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A generalization of Riemann-Roch theorem and certain algebra of meromorphic functions on symmetric Riemann surfaces

Dedicated to Professor Yukio Kusunoki on his sixtieth birthday

By

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Introduction

The theory of abelian integrals on arbitrary open Riemann surfaces has been attempted in several ways by putting restrictions on the boundary behavior of meromorphic functions and differentials under consideration (see references). In this direction, Kusunoki [6] introduced the notion of canonical potentials and gave a formulation of Riemann-Roch's theorem and Abel's theorem, which deal with the class of meromorphic functions whose real parts are canonical potentials (cf. [5], [7]). By using the notion of behavior spaces, Yoshida [21], Shiba [17] and others showed the extended theory corresponding to various classes of meromorphic functions. However the extended theory is yet limited in the point that the differentials used in the argument of behavior spaces are assumed to be semiexact. Further, in contrast with the classical theory, those classes are real vector spaces and the multiplication of two meromorphic functions in the concerned class does not always belong to that class. In order to improve these points we shall show in this paper a generalized Riemann-Roch theorem by using certain new behavior spaces over the complex number field with less restrictions. That is, we leave out the period conditions in Shiba's behavior spaces, which are required to make use of Riemann's period relation in proving the theory, and under the present conditions we are able to prove a Riemann-Roch theorem and an Abel's theorem without direct use of Riemann's period relation. To show a typical application of these theorems we introduce the symmetric behavior space on symmetric Riemann surfaces which was considered by Matsui [9] from a different point of view. Meromorphic functions subject to the symmetric behavior space are not only Dirichlet bounded in a neighbourhood of the ideal boundary by definition, but also bounded over there. Further, the concerned class of meromorphic functions in our Riemann-Roch theorem is closed by multiplications. At last some simple examples of meromorphic functions with symmetric behavior will be shown.

§1. Behavior spaces

Let R be an arbitrary open Riemann surface and $\Gamma=\Gamma(R)$ be the Hilbert space of square integrable complex differentials whose inner product is given by

$$(\omega_1, \omega_2)_R=\iint_R \omega_1 \wedge {}^*\bar{\omega}_2=i\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)dzd\bar{z},$$

where $\omega_i=a_i dz+b_i d\bar{z}$, $i=1, 2$, and ${}^*\omega_2=-a_2 i dz+b_2 i d\bar{z}$ is the conjugate differential of ω_2, z being a local parameter. We denote by Γ_h the subspace of Γ whose elements consist of harmonic differentials and by Γ_{eo} the closure in Γ of the family Γ_{eo}^1 of differentials of C^2-functions with compact supports. We introduce behavior spaces which will play the fundamental role in the following.

Definition. A subspace Γ_x of Γ_h is called a *behavior space* if the space ${}^*\Gamma_x=\{{}^*\omega;\ \omega\in\Gamma_x\}$ is the orthogonal complement of Γ_x in Γ_h.

If Γ_x is a behavior space, $\bar{\Gamma}_x=\{\bar{\omega};\ \omega\in\Gamma_x\}$ is also a behavior space, where $\bar{\omega}$ denotes the complex conjugate of ω.

Difinition. A meromorphic differential ψ has Γ_x-behavior if there exist a compact set K on R and differentials $\omega\in\Gamma_x$, $\omega_0\in\Gamma_{eo}$ such that $\psi=\omega+\omega_0$ on $R-K$.

At first we shall construct a specific kind of meromorphic differentials with Γ_x-behavior. For an oriented closed curve γ on R, we take a ring domain V_γ such that γ is a boundary component of V_γ and V_γ lies on the left side of γ. There is a function $f_\gamma\in C^1(R-\gamma)$ such that

$$f_\gamma=\begin{cases} 1 & \text{in (a neibourhood of } \gamma) \cap V_\gamma \\ 0 & \text{on } R-\bar{V}_\gamma. \end{cases}$$

Then $\theta_\gamma=df_\gamma$ is a closed differential in Γ. For a point p on R, we take a parametric disk $V(p)=\{z;\ |z|<1\}$ about p. There are real C^1-closed differentials on $R-\{p\}$ such that

$$\theta_{p,n}=\begin{cases} -\dfrac{1}{n}\, d\, \mathrm{Re}\dfrac{1}{z^n} & \text{on } V_{1/2}=\left\{z;\ |z|<\dfrac{1}{2}\right\} \\ 0 & \text{on } R-V(p),\ (n\geq 1), \end{cases}$$

$$\tilde{\theta}_{p,n}=\begin{cases} -\dfrac{1}{n}\, {}^*d\, \mathrm{Re}\dfrac{1}{z^n} & \text{on } V_{1/2} \\ 0 & \text{on } R-V(p),\ (n\geq 1). \end{cases}$$

Further for $q\in V_{1/2}$, there are real C^1-closed differentials on $R-\{p\}-\{q\}$ such that

$$\theta_{p,q}=\begin{cases} d\log\left|\dfrac{z}{z-z(q)}\right| & \text{on } V_{1/2} \\ 0 & \text{on } R-V(p), \end{cases}$$

$$\tilde{\theta}_{p,q}=\begin{cases} *\mathrm{d}\log\left|\dfrac{z}{z-z(q)}\right| & \text{on} \quad V_{1/2} \\ 0 & \text{on} \quad R-V(p). \end{cases}$$

