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Excursions in a cone for two-dimensional Brownian motion

By

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1. Introduction and preliminaries.

Let $\{B(t), 0\leqq t<\infty\}$ be the two-dimensional standard Brownian motion process with continuous paths on a probability space $(\Omega, \mathcal{F}, P_x, x\in \boldsymbol{R}^2)$, $P_x(B(0)=x)=1$. We also write P_0 by P. A most significant property of the Brownian paths is known as the *winding property: Let T be a finite Markov time of the two-dimensional Brownian motion process, then with probability one* $\{B(t), T\leqq t<T+\varepsilon\}$ *winds about $B(T)$ and cuts itself for every* $\varepsilon>0$ (Ito and McKean [9], 7.11). In this paper we will consider the contrary. Namely, when does occur a *non-winding* in the two-dimensional Brownian paths? We also determine the *law of a non-winding part* by giving the corresponding conditioned limit theorem for the Brownian motion.

Let $\boldsymbol{u}(\theta)$ be a unit vector $(\cos\theta, \sin\theta)$ in $\boldsymbol{R}^2$. For $0<\alpha<2\pi$ we set a circular cone

$$F=F(\alpha)=\{\boldsymbol{x}=r\boldsymbol{u}(\theta): r\geqq 0, 0\leqq\theta\leqq\alpha\}.$$

In the remainder of this section we consider an F such that the vertical angle is $0<\alpha\leqq\pi$ unless anything other is stated. Then F has the following property:

(1.1) *If $\boldsymbol{x}\in F$, then $F+\boldsymbol{x}\subset F$.*

Noting the continuity of the Brownian paths, we have from (1.1) the following lemma.

Lemma 1. *Let $0<\alpha\leqq\pi$. Then for every $0\leqq s<\infty$ there exists the largest interval among all of the closed ones $[\tau, \upsilon]$ satisfying*

$$(1.2)\quad 0\leqq\tau\leqq s\leqq\upsilon\leqq\infty \quad\text{and}\quad B(t)\in F+B(\tau) \quad \text{for}\quad \tau\leqq t\leqq\upsilon$$
$$(\text{or for } \tau\leqq t<\infty \text{ if } \upsilon=\infty).$$

Let $[\tau(s), \upsilon(s)]$ denote the largest interval, and call it an *F-excursion interval* (*straddling* s) if it does not degenerate to a single point $\{s\}$. We note that the largest interval does not exist in general, when (1.1) does not hold for $\pi<\alpha<2\pi$. Let $\mathcal{E}_F(B)$ be the random set of all F-excursion intervals. Clearly $\{B(t), \tau\leqq t<\upsilon\}$ does not wind about $B(\tau)$, if $[\tau, \upsilon]\in\mathcal{E}_F(B)$. First we consider the following theorem.

Theorem 1. *According as* $\alpha \leqq \pi/2$ *or* $\pi/2 < \alpha (\leqq \pi)$, *we have*

$$P(\mathcal{E}_F(B) \neq \phi) = 0 \quad or \quad 1.$$

In connection with Theorem 1 we recall Greenwood and Perkins [7] and Davis [3], in which they considered the excursions in or above square root boundaries for the one-dimensional Brownian motion. If we restate their problem in terms of the space-time Brownian motion, we see a connection between theirs and ours. In [14] we discussed a manner of non-winding for the halfplane (i.e. $F(\pi)$)-excursions.

In § 2 we prove Theorem 1.

Let ∂F be the boundary of F and set $\mathring{F} = F \backslash \partial F$. Let

$$\sigma_F(B) = \inf\{t > 0 : B(t) \notin \mathring{F}\} (\inf \phi = \infty).$$

In § 3 we consider a *limit theorem of the Brownian motion canditioned to stay in* $\mathring{F}$;

$$P_{\boldsymbol{x}}(B(\,.\,) \in * \mid \sigma_F(B) > 1), \quad \boldsymbol{x} \in \mathring{F}, \quad as \quad \boldsymbol{x} \longrightarrow \boldsymbol{0}$$

when the vertical angle is $0 < \alpha < 2\pi$ (Theorem 2). Especially, if $\pi/2 < \alpha \leqq \pi$, we identify the limit process with the *scaled Brownian F-meander* defined by

$$\tilde{B}(t) = (1-\tau(1))^{-1/2}\{B(\tau(1) + (1-\tau(1))t) - B(\tau(1))\} \tag{1.3}$$

(note that $\tau(1) < 1$ almost surely from (2.2) in § 2). See Theorem 3. In § 4 we show an asymptotic formula for the transition probability of the Brownian motion on $\mathring{F}$ with an absorbing boundary ∂F, which is an extension of a formula given in Spitzer [15] and plays a basic role in §§ 2 and 3.

