C)^{-r}f\|_{p,r}=\|f\|_p$, $f\in\boldsymbol{P}(E)$, imply that $(I-C)^r$ and $(I-C)^{-r}$ extend to isomorphic cperators $\boldsymbol{D}_{p,r}(E)\to L^p(E)$ and $L^p(E)\to\boldsymbol{D}_{p,r}(E)$ respectively. Therefore, we define the pairing ${}_{p,r}(f, g)_{q,-r}$ of $f\in\boldsymbol{D}_{p,r}(E)$ and $g\in\boldsymbol{D}_{q,-r}(E)$ by

$$(3.2)\qquad {}_{p,r}(f, g)_{q,-r}\equiv\int_W\langle(I-C)^r f(w), (I-C)^{-r}g(w)\rangle_E\mu(dw).$$

(The above value is equal to $\int_W \langle f(w), g(w)\rangle_E \mu(dw)$ if both f and g are in $\boldsymbol{P}(E)$). The assertion (i) follows from the fact that $L^p(E)^* = L^q(E)$.

(ii) Let $f \in \boldsymbol{P}(E_1)$ and $g \in \boldsymbol{P}(E_2)$. It is easy to verify that $D(f\otimes g) = (Df)\otimes g + f\otimes(Dg)$ and generally, $D^k(f\otimes g) = \sum_{j=0}^{k}\binom{k}{j}(D^j f)\otimes(D^{k-j}g)$. Noting $|f\otimes g|_{E_1\otimes E_2} = |f|_{E_1}|g|_{E_2}$, and using the norms in Theorem 2.4 (i),

$$\begin{aligned}\|f\otimes g\|_{r,k}^{(4)} &\equiv \|f\otimes g\|_r + \|D^k(f\otimes g)\|_r \\ &\leqq \|f\|_p\|g\|_q + \sum_{j=0}^{k}\binom{k}{j}\|D^j f\|_p\|D^{k-j}g\|_q \\ &\lesssim \Big(\sum_{j=0}^{k}\|D^j f\|_p\Big)\Big(\sum_{j=0}^{k}\|D^j g\|_q\Big) \equiv \|f\|_{p,k}^{(2)}\|g\|_{q,k}^{(2)}\end{aligned}$$ q. e. d.

Next, we extend the operators L, C and D.

Theorem 3.2. *Let $1<p<\infty$ and $r\in\boldsymbol{R}^1$. Then the operators L, C and D extend to unique bounded linear operators respectively, as follows;*

(i) $$L: \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{p,r-2}(E),$$

(ii) $$C: \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{p,r-1}(E),$$

(iii) $$D: \boldsymbol{D}_{p,r}(E) \longrightarrow \boldsymbol{D}_{p,r-1}(H\otimes E).$$

Proof. To show (ii), it is sufficient to show $\|(I-C)^{r-1}Cf\|_p \lesssim \|(I-C)^r f\|_p$ for $f\in\boldsymbol{P}(E)$. But it is clear, since $(I-C)^{-1}C = \sum_{n=0}^{\infty}\frac{\sqrt{n}}{\sqrt{1+n}}J_n = \sum_{n=1}^{\infty}\sqrt{\frac{1}{1+(1/n)}}J_n$ is L^p-bounded by Theorem 1.1.

(i) follows immediately from (ii).

The assertion (iii) is equivalent to the following;

$$\|(I-C)^r f\|_p \gtrsim \|(I-C)^{r-1}Df\|_p, \qquad f\in\boldsymbol{P}(E)$$

To show this, we set $S = \sum_{n=1}^{\infty}\frac{1+\sqrt{n-1}}{1+\sqrt{n}}J_n$. S commutes C and is bounded on $L^p(E)$, moreover we see that

$$D(I-C)^{r-1}S^{r-1}f = (I-C)^{r-1}Df, \qquad f\in\boldsymbol{P}(E).$$

Then,

$$\begin{aligned}\|(I-C)^r f\|_p &\gtrsim \|S^{r-1}(I-C)^r f\|_p = \|(I-C)^r S^{r-1}f\|_p \\ &\gtrsim \|C(I-C)^{-1}\cdot(I-C)^r S^{r-1}f\|_p \\ &= \|C(I-C)^{r-1}S^{r-1}f\|_p \\ &\sim \|D(I-C)^{r-1}S^{r-1}f\|_p \\ &= \|(I-C)^{r-1}Df\|_p.\end{aligned}$$ q. e. d.