Let $(\theta, \tilde{\theta})$ be any one of $(\theta_\gamma, \tilde{\theta}_\gamma)$, $(\theta_{p,n}, \tilde{\theta}_{p,n})$ and $(\theta_{p,q}, \tilde{\theta}_{p,q})$, where $\tilde{\theta}_\gamma=0$. The $\theta+{}^*\tilde{\theta}$ has a compact support and belongs to Γ. By the orthogonal decomposition $\Gamma=\Gamma_x+{}^*\Gamma_x+\Gamma_{eo}+{}^*\Gamma_{eo}$, write $\theta+{}^*\tilde{\theta}=\omega+\tau+\omega_0+{}^*\tau_0$, $\omega\in\Gamma_x$, $\tau\in{}^*\Gamma_x$, ω_0, $\tau_0\in\Gamma_{eo}$. Then $\theta-\omega-\omega_0=\tau+{}^*\tau_0-{}^*\tilde{\theta}$ is closed and coclosed in $R-\{p\}-\{q\}$. Hence $\phi=\theta-\omega-\omega_0$ is harmonic on $R-\{p\}-\{q\}$. Note that

$$\begin{aligned}\phi+i^*\phi&=\theta-\omega-\omega_0+i(\tilde{\theta}+{}^*\tau-\tau_0)\\ &=-\omega+i^*\tau-(\omega_0+i\tau_0)+\theta+i\tilde{\theta},\end{aligned}$$

where $-\omega+i^*\tau\in\Gamma_x$, $\omega_0+i\tau_0\in\Gamma_{eo}$ and $\theta+i\tilde{\theta}$ has a compact support. This shows that the meromorphic differential $\phi+i^*\phi$ has Γ_x-behavior. Thus we have a holomorphic differential with Γ_x-behavior $\psi_{\gamma,x}$ and meromorphic differentials with Γ_x-behavior $\psi_{p,n,x}(\psi_{p,q,x})$ whose singurarity is $\frac{1}{z^{n+1}}\,\mathrm{d}z$ at $p\left(\frac{1}{z}\,\mathrm{d}z\right.$ at p and $\frac{-1}{z-z(q)}\,\mathrm{d}z$ at $\left.q\right)$. We call these fundamental differentials with Γ_x-behavior.

Proposition 1. *If a meromorphic differential ψ has a Γ_x-behavior, then there exist a finite number of fundamental differentials $\psi_{\gamma_j,x}$, $\psi_{p_i,n,x}$, $\psi_{p_j,p_k,x}$ and complex numbers c_j, $c_{i,n}$, $c_{j,k}$ such that $\psi=\sum c_j\psi_{\gamma_j,x}+\sum c_{i,n}\psi_{p_i,n,x}+\sum c_{j,k}\psi_{p_j,p_k,x}$.*

Proof. Since ψ has Γ_x-behavior, there exist a compact regular region G and differentials $\omega\in\Gamma_x$ and $\omega_0\in\Gamma_{eo}$ such that $\psi=\omega+\omega_0$ on $R-G$. There are a finite number of poles of ψ and we denote the support by $p_1,\dots, p_m$. Remarking that

$$\int_{\partial G}\psi=\int_{\partial G}\omega+\omega_0=0,$$

we can choose $c_{i,n}$ and $c_{j,k}$ such that

$$\psi-\sum c_{i,n}\psi_{p_i,n,x}-\sum c_{j,k}\psi_{p_j,p_k,x}=\psi'$$

is a holomorphic differential. Clearly ψ' has a Γ_x-behavior and we write $\psi'=\omega'+\omega_0'$ on $R-G$, $\omega'\in\Gamma_x$, $\omega_0'\in\Gamma_{eo}$. Further we can choose $\{c_j\}$ such that $\psi'-\omega'-\omega_0'-\sum c_j\mathrm{d}f_{\gamma_j}=\sigma_0$ is exact on G and belongs to Γ_{eo}, where the sum about γ_j is taken over a homology basis of G. Let $\mathrm{d}f_{\gamma_j}=\omega_j+\tau_j+\omega_{j,0}$, $\omega_j\in\Gamma_x$, $\tau_j\in{}^*\Gamma_x$, $\omega_{j,0}\in\Gamma_{eo}$. From $\psi'=i^*\psi'$, we have

$$\begin{aligned}&\omega'+\omega_0'+\sigma_0+\sum c_j(\omega_j+\omega_{j,0}-i^*\tau_j)\\ &=i^*(\omega'+\omega_0'+\sigma_0)+\sum ic_j({}^*\omega_j+{}^*\omega_{j,0}+i\tau_j)\end{aligned}$$

and this vanishes. Thus $\psi'=\sum c_j(\tau_j+i^*\tau_j)=\sum c_j\psi_{\gamma_j,x}$ and

$$\psi=\sum c_j\psi_{\gamma_j,x}+\sum c_{i,n}\psi_{p_i,n,x}+\sum c_{j,k}\psi_{p_j,p_k,x}.$$

Remark. If the local parameters about $\{p_j\}$ which define the fundamental differentials are designated, the representation of ψ is unique.