2. Proof of Theorem 1.

We prove Theorem 1 in a similar way as [7]. But proof given here becomes simpler, because we have Lemma 1, and because we do not use the random walk approximation. We start on defining sequences of renewal epochs of the Brownian motion. Let $\boldsymbol{x}_n = n^{-1}\boldsymbol{u}(\alpha/2)$, $n = 1, 2, \cdots$. For every n we define

$$\sigma_0^{(n)} = 0 \quad \text{and} \quad \sigma_k^{(n)} = \inf\{t > \sigma_{k-1}^{(n)} : B(t) \notin F + B(\sigma_{k-1}^{(n)}) - \boldsymbol{x}_n\}, \quad k = 1, 2, \cdots. \tag{2.1}$$

Clearly $\sigma_k^{(n)} < \infty$ for every k and $\sigma_k^{(n)} \to \infty$ as $k \to \infty$ with probability one. Moreover, as in Lemma 1, we get from (1.1) and from the sample path continuity the following:

Lemma 2. *For a fixed* $[\tau, \upsilon] = [\tau(s), \upsilon(s)]$, $0 \leqq s < \infty$, *let* $j_n = \max\{k : \sigma_k^{(n)} \leqq \tau\}$. *Then we have* $\sigma_{j_n}^{(n)} \leqq \tau \leqq \upsilon \leqq \sigma_{j_n+1}^{(n)}$ *for every* n, $\sigma_{j_n}^{(n)} \to \tau$ *and* $\sigma_{j_n+1}^{(n)} \to \upsilon$ *as* $n \to \infty$.

For $0 < \nu < 1$, $p_\nu(x, y)$ is a probability density on $\boldsymbol{R}^2$ with respect to the Lebesgue measure:

$$p_\nu(x, y)=\begin{cases} \pi^{-1}\nu\sin(\pi\nu)(1-x)^{\nu-1}(x+y)^{-\nu-1} \text{ for } 0<x<1 \text{ and } y>0 \\ 0 \text{ for remaining } x \text{ and } y. \end{cases}$$

We show the following:

(2.2) *If* $\alpha>\pi/2$, *the random vector* $(1-\tau(1), \upsilon(1)-1)$ *has the probability density* $p_{\pi/2\alpha}(x, y)$.

(2.3) *If* $\alpha\leqq\pi/2$, $\quad P(\tau(1)=\upsilon(1)=1)=1$.

Clearly Theorem 1 follows from (2.2) and (2.3). To prove them we write $\sigma_{j_n}^{(n)}=\tau^{(n)}(1)$ and $\sigma_{j_n+1}^{(n)}=\upsilon^{(n)}(1)$, where $[\tau, \upsilon]=[\tau(1), \upsilon(1)]$. Then by Lemma 2

(2.4) $$\tau^{(n)}(1)\longrightarrow\tau(1) \quad\text{and}\quad \upsilon^{(n)}(1)\longrightarrow\upsilon(1) \quad\text{as}\quad n\longrightarrow\infty.$$

Moreover, by the scaling property of the Brownian motion,

(2.5) $$(1-\tau^{(n)}(1), \upsilon^{(n)}(1)-1)\overset{d}{=}(1-n^{-2}\sigma_{\gamma(n^2)}^{(1)}, n^{-2}\sigma_{\gamma(n^2)+1}^{(1)}-1),$$

where $\gamma(a)=\max\{k: \sigma_k^{(1)}\leqq a\}$. Noting (4.3) in §4, we apply some renewal theorem on the right-hand side of (2.5): For $\alpha>\pi/2$ (2.2) follows from Dynkin [4], theorems 3 and 4 (see also Feller [6], XIV 3). We have (2.3) from Rogozin [12], theorem 2 when $\alpha=\pi/2$, and from, e.g., [6] XI, (4.16) when $\alpha<\pi/2$. This completes the proof.

3. Conditioned limit theorem related to an F-excursion.

Let $\{P(s, \cdot; t, *), 0<s\leqq t\leqq 1\}$ be a family of transition probabilities on $\mathring{F}$ defined by

(3.1) $$P(s, \boldsymbol{x}; t, d\boldsymbol{z})=P_{\boldsymbol{x}}(B(t-s)\in d\boldsymbol{z}; \sigma_F(B)>t-s)\frac{P_{\boldsymbol{z}}(\sigma_F(B)>1-t)}{P_{\boldsymbol{x}}(\sigma_F(B)>1-s)}$$

for $\boldsymbol{x}$ and $\boldsymbol{z}$ in $\mathring{F}$. Let $|\ .\ |$ denote the Euclidian norm in $\boldsymbol{R}^2$. We set $\mu=\pi/\alpha$. Let $\{\rho(t, *), 0<t\leqq 1\}$ be a family of measures on $\boldsymbol{R}^2$ defined by

(3.2) $$\rho(t, d\boldsymbol{x})=\frac{|\boldsymbol{x}|^{\mu}}{2^{\mu/2}\Gamma(\mu/2)t^{1/2+\mu}}\exp(-|\boldsymbol{x}|^2/2t)\sin(\mu\theta) \times P_{\boldsymbol{x}}(\sigma_F(B)>1-t)d\boldsymbol{x},\quad \boldsymbol{x}=|\boldsymbol{x}|u(\theta)\in\mathring{F}.$$

