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# Some deformations of codimension two discrete groups

Dedicated to Yukio Kusunoki on his sixtieth birthday

By

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**1.** Let $H^n$ be hyperbolic $n$-space, and let $\tilde{L}^n$ denote the group of isometries of $H^n$; the orientation preserving half of $\tilde{L}^n$ is denoted by $L^n$. A parabolic element of $L^n$ is a transformation which has exactly one fixed point in the Euclidean closure of $H^n$. If we normalize so that $H^n$ is the upper half-space, and the parabolic element $j$ has its fixed point at $\infty$, then, in its action on $\hat{E}^{n-1}=E^{n-1}\cup\{\infty\}=\partial H^n$, $j(x)=r(x)+b$, where $r$ is an orthogonal transformation, and $b$ is not in the range of $1-r$. If $r=1$, then $j$ is a *pure* parabolic transformation, or translation, while if $r\neq 1$, $j$ is *impure.*

The following question was posed by John Morgan (oral communication). Is there a cofinite volume discrete subgroup $G$ of $L^n$, containing pure parabolic elements, and a deformation $\tilde{G}$ of $G$ in some $L^m$, $m>n$, where the corresponding parabolic elements of $\tilde{G}$ are impure?

Note that if $m<4$, then every parabolic element of $L^m$ is pure.

For $m=n+1$, it follows from a theorem of Cheeger and Gromoll [1] that a discrete free abelian group of orientation preserving Euclidean motions of rank $n-1$, acting on $E^n$, contains only pure parabolic elements; we outline an elementary proof of this fact below. It follows that if $G$ is a discrete subgroup of $L^n$, of cofinite volume, and $G$ contains only pure parabolic transformations, then no deformation of $G$ in $L^{n+1}$ contains impure parabolic elements.

For higher codimension, we give examples to show that one can deform a pure parabolic subgroup into an impure one. These examples also serve as examples for the following. For every $n\geq 4$, and for every positive $k\leq n-3$, there is a family of non-conjugate discrete subgroups $\{G_\alpha\}$ of $\tilde{L}^n$, with the following properties. The family is parametrized by $(S^1)^k$; for $n=4$ and 5 the $G_\alpha$ all have the same limit set, a Euclidean sphere of dimension $k$; for all $n\geq 4$ and for almost all $\alpha$, the stabilizer in $G_\alpha$ of the hyperbolic $(k+1)$-plane spanning the limit set is the identity (in particular, for almost all $\alpha$, $G_\alpha$ is not conjugate in $\tilde{L}^n$ to a subgroup of $\tilde{L}^{k+1}$); and all the $G_\alpha$ have the same finite sided fundamental polyhedron, with the same combinatorial identifications.

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Examples of distinct discrete groups with the same fundamental polyhedron were first given by Wielenberg [3].

**2.** We start with a canonical extension of $L^n$ into $L^{n+1}$, and a corresponding extension of fundamental polyhedrons in $H^n$ to fundamental polyhedrons in $H^{n+1}$.

Consider $H^n$ as a subset of $\hat{E}^n$, then an element of $L^n$ is a conformal homeomorphism of $\hat{E}^n = \partial H^{n+1}$. Every conformal homeomorphism of $\hat{E}^n$ has a unique extension to an isometry of $H^{n+1}$.

Similarly, let $D^n$ be a fundamental poly hedron in $H^n$ for the discrete subgroup $G$ of $L^n$. Each side $s$ of $D^n$ lies on a hyperbolic hyperplane in $H^n$, which in turn lies on a hypersphere $H$ in $\hat{E}^n$. There is a unique hyperbolic hyperplane $P$ in $H^{n+1}$ whose Euclidean boundary is $H$. The union of all the hyperspheres corresponding to the sides of $D^n$ separates $\hat{E}^n$ into some number of components; one or two of them contain $D^n$ and its reflection in $\partial H^n$. Call this component, or union of two components, $\tilde{D}^n$. There is a unique polyhedron $D^{n+1}$ in $H^{n+1}$ whose boundary is $\tilde{D}^n$. Each side $s^{n+1}$ of $D^{n+1}$ contains, in its boundary, a side $s^n$ of $D^n$; if $f \in G$ maps $s^n$ onto $(s')^n$, then $f$, as an element of $L^{n+1}$, also maps $s^{n+1}$ onto $(s')^{n+1}$. It is clear that $D^{n+1}$ is a fundamental polyhedron for the action of $G$ on $H^{n+1}$.

**3.** For our first example, we start with a finitely generated torsion-free Fuchsian group $G$, of the first kind. We assume that $G$ has signature $(p, n)$, $n \neq 1$. Let $A_1, B_1, \ldots, A_p, B_p, C_1, \ldots, C_n$ be a standard set of generators for $G$, where the $A_m$ and $B_m$ are hyperbolic, the $C_m$ are parabolic, and these generators satisfy the one defining relation: $\prod [A_m, B_m] \prod C_k = 1$. Let $D^2$ be a fundamental polygon in $H^2$ for $G$, where the sides of $D^2$ are identified precisely by these generators.