Lemma 1. *Let G be a disjoint union of parametric closed disks $\{G_i\}$ which do not accumulate in R. Let a Jordan curve γ not meet G. If a meromorphic differential ψ is written as $\psi=\omega+\omega_0$, $\omega\in\Gamma_x$, $\omega_0\in\Gamma_{eo}$ on $R-G$, then*

$$\int_\gamma \psi=-\int_{\partial G}\Psi\psi_\gamma,$$

where $d\Psi=\psi$ on ∂G and $\psi_\gamma=\psi_{\gamma,\bar{x}}$ denotes the fundamental differential with $\bar{\Gamma}_x$-behavior. Further, if Ψ is a single valued meromorphic function on every G_i, then

$$\int_\gamma \psi=-2\pi i\sum_G \operatorname{Res}\Psi\psi_\gamma.$$

Proof. Note that

$$0=(\psi,\ {}^*\bar{\psi}_\gamma)_{R-G}=(\omega+\omega_0,\ {}^*\bar{\psi}_\gamma)_R-(\omega+\omega_0,\ {}^*\bar{\psi}_\gamma)_G,$$

where $\psi_\gamma=\bar{\tau}+i^*\bar{\tau}$, $\tau=df_\gamma-\omega_\gamma-\omega_{\gamma,0}\in{}^*\Gamma_x$, $\omega_\gamma\in\Gamma_x$, $\omega_{\gamma,0}\in\Gamma_{eo}$. We have

$$\begin{aligned}(\omega+\omega_0,\ {}^*\bar{\psi}_\gamma)_R&=(\omega+\omega_0,\ {}^*\tau+i\tau)_R=(\omega+\omega_0,\ {}^*\tau)_R\\&=(\omega+\omega_0,\ {}^*df_\gamma-{}^*\omega_\gamma-{}^*\omega_{\gamma,0})_R\\&=(\omega+\omega_0,\ {}^*df_\gamma)_R\\&=\int_{\partial(R-\gamma)}f_\gamma(\omega+\omega_0)\\&=\int_\gamma\psi.\end{aligned}$$

On the other hand, remarking that $\int_{\partial G_i}\psi_\gamma=0$, we have

$$\begin{aligned}(\omega+\omega_0,\ {}^*\bar{\psi}_\gamma)_G&=-\iint_G d(W+W_0)\psi_\gamma\\&=-\int_{\partial G}\Psi\psi_\gamma\end{aligned}$$

where $dW=\omega$, $dW_0=\omega_0$ on G and $W+W_0=\Psi$. Thus the statement follows.

§2. A Riemann Roch type theorem

Let $\{V_j\}$ be a family of parametric disks ($\bar{V}_j\cap\bar{V}_k=\emptyset$, $j\neq k$) which do not accumulate in R, and set $V=\cup V_j$. Let $\delta=\delta_q/\delta_p=q_1\cdots q_m/p_1\cdots p_n(\cup\{q_i\}\cap\cup\{p_j\}=\emptyset)$ be a finite or an infinite divisor whose support is in V. We consider the following vector spaces over $\boldsymbol{C}$ of meromorphic functions and meromorphic differentials.

$M(1/\delta_p;\ \Gamma_x)=\{F;\ F$ is a multi-valued meromorphic functions on R whose divisor is multiple of $1/\delta_p$ and $dF=\omega+\omega_0$ on $R-V$, $\omega\in\Gamma_x$, $\omega_0\in\Gamma_{eo}\}$.

$S(\delta;\ \Gamma_x)\{f\in M(1/\delta_p;\ \Gamma_x);\ f$ is single valued and the divisor of f is multiple of $\delta\}$.

$D(1/\delta_q;\ \bar{\Gamma}_x)=\{\psi;\ \psi$ is a meromorphic differential with $\bar{\Gamma}_x$-behavior and the divisor of ψ is multiple of $1/\delta_q\}$.

$D(1/\delta;\ \bar{\Gamma}_x)=\{\psi \in D(1/\delta_q;\ \bar{\Gamma}_x)$; The divisor of ψ is multiple of $1/\delta\}$.
When $\deg \delta_q \neq 0$, two elements F_1 and F_2 of $M(1/\delta_p;\ \Gamma_x)$ is identified if and only if F_1-F_2 is a constant. The $M(1/\delta_p;\ \Gamma_x)$ is independent of the choice of V whenever it contains the support of δ_p.

Theorem 1. (*Riemann-Roch type*)

$$\dim \frac{M(1/\delta_p;\ \Gamma_x)}{S(\delta;\ \Gamma_x)} = \dim \frac{D(1/\delta_q;\ \bar{\Gamma}_x)}{D(1/\delta;\ \bar{\Gamma}_x)}.$$

If δ_p *is a finite divisor, then*

$$\dim S(\delta;\ \Gamma_x)=\deg \delta_p+1-\min(\deg \delta_q,\ 1)-\dim \frac{D(1/\delta_q;\ \bar{\Gamma}_x)}{D(1/\delta;\ \bar{\Gamma}_x)}.$$

Proof. As we have no period conditions as in Shiba [17], we need a slightly different argument. In place of the bilinear relation we shall use inner products with the same role as the bilinear relations. Now we consider the bilinear form

$$h(F,\ \psi)=2\pi i \sum \operatorname*{Res}_{p_i} F\psi,$$

which is defined on the product space $M(1/\delta_p;\ \Gamma_x)\times D(1/\delta_q;\ \bar{\Gamma}_x)$. Although F is multi-valued, $\operatorname*{Res}_{p_i} F\psi$ is well defined. For $F\in M(1/\delta_p;\ \Gamma_x)$ there exist differentials $\omega \in \Gamma_x$ and $\omega_0 \in \Gamma_{eo}$ such that $dF=\omega+\omega_0$ on $R-V$. By Proposition 1 we can write