Then $\{\rho(t, *), 0<t\leqq 1\}$ *is a probability entrance law for* $\{P(s, \cdot; t, *), 0<s\leqq t\leqq 1\}$, *that is, a family of probability measures on* $\mathring{F}$ *satisfying*

(3.3) $$\rho(t, *)=\int_{\mathring{F}}\rho(s, d\boldsymbol{x})P(s, \boldsymbol{x}; t, *)\qquad \textit{for every}\quad 0<s\leqq t\leqq 1.$$

By (4.1) and (4.2) in §4 we have

$$P_{\boldsymbol{x}}(B(s)\in *\,|\,\sigma_F(B)>1)\overset{d}{\longrightarrow}\rho(s, *)\qquad\text{in } \mathring{F} \text{ as }\quad \mathring{F}\ni\boldsymbol{x}\longrightarrow\boldsymbol{0}.$$

Let f be a bounded continuous function on $\mathring{F}$. Then

$$\int_{\mathring{F}} P_x(B(t)\in d\boldsymbol{z} \mid \sigma_F(B)>1)f(\boldsymbol{z})$$
(3.4)
$$=\int_{\mathring{F}} P_x(B(s)\in d\boldsymbol{y} \mid \sigma_F(B)>1)\int_{\mathring{F}} P(s, \boldsymbol{y};\ t, dz)f(z)$$

for every $\boldsymbol{x}$ in $\mathring{F}$, and $\int_{\mathring{F}} P(s, \cdot\,;\ t, dz)f(z)$ is easily shown to be a bounded continuous function on $\mathring{F}$ too. Then (3.3) follows by letting $\boldsymbol{x}\to\boldsymbol{0}$ in (3.4).

On the space of continuous functions $\boldsymbol{C}[a, b]=\boldsymbol{C}([a, b]\to\boldsymbol{R}^2)$, $0\leqq a\leqq b<\infty$, we define the topology of the uniform convergence, and give the Borel field $\mathcal{C}[a, b]$. The main result of this section is the following theorems.

Theorem 2. *Let $0<\alpha<2\pi$. A family of conditional probabilities on* $(\boldsymbol{C}[0, 1], \mathcal{C}[0, 1])$

(3.5) $$P_x(B(\,.\,)\in * \mid \sigma_F(B)>1),\ \boldsymbol{x}\in\mathring{F} \quad \textit{with} \quad |\boldsymbol{x}|\leqq 1$$

converges weakly in $\boldsymbol{C}[0, 1]$ as $\boldsymbol{x}\to\boldsymbol{0}$. The finite dimensional distribution of the limit law $\widetilde{W}$ is given as follows:

(3.6)
$$\widetilde{W}(w\in\boldsymbol{C}[0, 1]:\ w(t_1)\in d\boldsymbol{x}_1;\ w(t_2)\in d\boldsymbol{x}_2;\ \cdots;\ w(t_n)\in d\boldsymbol{x}_n)$$
$$=\rho(t_1, d\boldsymbol{x}_1)P(t_1, \boldsymbol{x}_1;\ t_2, d\boldsymbol{x}_2)\cdots P(t_{n-1}, \boldsymbol{x}_{n-1};\ t_n, d\boldsymbol{x}_n),$$

for every $0<t_1<t_2<\cdots<t_n\leqq 1$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ in $\mathring{F}$.

Combining Theorems 1 and 2, we get the following theorem.

Theorem 3. *If $\pi/2<\alpha\leqq\pi$, we have $\widetilde{W}(A)=P(\widetilde{B}(\,.\,)\in A)$ for all A in $\mathcal{C}[0, 1]$.*

Proof of Theorem 2. Firstly we consider the convergence of f.d.d's with (3.6), that is, as $\boldsymbol{x}\to\boldsymbol{0}$

$$P_x(B(t_1)\in d\boldsymbol{x}_1;\ B(t_2)\in d\boldsymbol{x}_2;\ \cdots;\ B(t_n)\in d\boldsymbol{x}_n \mid \sigma_F(B)>1) \xrightarrow{d}$$
$$\rho(t_1, d\boldsymbol{x}_1)P(t_1, \boldsymbol{x}_1;\ t_2, d\boldsymbol{x}_2)\cdots P(t_{n-1}, \boldsymbol{x}_{n-1};\ t_n, d\boldsymbol{x}) \text{ in } \mathring{F}^n.$$

This can be proved in a same way as (3.3) given above. Hence the remaining task is to show the tightness of (3.5) in $\boldsymbol{C}[0, 1]$. To do this we introduce the modulus of continuity of $w\in\boldsymbol{C}[a, b]$

$$\omega_w(\delta;\ a, b)=\sup\{|w(t)-w(s)|:\ a\leqq s<t\leqq b,\ t-s<\delta\},\quad 0<\delta\leqq b-a,$$

and it is enough to show the following two conditions: *For each positive η, there exists a λ such that*

(3.7) $$P_x(|B(0)|>\lambda \mid \sigma_F(B)>1)\leqq\eta \qquad \textit{for all} \quad \boldsymbol{x}\in\mathring{F} \textit{ with } |\boldsymbol{x}|\leqq 1.$$

For each positive ε,

(3.8) $$\lim_{\delta\to 0+}\limsup_{\mathring{F}\ni x\to 0} P_x(\omega_B(\delta;\ 0, 1)>\varepsilon \mid \sigma_F(B)>1)=0$$

(Billingsley [1], theorem 8.2).