Now we will discuss the dual operator D^* of D, which is the analogue of "$-$div" in the finite dimensional case. Let $1<p<\infty$ and $r\in\boldsymbol{R}^1$. Since D maps

$\boldsymbol{D}_{q,-r+1}(E)$ into $\boldsymbol{D}_{q,-r}(H\otimes E)$ where $\frac{1}{p}+\frac{1}{q}=1$, D^* maps $\boldsymbol{D}_{p,r}(H\otimes E)$ into $\boldsymbol{D}_{p,r-1}(E)$, and by definition, we have

$$_{p,r}(f, Dg)_{q,-r}={}_{p,r-1}(D^*f, g)_{q,-r+1}, \quad \text{for} \quad f\in \boldsymbol{D}_{p,r}(H\otimes E) \quad \text{and} \quad g\in \boldsymbol{D}_{q,-r+1}(E).$$

Theorem 3.3. (i) *D^* is a bounded linear operator*: $\boldsymbol{D}_{p,r}(H\otimes E)\to \boldsymbol{D}_{p,r-1}(E)$.
(ii) $D^*D=-L$.
(iii) *If $f\in \boldsymbol{P}(H)$ has the form*

$$f(w)=\sum_{i=1}^{n} f_i(w)l_i, \qquad f_i\in \boldsymbol{P}, \qquad l_i\in W^*, \tag{3.3}$$

then

$$D^*f(w)=-\mathrm{trace}\ Df(w)+{}_{W^*}(f(w), w)_W. \tag{3.4}$$

(iv) *If $f\in \boldsymbol{P}(H\otimes E)$ has the form $f(w)=\sum_{i=1}^{m} f_i(w)\otimes e_i$, where $f_i\in \boldsymbol{P}(H)$ having the form* (3.3) *and $e_i\in E$, then we have $D^*f(w)=\sum_{i=1}^{m}(D^*f_i(w))e_i$ where $D^*f_i(w)$ is given by* (3.4).

Proof. (i) follows directly from Theorem 3.1 (i) and Theorem 3.2 (iii). As for (ii), it sufficies to note that for $f, g\in \boldsymbol{P}$,

$$\int_W f(w)(-Lg(w))\mu(dw)=\int_W \langle Df(w), Dg(w)\rangle_H \mu(w)=\int_W f(w)\cdot D^*Dg(w)\mu(dw).$$

Then (iii) and (iv) are clear from (ii) and (1.7). q.e.d.

Definition 3.2. Let $\boldsymbol{D}_{+\infty}(E)\equiv\cap\{\boldsymbol{D}_{p,r}(E)\,;\ 1<p<\infty,\ r\in \boldsymbol{R}^1\}$

and $$\boldsymbol{D}_{-\infty}(E)\equiv\cup\{\boldsymbol{D}_{p,r}(E)\,;\ 1<p<\infty,\ r\in \boldsymbol{R}^1\}.$$

If $E=\boldsymbol{R}^1$, we denote them symply by $\boldsymbol{D}_{+\infty}$ and $\boldsymbol{D}_{-\infty}$ respectively.

$\boldsymbol{D}_{+\infty}(E)$ is a complete countably normed space and hence $\boldsymbol{D}_{-\infty}(E)$ is its dual space by Theorem 3.1 (i). $\boldsymbol{D}_{+\infty}$ is an algebra by Theorem 3.1 (ii).