$$\psi=\sum c_\gamma \psi_{\gamma,\bar{x}}+\sum c_{i,n}\psi_{q_i,n,\bar{x}}+\sum c_{j,k}\psi_{q_i,q_k,\bar{x}},$$

further write that on $R-V'$ ($V'=V\cup$ {a simply connected region having the poles})

$$\psi_{\gamma,\bar{x}}=\bar{\tau}_\gamma+i^*\bar{\tau}_\gamma,\quad \tau_\gamma \in {}^*\Gamma_x,$$

$$\psi_{q_i,n,\bar{x}}=\bar{\sigma}_{i,n}+\sigma_{0,i,n},\quad \sigma_{i,n}\in \Gamma_x \quad \text{and} \quad \sigma_{0,i,n}\in \Gamma_{eo}$$

$$\psi_{q_i,q_k,\bar{x}}=\bar{\sigma}_{j,k}+\sigma_{0,j,k},\quad \sigma_{j,k}\in \Gamma_x \quad \text{and} \quad \sigma_{0,j,k}\in \Gamma_{eo}.$$

Set

$$\phi=\sum c_\gamma(\bar{\tau}_\gamma+i^*\bar{\tau}_\gamma)+\sum c_{i,n}(\bar{\sigma}_{i,n}+\sigma_{0,i,n})+\sum c_{j,k}(\bar{\sigma}_{j,k}+\sigma_{0,j,k})$$

on R then $\phi=\psi$ on $R-V'$. We have

$$\begin{aligned}
0&=(dF,\ {}^*\bar{\psi})_{R-V'}\\
&=(\omega+\omega_0,\ {}^*\bar{\phi})_{R-V'}\\
&=(\omega,\ \sum \bar{c}_\gamma{}^*\tau_\gamma)_R-(\omega+\omega_0,\ {}^*\bar{\phi})_{V'}\\
&=\sum c_\gamma \int_\gamma \omega+\int_{\partial V'} w'\phi \quad (dw'=\omega+\omega_0,\ \sum\left|\int_{\partial V_i} w'\phi\right|\le \|dw'\|\,\|\phi\|)\\
&=\sum c_\gamma \int_\gamma dF+\int_{\partial V'} F\psi.
\end{aligned}$$

Thus $2\pi i \sum \operatorname{Res} F\psi = -\sum c_\gamma \int_\gamma dF$, and

$$h(F, \psi) = 2\pi i \sum \operatorname*{Res}_{p_j} F\psi = -\sum c_\gamma \int_\gamma dF - 2\pi i \sum \operatorname*{Res}_{q_j} F\psi.$$

Since $\sum_\gamma$ and $\sum_{q_j}$ are finite sums, $h(F, \psi)$ has a finite value. For $f \in S(\delta; \Gamma_x)$ and $\psi \in D(1/\delta_q; \bar{\Gamma}_x)$, it is clear that $\int_\gamma df = 0$ for any cycle γ and that $\sum_{q_j} \operatorname{Res} f\psi = 0$. Therefore $S(\delta; \Gamma_x)$ is contained in the right kernel of $h(\cdot, \cdot)$. Conversely let $F \in M(1/\delta_p; \Gamma_x)$ satisfies that $h(F, \psi) = 0$ for any $\psi \in D(1/\delta_q; \bar{\Gamma}_x)$. Since $\psi_\gamma \in D(1/\delta_q; \bar{\Gamma}_x)$, $\int_\gamma dF = -h(F, \psi_\gamma) = 0$. Thus F is single valued. If $\deg \delta_q = 0$, F belongs to $S(\delta; \Gamma_x)$. Let $\deg \delta_q \geq 1$. For $\psi_{q_i,n,\bar{x}} \in D(1/\delta_q; \bar{\Gamma}_x)$,

$$0 = h(F, \psi_{q_i,n,\bar{x}}) = -2\pi i \operatorname*{Res}_{q_i} F\psi_{q_i,n,\bar{x}}$$

and for $\psi_{q_j,q_k} \in D(1/\delta_q; \bar{\Gamma}_x)$,

$$0 = h(F, \psi_{q_j,q_k,\bar{x}}) = -2\pi i (F(q_j) - F(q_k)).$$

It follows that $F - F(q_1) \in S(\delta; \Gamma_x)$ and the right kernel of $h(\cdot, \cdot)$ is spaned by $S(\delta; \Gamma_x)$ and constants. The $D(1/\delta; \bar{\Gamma}_x)$ is contained in the left kernel of $h(\cdot, \cdot)$, because $h(F, \psi) = \sum_{p_i} \operatorname{Res} F\psi = 0$ for $\psi \in D(1/\delta; \bar{\Gamma}_x)$ and $F \in M(1/\delta_p; \Gamma_x)$. Let $\psi \in D(1/\delta_q; \bar{\Gamma}_x)$ satisfy that $h(F, \psi) = 0$ for any $F \in M(1/\delta_p; \Gamma_x)$. Take $\Psi_{p_j,n,x} \in M(1/\delta_p; \Gamma_x)$ $(d\Psi_{p_j,n,x} = \psi_{p_j,n,x})$. Then

$$0 = h(\Psi_{p_j,n,x}, \psi) = \operatorname*{Res}_{p_j} \Psi_{p_j,n,x}\psi,$$

and $\psi \in D(1/\delta; \bar{\Gamma}_x)$. This shows that $D(1/\delta; \bar{\Gamma}_x)$ is the left kernel of $h(\cdot, \cdot)$. The first statement follows. When δ_p is a finite divisor $p_1^{\mu_1} \cdots p_n^{\mu_n}$, $M(1/\delta_p; \Gamma_x)$ is spaned by $\{\Psi_{p_j,k,x}; d\Psi_{p_j,k,x} = \psi_{p_j,k,x},\ j = 1, \dots, n,\ 1 \leq k \leq \mu_j\}$ and a constant 1. From the convention $\dim M(1/\delta_p; \Gamma_x) = \deg \delta_p + 1 - \min(\deg \delta_q, 1)$. Thus the second statement follows.

