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A simple extension of a von Neumann regular ring

By

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In this note all rings are commutative rings with units. Let R be a (von Neumann) regular ring (i.e. an absolutely flat ring in Bourbaki's sense) and let $R[\alpha]$ be a reduced simple extension of R. Let $R[x]$ be a polynomial ring of a variable x over R, I the kernel of the canonical homomorphism T of $R[x]$ onto $R[\alpha]$ such that $T(x)=\alpha$.

In this note, first we shall give some conditions on I for $R[\alpha]$ to be quasi-regular. (Following [3], we say that a ring is quasi-regular if its total quotient ring is regular.) And then we shall give some results relating to the dondition (F) in [1]. Throughout this paper, we use the above notation. In particular, I is a semi-prime ideal of $R[x]$ with $I \cap R=(0)$.

We begin with an easy lemma.

Lemma 1. *Every maximal ideal of* $R[x]$ *is of height* 1 *and contains unique minimal prime ideal of the form* $\mathfrak{m}R[x]$ *where* $\mathfrak{m}$ *is a maximal ideal of* R.

The proof is easy and we omit it.

Corollary 2. *Every localization of* $R[\alpha]$ *at a maximal ideal is an integral domain.*

Proposition 3. $R[\alpha]$ *is quasi-regular if and only if* $R[\alpha]$ *is a p.p. ring, that is, every principal ideal of* $R[\alpha]$ *is projective as an* $R[\alpha]$*-module.*

Proof. This follows from Corollary 2 and [2].

Remark 4. In the proof of Proposition 1 in [2], it is shown that if a ring R' is a p.p. ring, then every idempotent in $Q(R')$ (=total quotient ring of R') is contained in R'.

For an $f(x) \in R[x]$, we denote by $c(f)$ the ideal of R generated by coefficients of $f(x)$. For an ideal J of $R[x]$ we denote by $C(J)$ the ideal of R generated by $\{c(f) \mid f(x) \in J\}$.

Theorem 5. *Assume that* $C(I)=R$. *Then* $R[\alpha]$ *is regular and is integral over*

R. Therefore the pair $(R, R[\alpha])$ *satisfies the condition* (*F*) *in* [1], *that is, for any ring* R'' *between* R *and* $R[\alpha]$, $R[\alpha]$ *is* R''*-flat.*

Proof. The regularity of $R[\alpha]$ is clear. Since $R[\alpha]$ is regular ,there is an idempotent e in $R[\alpha]$ such that $\alpha+e$ is not a zero-divisor, hence, a unit in $R[\alpha]$. Since $R[e]$ is regular and integral over R and since $R[\alpha]=R[e][\alpha+e]$, replacing R and α with $R[e]$ and $\alpha+e$, respectively, we may assume that α is a unit in $R[\alpha]$. Then it is well-known that α is integal over R (cf. [4]).

The latter part of the theorem follows from Proposition 3.1 in [1].

Theorem 6. *If* $C(I)$ *is principal, then* $R[\alpha]$ *is quasi-regular.*

Proof. Let $C(I)=eR$ with $e^2=e\in R$. By Theorem 5, we may assume $e\neq 1$. Then there is an $f(x)\in I$ such that $c(f)=eR$. It is clear that we can write $f(x)=e\bar{f}(x)$ with $c(\bar{f})=R$. Let $\mathfrak{M}$ be a maximal ideal of $R[x]$ containing $C(I)$. Then it is sufficient to show that $\mathfrak{M}$ contains non-zero-divisor modulo I (recall that we assume that $R[\alpha]$ is reduced). Since $\mathfrak{M}$ is maximal, there is a $g(x)$ in $\mathfrak{M}$ such $c(g)=R$. Then $h(x)=\bar{f}(x)g(x)+e$ is contained in $\mathfrak{M}$. We need to prove that $h(x)$ is not a zero-divisor modulo I. Let $k(x)$ be a polynomial such that $h(x)k(x)\in I$. Then

$$\bar{f}(x)g(x)k(x)+ek(x)\in I\subseteqq C(I)R[x]=eR[x],$$

whence $\bar{f}(x)g(x)k(x)\in eR[x]$, that is, $c(\bar{f}gk)\subseteqq eR$. Since $c(\bar{f})=c(g)=R$, we have $c(k)\subseteqq eR$. Hence we can write $k(x)=e\bar{k}(x)$ with $\bar{k}(x)\in R[x]$. Then

$$h(x)k(x)=eh(x)\bar{k}(x)=f(x)g(x)\bar{k}(x)+e^2\bar{k}(x)$$

is in I. Since $f(x)\in I$, $e^2\bar{k}(x)=k(x)$ is also contained in I which implies that $h(x)$ is not a zero-divisor modulo I.