Let $D^3$ and $D^4$ be the extensions of $D^2$ to $H^3$ and $H^4$ as above; we regard $G$ as acting on $H^4$ (i.e., $H^2 = \{(x, t) | t > 0\} = \{(x, t, 0, 0) \in H^4\}$. For every $\theta$, let $r(\theta)$ be the rotation of $E^3$, through an angle $\theta$, about the axis $R = \{x_2 = x_3 = 0\}$. Note that $\hat{R} = R \cup \{\infty\} = \Lambda(G)$, the limit set of $G$. Hence, as maps of $\hat{E}^3$, and also $H^4$, every such rotation commutes with every element of $G$.

Let $g$ be a generator of $G$, where $g$ maps the side $s$ of $D^2$ to the side $s'$. The extension of $D^2$ to $\hat{E}^3$, call it $\hat{D}^3$, can be obtained from $D^2$ by rotating the boundary about the line $R$. Call the sides of $\hat{D}^3$ by the same names as the corresponding sides of $D^2$. One sees at once that for every $\theta$, $r(\theta) \circ g(s) = s'$.

Choose arbitrary angles of rotation $\theta_m$, $\theta'_m$, $m = 1, \ldots, p$, and $\varphi_m$, $m = 1, \ldots, n$, where $\sum \varphi_m = 0$. Define the homomorphism $\rho: G \to L^4$ by $\rho(A_m) = r(\theta_m) \circ A_m$, $\rho(B_m) = r(\theta'_m) \circ B_m$, and $\rho(C_m) = r(\varphi_m) \circ C_m$. The sides of $D^4$ are pairwise identified by the generators of $\rho(G)$; the one relation is satisfied, and it is easy to see that $\rho(C_m)$ is again parabolic. Hence, by Poincaré's polyhedron theorem, (see [2] for proof), $\rho(G)$ is discrete, and $\rho$ is an isomorphism.

We remark that, independent of the choice of rotations, $\rho(G)$ and $G$ have the same limit set, $\hat{R}$.

**4.** For our next example, let $b_1, \ldots, b_{n-1}$ be the standard basis vectors in $E^{n-1}$, and let $\hat{b}_1, \ldots, \hat{b}_{n-1}$ be the standard dual basis. Let $j_m(x) = x + \sqrt{2} b_m$, and let $J =$

$\langle j_1,\ldots, j_{n-1}\rangle$, the group generated by the $j_m$. In $E^{n-1}$, choose the standard fundamental polyhedron for $J$ to be bounded by the hyperplanes $s_m=\{x|\hat{b}_m(x)=1/\sqrt{2}\}$, and $s'_m=\{x|\hat{b}_m(x)=-1/\sqrt{2}\}$. These hyperplanes $s_m$ and $s'_m$ have canonical extensions to $H^n$, where they bound a fundamental polyhedron for $J$. We call the extensions and the original hyperplanes by the same names.

Let $g$ be the reflection in the unit sphere in $E^n$; i.e., $g(x)=x/|x|^2$, and let $s_{2n-1}$ be the intersection of the Euclidean unit sphere in $E^n$ with $H^n$. The hyperbolic hyperplanes $s_1, s'_1,\ldots, s_{n-1}, s'_{n-1}$, and $s_{2n-1}$ bound a polyhedron $D^n\subset H^n$, whose sides are identified by the elements $j_1,\ldots, j_{n-1}$, and $g$; that is, if we denote the side of $D^n$ which lies on $s_k$ by $\tilde{s}_k$, then $j_m(\tilde{s}_m)=\tilde{s}'_m$, and $g(\tilde{s}_{2n-1})=\tilde{s}_{2n-1}$. Let $G=\langle J, g\rangle$, the group generated by $J$ and $g$. Of course, $g$ is an involution, and one easily sees that $G$ has the relations indicated by the sum of the angles at each cycle of edges of $D^n$; that is, $G$ satisfies the relations: $j_m\circ j_k=j_k\circ j_m$, $g^2=1$, $(j_m\circ g\circ j_m^{-1}\circ g)^2=1$. Hence by Poincaré's polyhedron theorem, $G$ is discrete, and we have listed all the relations in $G$.