Now remembering classical Weierstrass points, we define the following.

Definition. An positive integer n is called a Γ_x gap value at p if $S\left(\frac{1}{p^n}; \Gamma_x\right) = S\left(\frac{1}{p^{n-1}}; \Gamma_x\right)$, otherwise n is called a Γ_x non gap value at p. A point $p \in R$ is called a Γ_x Weierstrass point if all positive integers not exceeding the genus $g(\leq \infty)$ of R are Γ_x gap values at p.

By Theorem 1 n is a Γ_x gap value at p if and only if $D(p^n; \bar{\Gamma}_x) \neq D(p^{n-1}; \bar{\Gamma}_x)$. By usual argument we have

Proposition 2. *There exists a family of linealy independent holomorphic differentials $\psi_1, \dots, \psi_g$ with the $\bar{\Gamma}_x$-behavior such that the order μ_i of zero of ψ_i at p satisfy*

$$0 \leq \mu_1 < \mu_2 < \cdots < \mu_g.$$

Then the integers $\mu_1+1, \mu_2+1, \ldots, \mu_g+1$ *are whole* Γ_x *gap values at* p.

§3. An Abel type theorem

Let δ_p and δ_q be finite or infinite divisors on R whose supports have no intersection and are contained in $V_{i,1/2}\left(V_{i,1/2}=\left\{z_i;\ |z_i|<\frac{1}{2}\right\}\subset V_i\right)$. Assume that the restrictions to each $V_{i,1/2}$ of δ_p, δ_q have the same degree. Write them as $p_{i,1}\cdots p_{i,n}$ and $q_{i,1}\cdots q_{i,n}$, where $p_{i,j}$ (resp. $q_{i,j}$) may coincide with $p_{i,k}$ (resp. $q_{i,k}$) for $j\neq k$. In our assertion of Abel's theorem we need an assumption that there exists a closed C^1-differential θ in $R-\cup(p_{i,k}\cup q_{i,j})$ such that

$$\theta=\begin{cases} \mathrm{d}\sum\log(z_i-z_i(p_{i,j}))/(z_i-z_i(q_{i,j})) & \text{on} \quad V_{i,1/2} \\ 0 & \text{on} \quad R-V \end{cases}$$

and $(\theta, \theta)_{R-\cup V_{i\ 1/2}}<\infty$.

Theorem 2. (*Abel type*) *The following two conditions are equivalent.*
(1) *There exists a single valued meromorphic function* f *such that* (*i*) *the divisor of* f *is* δ, (*ii*) $\mathrm{d}\log f=\omega+\omega_0$ *on* $R-V$ *for some* $\omega\in\Gamma_x$, $\omega_0\in\Gamma_{eo}$.
(2) *Let* C *be a chain in* V *such that* $\partial C=\sum(p_{i,j}-q_{i,j})$. *Then* $\int_C\psi_{\gamma,\bar{x}}$ *is an integer for every Jordan curve* γ *not meet* V.

Proof. Let f be a meromorphic function in (1). Then we have

$$\begin{aligned}\int_C\psi_{\gamma,\bar{x}}&=\sum_V \operatorname{Res}\Psi_{\gamma,\bar{x}}\,\mathrm{d}\log f \qquad (\mathrm{d}\Psi_{\gamma,\bar{x}}=\psi_{\gamma,\bar{x}})\\ &=\frac{1}{2\pi i}\sum_i\int_{\partial V_i}\Psi_{\gamma,\bar{x}}\,\mathrm{d}\log f \quad \text{(absolutely convergent)}\\ &=\frac{-1}{2\pi i}\sum_i\int_{\partial V_i}\log f\,\psi_{\gamma,\bar{x}}.\end{aligned}$$

Therefore, by Lemma 1,

$$\int_C\psi_{\gamma,\bar{x}}=\frac{1}{2\pi i}\int_\gamma \mathrm{d}\log f=\frac{1}{2\pi}\int_\gamma \mathrm{d}\arg f$$

and this is an integer. Let show the converse. By the assumption $\theta-i{}^*\theta$ belongs to Γ. From the orthogonal decomposition we can write $\theta-i{}^*\theta=\omega+\tau$, $\omega\in\Gamma_x+\Gamma_{eo}$, $\tau\in{}^*\Gamma_x+{}^*\Gamma_{eo}$. Then $\psi=\theta-\omega=\tau+i{}^*\theta$ is closed and coclosed, hence is harmonic in $R-\cup(p_{i,j}\cup q_{i,j})$. The $\phi=(\psi+i{}^*\psi)/2$ is a meromorphic differential which $\phi=(-\omega+i{}^*\tau)/2$ on $R-V$. Hence by Lemma 1

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma\phi&=\frac{-1}{2\pi i}\sum_i\int_{\partial V_i}\Phi\psi_{\gamma,\bar{x}} \quad (\mathrm{d}\Phi=\phi)\\ &=\frac{1}{2\pi i}\sum_i\int_{\partial V_i}\Psi_{\gamma,\bar{x}}\phi \quad (\mathrm{d}\Psi_{\gamma,\bar{x}}=\psi_{\gamma,\bar{x}})\end{aligned}$$

$$= \sum_V \operatorname{Res} \Psi_{\gamma,\bar{x}}\phi$$

$$= \int_C \psi_{\gamma,\bar{x}}$$

and by condition (2) $\frac{1}{2\pi i}\int_\gamma \phi$ is an integer for every Jordan curve γ. Thus $f(p)=\exp\int^p \phi$ is a single valued meromorphic function and f satisfies (i) and (ii), because $d\log f=\phi$ and $\phi=-\omega+i^*\tau$ on $R-V$.