Corollary 7. *If* I *is finitely generated then* $R[\alpha]$ *is quasi-regular.*

If $R[\alpha]$ is quasi-regular, every idempotent in $Q(R[\alpha])$ is contained in $R[\alpha]$ as stated in Remark 4. In addition, if I is generated by linear polynomials, that is, polynomials of degree 1, then we have the following result.

Proposition 8. *Assume that* $R[\alpha]$ *is quasi-regular. If* I *is generated by linear polynomials, then every idempotent of* $Q(R[\alpha])$ *is contained in* R.

To prove the proposition, we need the following lemma.

Lemma 9. *Let* $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$ *be a polynomial in* $R[x]$ *such that* $f(\alpha)$ *is integral over* R. *Then* $a_i\in C(I)$ *and* $a_i\alpha^r$ *is integral over* R *for every* i *with* $0\leq i\leq n-1$ *and every integer* $r\geq 0$.

Proof. Since $f(\alpha)$ is integral over R, there are an integer m and $b_j\in R$ such that

$$\begin{aligned}0&=f(\alpha)^m+b_1f(\alpha)^{m-1}+\cdots+b_m\\&=a_0^m\alpha^{mn}+(\text{lower terms of }\alpha),\end{aligned}$$

that is,

$$a_0^m x^{mn} + (\text{lower terms of } x)$$

is contained in I. Therefore $a_0^m \in C(I)$ and, hence, $a_0 \in C(I)$ and $a_0\alpha$ is integral over R. Let $a_0 = e_0 u_0$ with $e_0^2 = e_0$ and u_0 a unit in R. Then $a_0\alpha^r = u_0^{1-r}(a_0\alpha)^r$ and is integral over R. Since $f(\alpha) - a_0\alpha^n$ is also integral over R, by induction on i, we have that $a_i \in C(I)$ and $a_i\alpha^r$ is integral over R for every i with $0 \leq i \leq n-1$ and for every r.

Proof of Proposition 8. By Remark 4, it is sufficient to show that every idempotent in $R[\alpha]$ is contained in R. Let $\phi(\alpha)$ be an idempotent of $R[\alpha]$ with $\phi(x) \in R[x]$. Since $\phi(\alpha)$ is integral over R, setting

$$\phi(\alpha) = a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_n$$

we have $a_i \in C(I)$ for every i with $0 \leq i \leq n-1$ by Lemma 9. Since I is generated by linear polynomials, there is a $b_0 \in R$ such that $a_0 x + b_0 \in I$ and $b_0 \in a_0 R$. Then

$$\phi(x) = (a_0 x + b_0)x^{n-1} + (a_1 - b_0)x^{n-1} + \cdots + a_n.$$

Since $a_1 - b_0 \in C(I)$, applying the above process inductively, we see that there is an $e \in R$ such that $\phi(x) - e \in I$. Since $\phi^2(x) - \phi(x) \in I$, e is an idempotent as is easily seen. Hence $\phi(\alpha) = e$ is contained in R.

Corollary 10. *If I is a principal ideal generated by $ax+b$ with $a \neq 0$ and $b \in aR$, then $R[\alpha]$ is quasi-regular and the pair $(R, Q(R[\alpha]))$ satisfies the condition (F).*

Proof. It is easy to see that I is semi-prime and $I \cap R = \{0\}$. Then by Theorem 6 $R[\alpha]$ is quasi-regular. Since every idempotent in $Q(R[\alpha])$ is contained in R, the pair $(R, Q(R[\alpha]))$ satisfies (F) by Proposition 3.2 in [1].

It is natural to ask under what additional conditions on I (or on $C(I)$) $R[\alpha]$ is quasi-regular in case that $C(I)$ is not principal. Though we do not have any answer yet, the following proposition reduces the problem in case that I is generated by linear polynomials.