Extend the action of $G$ to $E^{n+1}$, and write a point in $E^{n+1}$ as $(x, z)$, where $x\in E^{n-1}$, and $z\in \boldsymbol{C}$. Define the rotation $r(\theta)$ by $r(\theta)(x, z)=(x, e^{i\theta}z)$. Choose some set of rotation angles $\theta_1,\ldots, \theta_{n-1}$, and set $\tilde{j}_m=r(\theta_m)\circ j_m$, where, as usual, $j_m(x, z)=(j_m(x), z)$. Then, using the canonical extension, $\tilde{G}=\langle \tilde{j}_1,\ldots, \tilde{j}_{n-1}, g\rangle$ acts on $H^{n+2}$. Use the canonical extension of $D^n$ to $D^{n+2}$, and observe that, since every $r(\theta)$ commutes with every $j_m$ and with $g$, the sides of $D^{n+2}$ are identified by the generators of $\tilde{G}$. One easily verifies that the map $\rho$: $G\to\tilde{G}$, mapping $j_m$ to $\tilde{j}_m$, and $g$ to $g$, is a homomorphism. Hence we can again apply Poincaré's theorem to conclude that $\tilde{G}$ is discrete, that $\rho$ is an isomorphism, and that $D^{n+2}$ is a fundamental polyhedron for the action of $\tilde{G}$ on $H^{n+2}$.

Of course $D^{n+2}$ is also a fundamental polyhedron for the action of $G$ on $H^{n+2}$, so we have produced a deformation space of discrete groups, of real dimension $n-1$, where all the groups in the space have the same fundamental polyhedron, with the same combinatorial identifications of the sides. It is clear that the orientation preserving halves of these groups have the same properties.

**5.** Now let $G$ be a discrete subgroup of $L^n$ of cofinite volume, where every parabolic element of $G$ is pure, and let $\tilde{G}$ be a deformation of $G$ in $L^{n+1}$. We outline a proof of the fact that every parabolic element of $\tilde{G}$ is pure.

We start with the remark that every element of $L^n$ either has a fixed point in the interior of $H^n$ (i.e., it is elliptic), or it has exactly one fixed point on the sphere at $\infty$ (i.e., it is parabolic), or it has exactly two fixed points on the sphere at $\infty$ (i.e., it is loxodromic).

It is easy to see that since $G$ has cofinite volume, and every parabolic element of $G$ is pure, $G$ is either cocompact, in which case it contains no parabolic elements, or every subgroup of $G$ containing only parabolic elements is free Abelian, of rank $n-1$. Hence it suffices to prove that if $G$ is a group of Euclidean isometries, acting on $E^n$, where $G$ is free Abelian of rank $n-1$, then every element of $G$ is pure.

We next remark that we can normalize $G$ so that for some parabolic element

$j(x)=r(x)+b$, $r(b)=b$. Let $j'(x)=r(x)+b'$ be some parabolic element of $G$. Since $j'$ has no finite fixed point, $b'$ is not in the range of $1-r$. Since the range and null-space of $1-r$ are orthogonal, there is a translation $t$, so that $j=t\circ j'\circ t^{-1}$ has the form $j(x)=r(x)+b$, where $r(b)=b$.

Choose a set of free generators $j_1,\dots,j_{n-1}$ for $G$; write $j_m(x)=r_m(x)+b_m$, and normalize so that $r_1(b_1)=b_1$. Assume that $r_1\neq 1$.

Since $j_1$ and $j_m$ commute, $r_1$ and $r_m$ commute, and we also obtain

$$(r_1-1)(b_m)=(r_m-1)(b_1).$$

Notice that

$$\begin{aligned}(r_1-1)^2(b_m)&=(r_1-1)(r_m-1)(b_1)\\&=(r_m-1)(r_1-1)(b_1)\\&=0.\end{aligned}$$

Since the range and nullspace of $r_1-1$ are orthogonal, $r_1(b_m)=b_m$.

Let $N_1$ be the nullspace of $r_1-1$. We have shown that every $b_m\in N_1$, and one easily sees that $N_1$ is preserved by every $r_m$. Hence $G$ acts on $N_1$, and, in its action on $N_1$, $j_1$ is pure. Regard $N_1$ as a Euclidean space, note that $G$ acts as a group of parabolic elements on $N_1$, normalize so that $r_2(b_2)=b_2$, note that this renormalization leaves $j_1$ as a pure translation, and continue inductively. At the end, we have an $m$-dimensional Euclidean space $N$, on which $G$ acts as a group of pure translations.

Since every element of $G$ preserves orientation, the codimension of $N_1>1$; i.e., the dimension of $N$ is $\leq n-2$. Also since for each $j_m$, $b_m$ is not in the range of $1-r_m$, the dimension of $N$ is $\geq 1$. Since $G$ is discrete, and of rank $n-1$, there is a non-trivial element $j\in G$ with $j|N=1$.

Write $E^n=N\oplus M$, where $M$ is orthogonal to $N$. Then $j$ stabilizes both $N$ and $M$, and, since all the vectors $b_m$ lie in $N$, $j|M$ is orthogonal, and so $j$ has finite order. This contradicts the assumption that $J$ is free Abelian.

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