Clearly we have (3.7). Let us consider (3.8). Let D_ε be the closed disc of radius ε with the centre $\mathbf{0}$. Since

$$\omega_w(\delta;\ 0,\ 1) \leqq 2\varepsilon + \omega_w(\delta;\ \min\{\sigma_{D_\varepsilon}(w),\ 1\},\ 1) \qquad \text{for} \quad w \in \boldsymbol{C}[0,\ 1],$$

$P_{\boldsymbol{x}}(\omega_B(\delta;\ 0,\ 1) > 3\varepsilon \mid \sigma_F(B) > 1)$, $\boldsymbol{x} \in \mathring{F} \cap D_\varepsilon$, is bounded by

$$P_{\boldsymbol{x}}(\omega_B(\delta;\ \min\{\sigma_{D_\varepsilon}(B),\ 1\},\ 1) > \varepsilon \mid \sigma_F(B) > 1). \tag{3.9}$$

By the strong Markov property, (3.9) is equal to

$$\begin{aligned} &E_{\boldsymbol{x}}[1(\sigma_{D_\varepsilon}(B) < 1;\ \sigma_F(B) > \sigma_{D_\varepsilon}(B)) P_{B(\sigma_{D_\varepsilon})}(\sigma_F(B) > 1-s)_{|s=\sigma_{D_\varepsilon}(B)} \\ &\times P_{B(\sigma_{D_\varepsilon})}(\omega_B(\delta;\ 0,\ 1-s) > \varepsilon \mid \sigma_F(B) > 1-s)_{|s=\sigma_{D_\varepsilon}(B)}] / P_{\boldsymbol{x}}(\sigma_F(B) > 1), \end{aligned} \tag{3.10}$$

where $1(A)$ is a indicator function on a set $A \subseteqq \Omega$. If we have

$$K_\varepsilon(\delta) = \sup\{P_{\boldsymbol{x}}(\omega_B(\delta;\ 0,\ 1) > \varepsilon \mid \sigma_F(B) > t) :\ 0 < t \leqq 1,\ \boldsymbol{x} \in \mathring{F} \cap \partial D_\varepsilon\} \longrightarrow 0 \tag{3.11}$$

as $\delta \to 0$ for every $\varepsilon > 0$, then (3.8) follows from (3.9) and (3.10).

For (3.11) it is enough to prove the tightness, and hence the relative compactness of a family of probability measures

$$\mathcal{P}_\varepsilon = \{W_{\boldsymbol{x},t}(*) = P_{\boldsymbol{x}}(B(\ .\ ) \in * \mid \sigma_F(B) > t) :\ 0 < t \leqq 1,\ \boldsymbol{x} \in \mathring{F} \cap \partial D_\varepsilon\}$$

on $\boldsymbol{C}[0,\ 1]$ ([1], theorem 6.2). Clearly $W_{\boldsymbol{x},t} \overset{w}{\to} W_{\boldsymbol{x}_0, t_0}$ in $\boldsymbol{C}[0,\ 1]$ if $\boldsymbol{x} \to \boldsymbol{x}_0 \in \mathring{F}$ and $t \to t_0 \geqq 0$. Therefore, if we have

(3.12) $W_{\boldsymbol{x},t}$ *converges weakly in* $\boldsymbol{C}[0,\ 1]$ *as* $\boldsymbol{x} \to \boldsymbol{x}_0 \in \partial F \setminus \{\mathbf{0}\}$ *and* $t \to t_0 \geqq 0$,

the relative compactness of $\mathcal{P}_\varepsilon$ follows.

Now we formulate (3.12) in a following slightly general form.

Lemma 3. *Let S be a closed domain in $\boldsymbol{R}^2$ which has the piecewise smooth boundary ∂S and satisfies*

$$S \cap D_a = F(\pi) \cap D_a \tag{3.13}$$

for some $a > 0$. Then conditional probability $P_{\boldsymbol{x}}(B(\ .\ ) \in * \mid \sigma_S(B) > t)$ $0 < t \leqq 1$, $\boldsymbol{x} \in \mathring{S}$, converges weakly in $\boldsymbol{C}[0,\ 1]$ as $\boldsymbol{x} \to \mathbf{0}$ and $t \to c \geqq 0$.

Here we only mention an outline of the proof. We first remark on *S-excursion intervals*. Note that we do not assume (1.1) but (3.13) for S. Then for every Brownian path, we can determine a sequence of S-excursion intervals by modifying the $F(\pi)$-excursion intervals in an obvious way. Let $\mathcal{E}_S(B)$ denote the random set of all S-excursion intervals.