Proposition 11. *Let $\bar{R}$ e the integral closure of R in $R[\alpha]$ and let $\bar{R}[x] = R[x] \otimes_R \bar{R}$. We denote by $\bar{I}$ the kernel of the canonical homomorphism $\bar{T}$ of $\bar{R}[x]$ onto $\bar{R}[\alpha]$ $(= R[\alpha])$ such that $\bar{T}(x) = \alpha$. Then $\bar{I}$ is generated by linear polynomials and $C(\bar{I}) = C(I)\bar{R}$.*

Proof. It is clear that $\bar{I}$ is generated by linear polynomials of the form $\phi(\alpha)x - \psi(\alpha)$ where $\phi(\alpha)$, $\psi(\alpha) \in \bar{R}$ with $\phi(x)$, $\psi(x) \in R[x]$ and $\phi^2(\alpha) = \phi(\alpha)$, $\psi(\alpha) \in \phi(\alpha)\bar{R}$. Let $\phi(x) = \sum_{i=0}^{n} a_i x^{n-i}$ with $a_i \in R$ and $a_0 \neq 0$. Then by Lemma 9, $a_i \in C(I)$ and $a_i\alpha \in \bar{R}$ for every i with $0 \leq i \leq n-1$. Then $a_i x - a_i\alpha \in \bar{I}$ and

$$\phi(\alpha)x - \psi(\alpha) = \sum_{i=0}^{n-1} e_i\alpha^{n-i}(a_i x - a_i\alpha)$$
$$+ a_n x + \sum_{i=0}^{n-1} a_i\alpha^{n+1-i} - \psi(\alpha),$$

where e_i's are idempotents in R such that $a_i = e_i u_i$ with u_1 units in R. Since $a_i\alpha^r$ and, therefore, $e_i\alpha^r$ are contained in $\bar{R}$ for every r by Lemma 9, we see that $a_n x + \sum_{i=0}^{n-1} a_i\alpha^{n+1-i} - \psi(\alpha) \in \bar{I}$. Then it is easy to see that $a_n \in C(I)$. Therefore $\phi(\alpha) = \sum_{i=0}^{n-1} a_i e_i \alpha^{n-i} + a_n$ is contained in $C(I)\bar{R}$ and we have $C(\bar{I}) \subseteqq C(I)\bar{R}$. Since the converse inclusion is obvious, we see that $C(\bar{I}) = C(I)\bar{R}$.

Examples 12. Let $\{k_\lambda\}_{\lambda\in\Lambda}$ be a family of infinitely many fields and let $R = \prod_{\lambda\in\Lambda} k_\lambda$. We denote by e_λ the idempotent of R such that λ-th component is 1 and the others are 0.

(1) Let n be a natural number and let $f_\lambda(x)$ be a square-free polynomial of $k_\lambda[x]$ such that $0 < \deg f_\lambda(x) \leq n$ for every $\lambda \in \Lambda$. Let I be the ideal of $R[x]$ generated by $\{f_\lambda(x)\}_{\lambda\in\Lambda}$. It is easy to see that I is semi-prime and $I \cap R = \{0\}$. We show that $R[\alpha] = R[x]/I$ is not quasi-regular. By our assumption, there is a polynomial $f(x)$ in $R[x]$ such that $e_\lambda f(x) = f_\lambda(x)$ for every λ. Then $C(f) = R$ and $\mathrm{Ann}_{R[\alpha]} f(\alpha) = C(I)R[\alpha]$. Since $C(I) = \sum_{\lambda\in\Lambda} k_\lambda$ and is not principal, $R[\alpha]$ is not quasi-regular by Proposition 3.

(2) Let $g_\lambda(x)$ be an irreducible polynomial in $k_\lambda[x]$ for every $\lambda \in \Lambda$. Assume that for every integer n, the set of λ with $\deg g_\lambda(x) \leq n$ is a finite set. Then the ideal I of $R[x]$ generated by $\{g_\lambda(x)\}_{\lambda\in\Lambda}$ is semi-prime and $I \cap R = \{0\}$. We show that $R[\alpha] = R[x]/I$ is quasi-regular. First we remark that $C(I) = \sum_{\lambda\in\Lambda} k_\lambda$.

Let M be a maximal ideal of $R[x]$ containing $C(I)$. We need to show that M contains a non-zero-divisor modulo I. There is a $g(x)$ in M such that $c(g) = R$. Assume that $g(x)$ is a zero-divisor modulo I. Then the set of λ with $g_\lambda(x) | e_\lambda g(x)$ is a finite set, say, $\{\lambda_1, \ldots, \lambda_r\}$. Then $g(x) + e_{\lambda_1} + \cdots + e_{\lambda_r}$ is a non-zero-divisor modulo I and is contained in M. Hence $R[\alpha]$ is quasi-regular.

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