Remark. The f is uniquely determined except for non zero multiplicative constants. If f_1 is another function in (1), $\sigma=d\log f-d\log f_1$ belongs to $\Gamma_x+\Gamma_{eo}$. Then $(\sigma, \sigma)=(\sigma, i^*\sigma)=0$ and $\sigma=0$.

§4. Symmetric behavior spaces on symmetric Riemann surfaces

Let R be a symmetric Riemann surface i.e. there is an anticonformal mapping J from R to R so called involution, and the composite mapping $J\circ J$ is an identity mapping. Let V be a parametric disk and z be the local parameter. Then $J(V)$ is also a parametric disk and denote the local parameter by $w(J: z\to w)$. For a $\omega\in\Gamma_h(R)$ there exist harmonic functions $f(z)$ (on V) and $\tilde{f}(w)$ (on $J(V)$) such that

$$\omega=\begin{cases} df=f_z dz+f_{\bar{z}}d\bar{z} & \text{on } V \\ d\tilde{f}=\tilde{f}_w dw+\tilde{f}_{\bar{w}}d\bar{w} & \text{on } J(V). \end{cases}$$

We define $J^\#(\omega)$ the pull back of ω by J;

$$J^\#(\omega)=\begin{cases} d(\tilde{f}\circ J)=\tilde{f}_{\bar{w}}\bar{w}_z dz+\tilde{f}_w w_{\bar{z}}d\bar{z} & \text{on } V \\ d(f\circ J)=f_{\bar{z}}\bar{z}_w dw+f_z z_{\bar{w}}d\bar{w} & \text{on } J(V). \end{cases}$$

Note that $J^\#(J^\#(\omega))=d(f\circ J\circ J)=df$ on V, $=d(\tilde{f}\circ J\circ J)=d\tilde{f}$ on $J(V)$, and for $^*\omega=-if_z dz+if_{\bar{z}}d\bar{z}$ on V, $=-i\tilde{f}_w dw+i\tilde{f}_{\bar{w}}d\bar{w}$ on $J(V)$

$$J^\#(^*\omega)=\begin{cases} i\tilde{f}_{\bar{w}}\bar{w}_z dz-i\tilde{f}_w w_{\bar{z}}d\bar{z} & \text{on } V \\ if_{\bar{z}}\bar{z}_w dw-if_z z_{\bar{w}}d\bar{w} & \text{on } J(V). \end{cases}$$

So we can define a linear mapping $J^\#: \Gamma_h\to\Gamma_h$ and get the following.

Lemma 2. *For* $\omega\in\Gamma_h$

$$J^\#(J^\#(\omega))=\omega,\ J^\#(^*\omega)=-^*J^\#(\omega).$$

Set

$$\Gamma_s=\{\omega+J^\#(\omega);\ \omega\in\Gamma_h\},$$

$$\Gamma_t=\{\omega-J^\#(\omega);\ \omega\in\Gamma_h\}.$$

Then $J^\#(\Gamma_s)=\Gamma_s$, $J^\#(\Gamma_t)=\Gamma_t$, and $^*\Gamma_s=\{^*\omega;\ \omega\in\Gamma_s\}=\Gamma_t$, because by Lemma 2 $J^\#(\omega+J^\#(\omega))=J^\#(\omega)+\omega$, $J^\#(\omega-J^\#(\omega))=J^\#(\omega)-\omega$ and $^*\omega+{}^*J^\#(\omega)={}^*\omega-J^\#(^*\omega)$.

Lemma 3. *For ω_1 and ω_2 in Γ_h*

$$(\omega_1, J^{\#}(\omega_2))=(J^{\#}(\omega_1), \omega_2) \text{ and } (J^{\#}(\omega_1), J^{\#}(\omega_2))=(\omega_1, \omega_2).$$

Proof. Write $\omega_i=a_i(z)dz+b_i(z)d\bar{z}$ on V, $=\tilde{a}_i(w)dw+\tilde{b}_i(w)d\bar{w}$ on $J(V)(i=1, 2)$. we have

$$(J^{\#}(\omega_1), \omega_2)=\iint_R (\tilde{b}_1\bar{w}_z dz+\tilde{a}_1 w_{\bar{z}} d\bar{z})\wedge\overline{(-ia_2dz+ib_2d\bar{z})}$$

$$=i\iint_R (\tilde{b}_1\bar{a}_2\bar{w}_z+\tilde{a}_1\bar{b}_2 w_{\bar{z}})\, dz\wedge d\bar{z},$$

$$(\omega_1, J^{\#}(\omega_2))=\iint_R (\tilde{a}_1 dw+\tilde{b}_1 d\bar{w})\wedge\overline{(-ib_2\bar{z}_w dw+ia_2 z_{\bar{w}} d\bar{w})}$$

$$=i\iint_R (\tilde{a}_1\bar{b}_2 z_{\bar{w}}+\tilde{b}_1\bar{a}_2\bar{z}_w)\, dw\wedge d\bar{w}$$

$$=i\iint_R (\tilde{a}_1\bar{b}_2 w_{\bar{z}}+\tilde{b}_1\bar{a}_2\bar{w}_z)\, dz\wedge d\bar{z}.$$

Thus $(\omega_1, J^{\#}(\omega_2))=(J^{\#}(\omega_1), \omega_2)$ and $(J^{\#}(\omega_1), J^{\#}(\omega_2))=(J^{\#}(J^{\#}(\omega_1)), \omega_2)=(\omega_1, \omega_2)$.