Let $c > 0$. Let $[\hat{\tau}_c,\ \hat{\upsilon}_c]$ be the first element of $\mathcal{E}_S(B)$ satisfying $\upsilon - \tau > c$, and set

$$\hat{B}_c(s) = B(\hat{\tau}_c + s) - B(\hat{\tau}_c), \qquad 0 \leqq s \leqq 1.$$

Then, in a similar way as Bolthausen [2] or Shimura [13], we get

$$P_x(B(\,.\,)\in *\,|\,\sigma_S(B)>t) \longrightarrow P(\hat{B}_c(\,.\,)\in *) \text{ as } \boldsymbol{x}\longrightarrow \boldsymbol{0} \text{ and } t\longrightarrow c$$

weakly in $\boldsymbol{C}[0, 1]$.

Let $c=0$. Noting $\hat{\tau}_c\to 0$ as $c\to 0+$, we have

$$P_x(B(\,.\,)\in *\,|\,\sigma_S(B)>t) \longrightarrow P(B(\,.\,)\in *) \text{ as } \boldsymbol{x}\longrightarrow \boldsymbol{0} \text{ and } t\longrightarrow 0$$

weakly in $\boldsymbol{C}[0, 1]$. Hence we conclude the lemma.

Proof of Theorem 3. Let $\tau^{(n)}(1)$ be those defined in § 2, and set

$$\tilde{B}^{(n)}(t)=(1-\tau^{(n)}(1))^{-1/2}\{B(\tau^{(n)}(1)+(1-\tau^{(n)}(1))t)-B(\tau^{n)}(1))\}.$$

By Lemma 2 and Theorem 1 we have

$$\lim_{n\to\infty} E[1(a<\tau^{(n)}(1)<b)g(\tilde{B}^{(n)}(\,.\,))]=E[1(a<\tau(1)<b)g(\tilde{B}(\,.\,))]$$

for every $0<a<b<1$ and bounded continuous function g on $\boldsymbol{C}[0, 1]$. By the scaling property of the Brownian motion, we show that $E[1(a<\tau^{(n)}(1)<b)$ $g(\tilde{B}^{(n)}(\,.\,))]$ is equal to

$$\int_a^b P(\tau^{(n)}(1)\in ds)E_{n^{-1}(1-s)^{-1/2}u(\alpha/2)}[g(B(\,.\,))\,|\,\sigma_F(B)>1]$$

Hence by Theorem 2 we have

$$E[1(a<\tau(1)<b)g(\tilde{B}(\,.\,))]=P(a<\tau(1)<b)\int_{C[0,1]} g(w)\widetilde{W}(dw). \tag{3.14}$$

Letting $a\downarrow 0$ and $b\uparrow 1$ in (3.14), we conclude the theorem.

4. Extension of Spitzer's formula.

In the previous sections 2 and 3 we used the following asymptotic formulas for the transition probability of an absorbing Brownian motion on $\mathring{F}=\mathring{F}(\alpha)$, $0<\alpha<2\pi$. *For every bounded continuous function f on $\mathring{F}$ and $s>0$, we have*

$$\lim_{r\to 0+}\sup_{0<\theta<\alpha} |\{r^{\mu}\sin(\mu\theta)\}^{-1}E_{ru(\theta)}[f(B(s))1(\sigma_F(B)>s)] - \frac{2^{1-\mu}}{\pi\Gamma(\mu)s^{1/2+\mu}}\int_{\mathring{F}} f(z)\exp(-|z|^2/2s)|z|^{\mu}\sin(\mu\zeta)dz|=0, \tag{4.1}$$

where $\mu=\pi/\alpha$ and $z=|z|u(\zeta)$. Especially, if we set $f=1$, we have

$$\lim_{r\to 0+}\sup_{0<\theta<\alpha} |\{r^{\mu}\sin(\mu\theta)\}^{-1}P_{ru(\theta)}(\sigma_F(B)>s)-\frac{2^{2-3\mu/2}}{\pi^{1/2}\Gamma(1/2+\mu/2)s^{\mu/2}}|=0. \tag{4.2}$$

Combining the scaling property of the Brownian motion with (4.5) below, we get

$$\lim_{t\to\infty} t^{\mu/2}P_x(\sigma_F(B)>t)=\frac{2^{2-3\mu/2}}{\pi^{1/2}\Gamma(1/2+\mu/2)}|\boldsymbol{x}|^{\mu}\sin(\mu\theta) \tag{4.3}$$

for every $\boldsymbol{x}=|\boldsymbol{x}|\boldsymbol{u}(\theta)$ in $\mathring{F}$.

In [15] Spitzer obtained an explicit form of an integral transform of $u(r, \theta, t)=P_{ru(\theta)}(\sigma_F(B)>t)$ (formula (2.6) in [15]). Formula (4.3) will be given from it through some Tauberian argument. But it seems difficult to extend his formula to get (4.1). Here we will prove (4.1), applying the Hilbert-Schmidt expansion for transition probabilities.