As we have seen in Theorem 3.2, D extends to a continuous linear operator mapping $\boldsymbol{D}_{p,r}(E)$ into $\boldsymbol{D}_{p,r-1}(H\otimes E)$, for all $1<p<\infty$ and $r\in \boldsymbol{R}^1$, and since such extentions are consistent, i.e., the diagram

$$\begin{array}{ccc} \boldsymbol{D}_{q,s}(E) & \subsetneq & \boldsymbol{D}_{p,r}(E) \\ \downarrow D & & \downarrow D \\ \boldsymbol{D}_{q,s-1}(H\otimes E) & \subsetneq & \boldsymbol{D}_{p,r-1}(H\otimes E) \end{array}$$

is commutative for any $1<p\leqq q<\infty$ and $r\leqq s$, D is actually well-defined on the whole $\boldsymbol{D}_{-\infty}(E)$ taking value in $\boldsymbol{D}_{-\infty}(H\otimes E)$. Therefore it can be said that when we restrict the domain to $\boldsymbol{D}_{p,r}(E)$, D maps it continuously into $\boldsymbol{D}_{p,r-1}(H\otimes E)$. In particular, it maps $\boldsymbol{D}_{+\infty}(E)$ continuously into $\boldsymbol{D}_{+\infty}(H\otimes E)$. Similarly, the mappings L, C: $\boldsymbol{D}_{-\infty}(E)\to \boldsymbol{D}_{-\infty}(E)$ and D^*: $\boldsymbol{D}_{-\infty}(H\otimes E)\to \boldsymbol{D}_{-\infty}(E)$ are well-defined and the restricted ones L, C: $\boldsymbol{D}_{+\infty}(E)\to \boldsymbol{D}_{+\infty}(E)$ and D^*: $\boldsymbol{D}_{+\infty}(H\otimes E)\to \boldsymbol{D}_{+\infty}(E)$ are continuous.

4. The composition of Schwartz's distributions and Wiener functionals.

As we mentioned before, our Sobolev space $\boldsymbol{D}_{p,r}(E)$ with $r<0$ is no longer a space of Wiener functionals. It might be said to be a space of *generalized Wiener functionals.* Indeed, Watanabe [11] introduced the notion of the composition of Schwartz's distributions and *non-degenerate smooth* Wiener functionals, and as its application, discussed the smoothness of the laws of such functionals. Let us follow Watanabe's method in our framework.

Let $\mathcal{S}=\mathcal{S}(\boldsymbol{R}^d)$, $d\in\boldsymbol{N}$, be the Schwartz space of all rapidly decreasing C^∞-functions on $\boldsymbol{R}^d$, and $\mathcal{S}'=\mathcal{S}'(\boldsymbol{R}^d)$ be its dual space, i.e., the Schwartz space of all tempered distributions on $\boldsymbol{R}^d$. We endow $\mathcal{S}$ with the countable norms;

$$\|\varphi\|_k\equiv\|(1+|x|^2-\Delta)^k\varphi\|_\infty,\quad \varphi\in\mathcal{S},\quad k\in\boldsymbol{Z}, \tag{4.1}$$

where $|x|$ is the length of the vector $x\in\boldsymbol{R}^d$, Δ is the Laplcian on $\boldsymbol{R}^d$ and $\|\ \|_\infty$ is the maximum norm. By $\mathcal{T}_k$, we denote the completion of $\mathcal{S}$ by the norm $\|\ \|_k$. Then we have the following inclusion relation;

$$\mathcal{S}'=\bigcup_{k\in\boldsymbol{Z}}\mathcal{T}_k\supset\cdots\supset\mathcal{T}_{-2}\supset\mathcal{T}_{-1}\supset\mathcal{T}_0=C_\infty(\boldsymbol{R}^d)\supset\mathcal{T}_1\supset\mathcal{T}_2\supset\cdots\supset\bigcap_{k\in\boldsymbol{Z}}\mathcal{T}_k=\mathcal{S} \tag{4.2}$$