Proposition 3. *The Γ_s and the Γ_t are behavior spaces such that*

$$\Gamma_s+\Gamma_t=\Gamma_h.$$

Proof It is clear that $\Gamma_s+\Gamma_t=\Gamma_h$. By Lemma 3 we have

$$(\omega_1+J^{\#}(\omega_1), \omega_2-J^{\#}(\omega_2))$$
$$=(\omega_1, \omega_2)-(J^{\#}(\omega_1), J^{\#}(\omega_2))-(\omega_1, J^{\#}(\omega_2))+(J^{\#}(\omega_1), \omega_2)$$
$$=0.$$

This shows that Γ_s is orthogonal to Γ_t. Since $\Gamma_t={}^*\Gamma_s$, they are behavior spaces.

Definition. A meromorphic differential ψ on a symmetric Riemann surface R has the symmetric behavior if ψ has the Γ_s-behavior and a meromorphic function f has the symmetric behavior if df has the symmetric behavior.

A meromorphic function with the symmetric behavior has the following property.

Proposition 4. *Let f be a meromorphic function with the symmetric behavior. Then f is bounded in a neighbourhood of the ideal boundary.*

Proof. Since df has the symmetric behavior, there exist a compact regular region $\bar{G}(\bar{G}=J(\bar{G}))$ and C^1-differentials $\omega\in\Gamma_s$, $\omega_0\in\Gamma_{eo}$ such that $df=\omega+\omega_0$ on $R-G$. The restriction $\omega|_{R-\bar{G}}$ on $R-\bar{G}$ of ω is exact and it can be extended to C^1-differential ω' on R which is exact on R. For $\omega'=dW'$, set $\tilde{W}=(W'+W'\circ J)/2$. By the orthogonal decomposition we can write $d\tilde{W}=\tilde{\omega}+\tilde{\omega}_0$, $\tilde{\omega}\in\Gamma_{he}$, $\tilde{\omega}_0\in\Gamma_{eo}$. Since $d\tilde{W}=J^{\#}(d\tilde{W})=(J^{\#}(\tilde{\omega})+\tilde{\omega})/2+(J^{\#}(\tilde{\omega}_0)+\tilde{\omega}_0)/2$, the $\tilde{\omega}$ belongs to $\Gamma_s\cap\Gamma_{he}$.

Write $\tilde{\omega}=dS$ and $\tilde{\omega}_0+\omega_0=dS_0$, where $|S_0|$ is a Dirichlet potential. Then we can assume that for some potential P_0 $|S_0|\leq P_0$. Now we know $d(S+S_0)=df$ on $R-\bar{G}$, because $d(S+S_0)=d\tilde{W}+\omega_0$ and $d\tilde{W}=(\omega+J^*(\omega))/2=\omega$ on $R-\bar{G}$. Let S_1 and $S_{1,0}$ (resp. S_2 and $S_{2,0}$) be the real part (resp. the imaginary part) of S and S_0 respectively. Take constants n and m which satisfy $m\leq S_i+S_{i,0}\leq n$ on G, $i=1, 2$. Let $S_i\wedge n$ be the greatest harmonic minorant of $\min(S_i, n)$ and $S_i\vee m$ be the least harmonic majorant of $\max(S_i, m)$. Set $h_i=(S_i\wedge n)\vee m$. Since $h_i\circ J=h_i$, $dh_i\in\Gamma_s$. We can write

$$\min(S_i, n)=S_i\wedge n+P_{i,1},$$

$$\max(S_i\wedge n, m)=h_i-P_{i,2},$$

where $P_{i,1}$ and $P_{i,2}$ are potentials. Then

$$\begin{aligned}S_i\wedge n+P_{i,1}+P_0&=\min(S_i+P_0, n+P_0)\\&\geq\min(S_i+S_{i,0}, n)\\&\geq\min(S_i-P_0, n-P_0)\\&=S_i\wedge n+P_{i,1}-P_0.\end{aligned}$$

Further

$$\begin{aligned}\max\{\min(S_i+S_{i,0}, n), m)\}&\leq\max(S_i\wedge n+P_{i,1}+P_0, m)\\&\leq\max(S_i\wedge n, m)+P_{i,1}+P_0\\&=h_i-P_{i,2}+P_{i,1}+P_0\end{aligned}$$

and also

$$\begin{aligned}\max\{\min(S_i+S_{i,0}, n), m)\}&\geq\max(S_i\wedge n+P_{i,1}-P_0, m)\\&\geq\max(S_i\wedge n, m)-P_0-P_{i,1}\\&=h_i-P_{1,2}-P_{i,1}-P_0.\end{aligned}$$