Consider the two-dimensional Ornstein-Uhlenbeck process

$$U(t)=e^{-t/2}B(e^t-1), \qquad t\geqq 0 \tag{4.4}$$

on the probability space $(\Omega, \mathfrak{F}, P_x, x\in \boldsymbol{R}^2)$, which is a diffusion process on $\boldsymbol{R}^2$ with the generator

$$\mathcal{G}_U=(1/2)[\partial^2/\partial x^2+\partial^2/\partial y^2-x\partial/\partial x-y\partial/\partial y]$$

(refer, e. g., Knight [10], pp 97-98). Instead of showing (4.1) directly, we derive a corresponding formula for the transition probability of an absorbing Ornstein-Uhlenbeck process

$$\dot{Q}(t;\ \boldsymbol{x},\ d\boldsymbol{z})=P_{\boldsymbol{x}}(\sigma_F(U)>t;\ U(t)\in d\boldsymbol{z}),\ \boldsymbol{x},\ \boldsymbol{z}\in \mathring{F},$$

noting the relation (4.4) and the following property on F:

$$cF=\{c\boldsymbol{x}:\ \boldsymbol{x}\in F\}=F \qquad \textit{for every} \quad c>0. \tag{4.5}$$

Such the translation will simplify the derivation, because the absorbing Ornstein-Uhlenbeck process has only the discrete spectrum as given below.

For every $\nu\geqq 0$ let $L_j^{(\nu)}(x)$ be the generalized Laguerre polynomials defined by

$$L_j^{(\nu)}(x)=\sum_{k=0}^{j}\binom{j+\nu}{j-k}(-x)^k, \qquad j=0,\ 1,\ 2,\ \cdots.$$

We set

$$\lambda_j(\nu)=j+\nu/2, \qquad C_j(\nu)=j!\ \mu 2^{1-\nu}/\{\pi\Gamma(j+\nu+1)\}$$

and

$$\Phi_j(\boldsymbol{x};\ \nu)=r^{\nu}L_j^{(\nu)}(r^2/2)\sin(\nu\theta), \qquad \boldsymbol{x}=r\boldsymbol{u}(\theta).$$

Then we get the following lemma.

Lemma 4. *With respect to the measure* $N(d\boldsymbol{z})=\exp(-|\boldsymbol{z}|^2/2)d\boldsymbol{z}$, $\dot{Q}(t;\boldsymbol{x},d\boldsymbol{z})$ *for* $t>0$ *has a density* $\dot{Q}(t;\ \boldsymbol{x},\ \boldsymbol{z})$ *which is given by a series convegring uniformly on every bounded set in* $\mathring{F}$:

$$\dot{Q}(t;\ \boldsymbol{x},\ \boldsymbol{z})=\Sigma'_{(n,j)}C_j(n\mu)\exp\{-\lambda_j(n\mu)t\}\,\Phi_j(\boldsymbol{x};\ n\mu)\Phi_j(\boldsymbol{z};\ n\mu), \tag{4.6}$$

where the summation $\Sigma'_{(n,j)}$ *is taken over all pairs* $(n,\ j)$, $n=1,\ 2,\ \cdots,\ j=0,\ 1,\ 2,\ \cdots$ *such the order that* $\lambda_j(n\mu)$ *is nondecreasing.*

As a consequence of Lemma 4 we have the following:

Corollary to Lemma 4. *Let* f *be a bounded measurable function on* $\mathring{F}$. *Then for every* $r>0$ *we have*

$$\lim_{t\to\infty}\sup_{0<\theta<\alpha}\left|\frac{\exp(\mu t/2)}{\sin(\mu\theta)}\int_{\mathring{F}}\dot{Q}(t;\ r\boldsymbol{u}(\theta),\ dz)f(z)\right. \tag{4.7}$$
$$\left.-\frac{2^{1-\mu}r^{\mu}}{\pi\Gamma(\mu)}\int_{\mathring{F}}f(z)\exp(-|z|^2/2)|z|^{\mu}\sin(\mu\zeta)dz\right|=0,$$

where $z=|z|\boldsymbol{u}(\zeta)$.

By (4.4), (4.5) and by the scaling property of the Brownian motion, $r^{-\mu}E_{r\boldsymbol{x}}[f(B(s))1(\sigma_F(B)>s)]$ is equal to

$$(\mathrm{e}^t-1)^{\mu/2}\int_{\mathring{F}}\dot{Q}(t;\ s^{-1/2}\boldsymbol{x},\ dz)f\left(\left(\frac{s}{1-\mathrm{e}^{-t}}\right)^{1/2}z\right),$$

where $r=(\mathrm{e}^t-1)^{-1/2}$. Then (4.1) is an easy consequence of (4.7).