Now let $f=(f_1,\cdots,f_d)\in\boldsymbol{D}_{+\infty}(\boldsymbol{R}^d)$ and $\varphi\in\mathcal{S}$. It is clear that the composite function $\varphi(f)\in\boldsymbol{D}_{+\infty}$. Thus, f being *fixed,* we define the following mapping;

$$\Phi_f:\ \mathcal{S}\ni\varphi\longrightarrow\varphi(f)\in\boldsymbol{D}_{+\infty}. \tag{4.3}$$

Finally we put $\sigma_{ij}(w)\equiv\langle Df_i(w), Df_j(w)\rangle_H$, $w\in W$. With these definitions, we are able to state the main theorem of the section obtained by Watanabe [11].

Theorem 4.1. *If (σ_{ij}) is a strictly positive definite matrix for a.a. w (μ) and, setting $(\gamma_{ij})=(\sigma_{ij})^{-1}$, $\gamma_{ij}\in\bigcap_{1<p<\infty}L^p$ holds for $i, j=1,\cdots,d$, then the mapping Φ_f will extend to a unique continuous linear mapping as*

$$\Phi_f:\ \mathcal{T}_{-k}\longrightarrow\boldsymbol{D}_{p,-2k}, \tag{4.4}$$

for all $k\in\boldsymbol{N}$ and $1<p<\infty$. Consequently, it will be well-defined as a mapping $\mathcal{S}'\to\boldsymbol{D}_{-\infty}$.

Definition 4.1. Given $T\in\mathcal{S}'$, $\Phi_f(T)$, which is defined by Theorem 4.1, will be denoted by $T(f)$ and called *the composite of T and f.*

The proof of the theorem is essentially due to Malliavin's formula of integration by parts.

Lemma 4.1. *Let f satisfy the hypotheses of Theorem 4.1. Then, for any $g\in\boldsymbol{D}_{q,k}$, $1<q<\infty$ $k\in\boldsymbol{N}$, and $\partial^\alpha\equiv\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$, $\alpha_1+\cdots+\alpha_d\equiv|\alpha|=k$, there exists an $l^\alpha(g)\in L^r$, $1\leq r<q$, such that*

(4.5) $$\int_W \partial^\alpha \varphi(f)(w)g(w)\mu(dw)=\int_W \varphi(f)(w)l^\alpha(g)(w)\mu(dw)$$

for any $\varphi\in C_b^k(\boldsymbol{R}^d)\equiv\{\varphi\in C^k\,;\ \partial^\beta\varphi$ *is bounded and continuous,* $0\leqq|\beta|\leqq k\}$. *Furthermore, we have*

(4.6) $$\sup\{\|l^\alpha(g)\|_r\,;\ g\in\boldsymbol{D}_{q,k},\ \|g\|_{q,k}\leqq 1\}<+\infty\,.$$

Proof. (i) First, we prove the lemma when $k=1$ and $\partial^\alpha=\partial^i\equiv\partial/\partial x_i$. By the chain-rule, we obtain

$$\langle D(\varphi(f)),\ Df_j\rangle_H=\sum_{i=1}^d(\partial^i\varphi(f))\sigma_{ij}\,.$$

Then,

$$\partial^i\varphi(f)=\sum_{j=1}^d\langle D(\varphi(f)),\ Df_j\rangle_H\gamma_{ij}\,.$$

Hence we see that

$$\begin{aligned}\int_W\partial^i\varphi(f)g\ d\mu&=\sum_{j=1}^d\int_W\langle D(\varphi(f)),\ Df_j\rangle_H\gamma_{ij}g\ d\mu\\&=\sum_{j=1}^d\int_W\langle D(\varphi(f)),\ \gamma_{ij}g\ Df_j\rangle_H d\mu\\&=\sum_{j=1}^d\int_W\varphi(f)D^*(\gamma_{ij}g\ Df_j)d\mu\,.\end{aligned}$$

Compairing with (4.5), we can write down $l^\alpha(g)=l^i(g)$ explicitly as

(4.7) $$l^i(g)=\sum_{j=1}^d D^*(\gamma_{ij}g\ Df_j)\,.$$

Then let us examine if the right hand side belongs to L^r. To this end, we need two more formulas.