Thus we can write

$$\max\{\min(S_i+S_{i,0}, n), m\}=h_i+\tilde{S}_{i,0},$$

where $d\tilde{S}_{i,0}$ belongs to Γ_{eo}. Set $h=h_1+ih_2$ and $\tilde{S}_0=\tilde{S}_{1,0}+i\tilde{S}_{2,0}$. Then

$$d(S+S_0)=d(h+\tilde{S}_0)\quad\text{on}\quad G,$$

and

$$\|d(h+\tilde{S}_0)\|_{R-\bar{G}}\leq\|d(S+S_0)\|_{R-\bar{G}}=\|df\|_{R-\bar{G}}.$$

By the way

$$\begin{aligned}&(d(S+S_0-h-\tilde{S}_0), d(S+S_0))_{R-\bar{G}}\\&\quad=(d(S+S_0-h-\tilde{S}_0), i^*d(S+S_0))_R=0,\end{aligned}$$

hence

$$\|d(S+S_0)\|^2_{R-\bar{G}}=(d(h+\tilde{S}_0),\ d(S+S_0))_{R-\bar{G}}.$$

Therefore

$$\begin{aligned}0&\le \|d(S+S_0-h-\tilde{S}_0)\|^2\\ &=\|d(S+S_0)\|^2-2(d(h+\tilde{S}_0),\ d(S+S_0))+\|d(h+\tilde{S}_0)\|^2\\ &=\|d(h+\tilde{S}_0)\|^2-\|d(S+S_0)\|^2\le 0.\end{aligned}$$

Thus $d(S+S_0)=d(h+\tilde{S}_0)$. It follows that f is bounded on $R-\bar{G}$.

This boundedness allows that the multiplications of meromorphic functions with symmetric behavior have finite Dirichlet integrals in a neighbourhood of the ideal boundary. Further, we have

Proposition 5. *Let δ_1 and δ_2 be finite divisors. If $f_i\in S(\delta_i;\ \Gamma_s)$ $i=1, 2$, then $f_1f_2\in S(\delta_1\delta_2;\ \Gamma_s)$.*

Proof. We can write

$$df_i=dS_i+dW_i \quad \text{on} \quad R-V,$$

where S_i are harmonic and $S_i\circ J=S_i$ and W_i are Dirichlet potentials. The functions S_i and W_i have continuous extensions $\hat{S}_i$ and $\hat{W}_i$ to Royden's compactification R^* of R and $\hat{W}_i$ are zero on the harmonic boundary of R^* (cf. [4]). Since $(S_1+W_1)\times(S_2+W_2)$ has a finite Dirichlet integral, by the Royden's decomposition

$$(S_1+W_1)\,(S_2+W_2)=h+P,$$

where h is a harmonic function and P is a Dirichlet potential. Note that h is a solution $H_{\hat{S}_1\hat{S}_2}$ of generalized Dirichlet problem on R^* with boundary value $\hat{S}_1\hat{S}_2$ and $h\circ J=h$. Since $d(f_1f_2)=dh+dP$ on $R-V$, f_1f_2 belongs to $S(\delta_1\delta_2;\ \Gamma_s)$.

Similarly we can prove the following.

Proposition 6. *Let δ be a finite divisor and $f\in S(\delta;\ \Gamma_s)$. Let a complex number α be excluded from the boundary cluster set $\cap\{\overline{f(R-R_n)};\ R_n$ is a regular exhaustion of $R\}$. Then $1/(f-\alpha)$ has the symmetric behavior.*

As for the Γ_s gap values we have, by Proposition 5, the following.

Proposition 7. *If n is a Γ_s non gap value at p, then for every positive integer m the integer nm is also a Γ_s non gap value at p and $D(p^{nm};\ \Gamma_s)=D(p^{nm-1};\ \Gamma_s)$.*

At last we give examples of meromorphic functions with the symmetric behavior. Let $D_1=\{z\in \boldsymbol{C};\ |z|<1\}$, $D_2=\hat{\boldsymbol{C}}-E$ and $D_3=\hat{\boldsymbol{C}}-F$, where E is a compact set in $\boldsymbol{C}$ which is symmetric with respect to real axis and F is a compact proper subset in a unit circle. Then $\underline{J}_1(z)=\bar{z}$, $\underline{J}_2(z)=\bar{z}$ and $\underline{J}_3(z)=1/\bar{z}$ are anticonformal mappings on D_1, D_2 and D_3 respectively. Let R_i be an n sheeted full covering on D_i such that the branch points lie on a set B_i which is invariant by $\underline{J}_i$ and the lift J_i of $\underline{J}_i$ can be defined to give an anticonformal mapping on R_i. Further provide that $0 \in B_1$, $\infty \in B_2$ and $\infty \in B_3$ are isolated branch points of order $n-1$ and no accumulating point of B_3 lies out of the unit circle. A meromorphic function $f_1(z)=z+1/z$ has the symmetric behavior on R_1. If the genus of R_1 is larger than n, zero is a Γ_s Weierstrass point. When E consists of slits on $y=\pm\sqrt{3}x$ or $y=0$, $f_2(z)=z^3$ has also the symmetric behavior on R_2. On R_3 $f_3(z)=z$ is a meromorphic function with the symmetric behavior, which is unlike f_1 and f_2 at the following point: $f_1 \circ J_1 = \bar{f}_1$, $f_2 \circ J_2 = \bar{f}_2$ but $f_3 \circ J_3 \neq \bar{f}_3$. The reason why f_i has the symmetric behavior is $f_i \circ J = \bar{f}_i$ on the boundary.

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