Proof of Lemma 4. Note that the transition probability of the Ornstein Uhlenbeck process

$$Q(t;\ \boldsymbol{x},\ dz)=P_{\boldsymbol{x}}(U(t)\in dz),\ \boldsymbol{x},\ z\in\boldsymbol{R}^2$$

has the jointly continuous and symmetric density

$$Q(t;\ \boldsymbol{x},\ z)=\{2\pi(1-\mathrm{e}^{-t})\}^{-1}\exp\left\{-\frac{\mathrm{e}^{-t}(|\boldsymbol{x}|^2+|z|^2)-2\mathrm{e}^{-t/2}(\boldsymbol{x},\ z)}{2(1-\mathrm{e}^{-t})}\right\} \tag{4.8}$$

with respect to $N(dz)$ for every $t>0$, where $(\boldsymbol{x},\ z)$ denotes the inner product.

Firstly we show the following assertion: *With respect to* $N(dz)$, $\dot{Q}(t;\boldsymbol{x},dz)$ *for* $t>0$ *has a density* $\dot{Q}(t;\ \boldsymbol{x},\ z)$ *which is jointly continuous and symmetric in* $\boldsymbol{x}$ *and* z *in* $\mathring{F}$. *For every* $\boldsymbol{x}_0\in\partial F$ *and* $z\in\mathring{F}$

$$\dot{Q}(t;\ \boldsymbol{x},\ z)\longrightarrow 0\quad as\quad \boldsymbol{x}\longrightarrow\boldsymbol{x}_0. \tag{4.9}$$

To prove this we introduce a two-dimensional pinned Brownian motion starting at $\boldsymbol{x}$ and ending at time t at z;

$$B_{\boldsymbol{x}}^{t,z}(s)=\boldsymbol{x}+(s/t)(z-\boldsymbol{x})+(s-t)\int_0^s(u-t)^{-1}dB(u),\quad 0\leqq s<t \tag{4.10}$$

on the probability space $(\Omega,\ \mathcal{F},\ P)$ (Ikeda and Watanabe [8], p 229). Set

$$M(t;\ \boldsymbol{x},\ z)=P(\sigma_F(B_{\boldsymbol{x}}^{\mathrm{e}^t-1,\mathrm{e}^{t/2}z})>\mathrm{e}^t-1).$$

By (4.4) and (4.5) the desired transition density is given by

$$\dot{Q}(t;\ \boldsymbol{x},\ z)=M(t;\ \boldsymbol{x},\ z)Q(t;\ \boldsymbol{x},\ z)\quad\text{for } t>0 \text{ and } \boldsymbol{x},\ z \text{ in } \mathring{F}. \tag{4.11}$$

Indeed, the joint continuity of $\dot{Q}(t;\ \cdot,\ \cdot)$ and (4.9) follows from the expression (4.10). The symmetry of $\dot{Q}(t;\ \cdot,\ \cdot)$ is shown by a modification of Dynkin [5], lemma 14.1 (see also footnote in subsection 4.20).

Let

$$\dot{Q}_tf(\boldsymbol{x})=\int_{\mathring{F}}\dot{Q}(t;\ \boldsymbol{x},\ dz)f(z)\quad\text{for } f\in L^2(\mathring{F},\ N),\ t\geqq 0 \text{ and } \boldsymbol{x}\in\mathring{F}.$$

Secondly we show that $\dot{Q}_t$ *maps* $L^2(\mathring{F},\ N)$ *into* $L^2(\mathring{F},\ N)\cap\{f\in\boldsymbol{C}(\mathring{F}\to\boldsymbol{R}):f(\boldsymbol{x})\to 0$ *as* $\boldsymbol{x}\to\boldsymbol{x}_0$ *for every* $\boldsymbol{x}_0$ *in* $\partial F\}$.

By the symmetry of $\dot{Q}(t;\ .,\ .)$ and by (4.11)

$$\int_{\mathring{F}}\dot{Q}(t;\ \boldsymbol{x},\ \boldsymbol{z})^2 N(d\boldsymbol{z})=\dot{Q}(2t;\ \boldsymbol{x},\ \boldsymbol{x})<Q(2t;\ \boldsymbol{x},\ \boldsymbol{x}).$$

Therefore by (4.8)

$$\int_{\mathring{F}}\int_{\mathring{F}}\dot{Q}(t;\ \boldsymbol{x},\ \boldsymbol{z})^2 N(d\boldsymbol{z})N(d\boldsymbol{x})\leqq\int_{\mathring{F}}Q(2t;\ \boldsymbol{x},\ \boldsymbol{x})N(d\boldsymbol{x})<\infty.$$

Then we conclude the second assertion from the first one.