(4.8) $$D^*(f^1Df^2)=-\langle Df^1,\ Df^2\rangle_H+f^1Lf^2,\quad f^1,\ f^2\in\boldsymbol{D}_{-\infty}$$

(4.9) $$D\gamma_{ij}=-\sum_{m,n=1}^d\gamma_{im}\gamma_{jn}D\sigma_{mn}$$

The former is quite easy and the latter is found in Ikeda-Watanabe [2]. Now, applying these formulas, we have

$$\begin{aligned}l^i(g)&=-\sum_{j=1}^d\{\langle D(\gamma_{ij}g),\ Df_j\rangle_H+\gamma_{ij}g\ Lf_j\}\\&=-\sum_{j=1}^d\{\langle gD\gamma_{ij},\ Df_j\rangle_H+\langle\gamma_{ij}Dg,\ Df_j\rangle_H+\gamma_{ij}g\ Lf_j\}\\&=-\sum_{j=1}^d\Big\{-\sum_{m,n=1}^d g\gamma_{im}\gamma_{jn}\langle D\sigma_{mn},\ Df_j\rangle_H+\gamma_{ij}\langle Dg,\ Df_j\rangle_H\\&\qquad+\gamma_{ij}g\ Lf_j\Big\}.\end{aligned}$$

Since, $\langle D\sigma_{mn},\ Df_j\rangle_H=\langle D^2f_m,\ Df_n\otimes Df_j\rangle_{H\otimes H}+\langle D^2f_n,\ Df_m\otimes Df_j\rangle_{H\otimes H}$, the following estimate is valid;

$$|\langle D\sigma_{mn}, Df_j\rangle_H| \leqq (|D^2 f_m|_{H\otimes H}|Df_n|_H + |D^2 f_n|_{H\otimes H}|Df_m|_H)|Df_j|_H$$

Therefore,

$$(4.10)\qquad |l^i(g)| \leqq \sum_{j,m,n=1}^{d} |g|\,|\gamma_{im}|\,|\gamma_{jn}|(|D^2 f_m|_{H\otimes H}|Df_n|_H + |D^2 f_n|_{H\otimes H}|Df_m|_H)$$

$$\cdot|Df_j|_H + \sum_{j=1}^{d}(|\gamma_{ij}|\,|Dg|_H|Df_j|_H + |\gamma_{ij}|\,|g|\,|Lf_j|)$$

By the hypotheses, $|\gamma_{ij}|$, $|Df_j|_H$ and $|D^2 f_m|_{H\otimes H}$ belong to $\bigcap_{1<p<\infty} L^p$ and $|g|$, $|Dg|_H$ belong to L^q, consequently we see that $l^i(g)\in L^r$ and (4.6) is clear.

Next, we will show for the general $k\in \boldsymbol{N}$. Let us think the case when $k=2$ and $\partial^\alpha=\partial^i\partial^j$, for example. According to (4.5), we have $l^\alpha(g)=l^j(l^i(g))$ in this case. Then, when we estimate $|l^\alpha(g)|$ as (4.10), the estimation of $|g|$, $|Dg|_H$ and $|D^2 g|_{H\otimes H}$ will be important. But if $g\in \boldsymbol{D}_{q,2}$, all these are elements of L^q. Hence the assertion is valid. Iterating the same procedure, we can verify the lemma for all $k\in\boldsymbol{N}$. q.e.d.