Thirdly we consider the eigenvalue problem

$$\dot{Q}_t\Phi(\boldsymbol{x})=\exp(-\lambda t)\Phi(\boldsymbol{x}),\qquad \Phi\in L^2(\mathring{F},\ N)$$

for $t>0$. The eigenvalues and the corresponding eigenfunctions are given by

$$\exp\{-\lambda_j(n\mu)t\}\quad\text{and}\quad \Phi_j(\boldsymbol{x};\ n\mu),\ n=1,\ 2,\ \cdots,\ j=0,\ 1,\ 2,\ \cdots.$$

Since $\{\sin(n\mu\theta),\ n=1,\ 2,\cdots\}$ forms a complete orthogonal system for $L^2((0,\ \alpha),\ d\theta)$, and since so does $\{r^{n\mu}L_j^{(n\mu)}(r^2/2),\ j=0,\ 1,\ 2,\ \cdots\}$ for $L^2((0,\ \infty),\ r\exp(-r^2/2)dr)$ for every $n=1,\ 2,\ \cdots$ (Szegö [16], 5.7), $\{\Phi_j(\boldsymbol{x};\ n\mu),\ n=1,\ 2,\ \cdots,\ j=0,\ 1,\ 2,\ \cdots\}$ is shown to be a complete orthogonal system for $L^2(\mathring{F},\ N)$. Then by a small modification of Marcer's theorem (Riesz and Sz.-Nagy [11], § 98), we conclude the lemma.

Proof of Corollary to Lemma 4. We set

$$K(t;\ \boldsymbol{x},\ \boldsymbol{z})=\exp\{\lambda_0(\mu)t\}\,Q(t;\ \boldsymbol{x},\ \boldsymbol{z})-C_0(\mu)\Phi_0(\boldsymbol{x};\ \mu)\Phi_0(\boldsymbol{z};\ \mu).$$

In a similar way as in Uchiyama [17], p 80, we have from Lemma 4

$$K(t;\ \boldsymbol{x},\ \boldsymbol{z})\leqq\dot{Q}(\delta t;\ \boldsymbol{x},\ \boldsymbol{x})^{1/2}\dot{Q}(\delta t;\ \boldsymbol{z},\ \boldsymbol{z})^{1/2}\qquad\text{for all}\quad t>0,\ \boldsymbol{x},\ \boldsymbol{z}\in\mathring{F},$$

where $\delta=\min\{2/(\mu+2),\ 1/2\}$. Moreover we have from Lemma 4

$$\dot{Q}(t;\ \boldsymbol{x},\ \boldsymbol{x})\leqq\exp\{-\lambda_0(\mu)(t-1)\}\,\dot{Q}(1;\ \boldsymbol{x},\ \boldsymbol{x})$$

and

$$\int_{\mathring{F}}\dot{Q}(t;\ \boldsymbol{z},\ \boldsymbol{z})^{1/2}N(d\boldsymbol{z})=O(\exp(-\mu t/2))\quad\text{as}\quad t\longrightarrow\infty.$$

Therefore we conclude (4.7), if, in addition to the above three estimates, we have the following

$$\text{(4.12)}\qquad \dot{Q}(1;\ \boldsymbol{x},\ \boldsymbol{x})\leqq c_r\sin^2(\mu\theta)\qquad\text{for}\quad \boldsymbol{x}=r\boldsymbol{u}(\theta)\ \text{in}\ \mathring{F}.$$

where c_r is a constant which depends only on r.

For $\nu\geqq0$ let

$$q^{(\nu)}(t;\ r,\ s)=\sum_{j=0}^{\infty}d_j(\nu)\exp\{-\lambda_j(\nu)t\}\,r^{\nu}L_j^{(\nu)}(r^2/2)s^{\nu}L_j^{(\nu)}(s^2/2)$$

for positive t, r and s, where $d_j(\nu)=(\alpha/2)C_j(\nu)$. Note that this formula defines the transition density with respect to the measure $s\exp(-s^2/2)ds$ of the diffusion process on $(0,\ \infty)$ whose generator is given by

$$(1/2)\left\{\frac{d^2}{dr^2}+(r^{-1}-r)\frac{d}{dr}-(\nu r^{-1})^2\right\}.$$

Since the series (4.6) also converges absolutely, we can rewrite it as follows:

$$\dot{Q}(t;\ \boldsymbol{x},\ \boldsymbol{z})=\sum_{n=1}^{\infty}q^{(n\mu)}(t;\ r,\ s)\sin(n\mu\theta)\sin(n\mu\zeta) \tag{4.13}$$

for $\boldsymbol{x}=r\boldsymbol{u}(\theta)$ and $\boldsymbol{z}=s\boldsymbol{u}(\zeta)$ in $\mathring{F}$. By Schwarz' inequality and by an obvious inequality $q^{(\nu)}(t;\ r,\ r)\leqq q^{(0)}(t;\ r,\ r)$,

$$q^{(n\mu)}(1;\ r,\ r)\leqq\exp(-n\mu/4)q^{(0)}(1/2;\ r,\ r).$$

Hence by (4.13)

$$\dot{Q}(1;\ \boldsymbol{x},\ \boldsymbol{x})\leqq q^{(0)}(1/2;\ r,\ r)\sum_{n=1}^{\infty}\exp(-n\mu/4)\sin^2(n\mu\theta),$$

from which we get (4.12) easily. This completes the proof.

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Note added in proof. After submitting the paper the author learned that Krzysztof Burdzy proved in a different way a similar result to Theorem 1 for $d(\geqq 2)$-dimensional Brownian motion in his paper:

Brownian paths and cones, to appear in Ann. Probab.

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