Now that Lemma 4.1 is proved, the proof of Theorem 4.1 is an easy consequence of it.

Proof of Theorem 4.1. Let $q>1$ such that $(1/p)+(1/q)=1$. The preceding lemma implies that for any $k\in\boldsymbol{N}$, $g\in\boldsymbol{D}_{q,2k}$, there exists an $l_k(g)\in L^1$ such that

$$(4.11)\qquad \int_W \{(1+|x|^2-\Delta)^k\varphi(f)\}\,g\,d\mu=\int_W \varphi(f)l_k(g)d\mu\,,$$

for all $\varphi\in C_b^{2k}(\boldsymbol{R}^d)$, and

$$(4.12)\qquad K\equiv\sup\{\|l_k(g)\|_{L^1}\,;\ g\in\boldsymbol{D}_{q,2k},\ \|g\|_{q,2k}\leqq 1\}<+\infty\,.$$

Then, we have

$$\left|\int_W \varphi(f)g\,d\mu\right|=\left|\int_W (1+|x|^2-\Delta)^k\{(1+|x|^2-\Delta)^{-k}\varphi(f)\}\,g\,d\mu\right|$$

$$=\left|\int_W (1+|x|^2-\Delta)^{-k}\varphi(f)l_k(g)d\mu\right|$$

$$\leqq\|(1+|x|^2-\Delta)^{-k}\varphi\|_\infty\|l_k(g)\|_{L^1}\,.$$

Hence, by Theorem 3.1 (i), we conclude that

$$\|\varphi(f)\|_{p,-2k}\leqq K\|\varphi\|_{-k}\,,$$

which completes the proof. q.e.d.

Finally, as an application of Theorem 4.1, we shall prove the following theorem, which was first proved by Malliavin.

Theorem 4.2. *If f satisfies the hypotheses of Theorem* 4.1, *its distribution law, i.e., the induced measure $\mu_* f$ on $\boldsymbol{R}^d$, has a C^∞-density with respect to the Lebesgue measure.*

Proof. Let δ_x be the Dirac δ-function having the mass at $x \in \boldsymbol{R}^d$. The following facts are well-known;

(i) $\delta_x \in \mathcal{T}_{-m}$ if and only if $d/2 < m$,

(ii) the mapping $\boldsymbol{R}^d \ni x \mapsto \delta_x \in \mathcal{T}_{-m}$ is continuous if $d/2 < m$,

(iii) the mapping $\boldsymbol{R}^d \ni x \mapsto \delta_x \in \mathcal{T}_{-m-k}$ is $2k$-times continuously differentiable if $d/2 < m$. $(k \in \boldsymbol{N})$

Consequently, by Theorem 4.1, the mapping $\boldsymbol{R}^d \ni x \mapsto \delta_x(f) \in \boldsymbol{D}_{p,-2m-2k}$ is proved to be $2k$-times continuously differentiable, and further, the mapping $\boldsymbol{R}^d \ni x \mapsto {}_{p,-2m-2k}(\delta_x(f), g)_{q,2m+2k} \in \boldsymbol{R}^1$ is also $2k$-times continuously differentiable, where $(1/p)+(1/q)=1$ and $g \in \boldsymbol{D}_{q,2m+2k}$.

On the other hand, we can easily verify that

$$p(x) \equiv {}_{p,-2m-2k}(\delta_x(f), \mathbf{1})_{q,2m+2k}, \tag{4.13}$$

where $\mathbf{1}$ is the constant equal to 1, is the density of $\mu_* f$ with respect to the Lebesgue measure on $\boldsymbol{R}^d$. Since k is an arbitrary positive integer, we conclude that $p(x) \in C^\infty(\boldsymbol{R}^d)$. q.e.d.

Remark. After I have finished the whole manuscript, I came to know the paper [12] by M. Krée and P. Krée in which they obtained the results related to our Theorem 3.2 and Theorem 3.3.

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