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Half-canonical divisors on variable Riemann surfaces

To Professor Yukio Kusunoki on his sixtieth birthday

By

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1. Introduction and first statement of main result.

1.1. The study of holomorphic sections of the Teichmüller curve was initiated by Hubbard [16] (see also Earle-Kra [7], [8]). Hubbard proved that for $p=2$ the map $\pi\colon V_p \to T_p$ has precisely 6 holomorphic sections, the Weierstrass sections, while for $p \geqq 3$, π has no holomorphic sections. The base space T_p is a contractible domain of holomorphy. Nevertheless, Hubbard's result shows that for $p>2$ we cannot choose a point on each surface in a way that depends holomorphically on moduli. On the other hand, we can choose on every surface of genus $p \geqq 2$ a divisor class of degree one that depends holomorphically on moduli (see [5]).

1.2. Let n be a positive integer. Let $\pi_n\colon S_T^n(V_p) \to T_p$ be the fiber space whose fiber over $t \in T_p$ is $\pi_n^{-1}(t) = S^n(X_t)$, the n-fold symmetric product of the Riemann surface $X_t = \pi^{-1}(t)$ represented by t (see §§2 and 3 for details). The points of $S^n(X_t)$ can be identified with the integral divisors of degree n on X_t. A holomorphic section of π_n corresponds to a choice on each surface of an integral divisor of degree n that depends holomorphically on moduli.

In this paper we concentrate on the case $n=p-1$. For $n<p$ every divisor $D \in S^n(X)$ on a compact Riemann surface X of genus p is *special* in the sense that there exists on X a nontrivial abelian differential of the first kind that vanishes on D ($p-1$ is the largest integer with this property). A divisor $D \in S^{p-1}(X)$ is *half-canonical* if $2D$ is the divisor of a nontrivial abelian differential of the first kind. Similarly, a section s of π_{p-1} is *half-canonical* if $s(t)$ is a half-canonical divisor for all $t \in T_p$. We can now state our main result as

Theorem 1. *The map $\pi_{p-1}\colon S_T^{p-1}(V_p) \to T_p$, $p \geqq 2$, has a half-canonical holomorphic section if and only if $p=2$, 3, or 4. The number of such sections is precisely 6 for $p=2$, 28 for $p=3$, and 120 for $p=4$; that is, precisely the number of odd half-periods in the Jacobi variety.*

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Bers [1] has constructed holomorphic sections of π_{2p-2}. The general problem of determining all sections of π_n for arbitrary n is open and apparently quite difficult. In particular, it would be of interest to determine the lowest value of n for which π_n has a holomorphic section.[2]

1.3. The study of sections of $\pi=\pi_1$ involved the Kobayashi metric on Teichmüller space. In the present study, line bundles over the Teichmüller curve, Jacobi varieties, and the Riemann θ-function play a crucial role. In order to explain that role clearly we have devoted most of §§2, 3, and 4 to exposition of rather standard results. In addition, our method of proof leads us to consider Riemann surfaces with nodes (as introduced, for example, by Bers [3]), and §7 includes a discussion of some properties of the Bers deformation space. All this expository material has increased the length of this paper, but we trust it may be useful in other connections.

The authors thank Hershel Farkas for many fruitful discussions on classical function theory.

2. Holomorphic families, divisors, and line bundles.

2.1. The material in this section is standard, but we find it convenient to summarize it here. We start with a holomorphic family $\pi\colon V\to B$ of closed Riemann surfaces of genus $p\geqq 2$. This means that V and B are connected complex manifolds, π is a proper holomorphic map of V onto B, the derivative of π is surjective at every point of V, and each fiber $X_t=\pi^{-1}(t)$, $t\in B$, is a closed Riemann surface of genus p. Let n be any positive integer, and let $S^n(X_t)$ be the complex manifold obtained by taking the quotient of the n-fold Cartesian product X_t^n by the obvious action of the permutation group $\sum(n)$. Our first goal is to define a complex manifold $S_B^n(V)$ and a proper holomorphic map $\pi_n\colon S_B^n(V)\to B$ so that $\pi_n^{-1}(t)=S^n(X_t)$. The first step is to form

$$V_B^n=\{(x_1,\dots,x_n)\in V^n;\ \pi(x_i)=\pi(x_j) \quad \text{for} \quad i,j=1,\dots,n\}.$$

It is easy to verify that V_B^n is a complex manifold, that the map $\omega_n\colon V_B^n\to B$ defined by

$$\omega_n(x_1,\dots,x_n)=\pi(x_1) \qquad \text{for all} \quad (x_1,\dots,x_n)\in V_B^n$$

is proper and holomorphic, with a surjective derivative at every point, and that $\omega_n^{-1}(t)=X_t^n$ for all t in B.

The left action of $\sum(n)$ on V_B^n is given by $\sigma(x_1,\dots,x_n)=(x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(n)})$ for all $\sigma\in\sum(n)$ and $(x_1,\dots,x_n)\in V_B^n$.

Proposition. *The quotient space $S_B^n(V)=V_B^n/\sum(n)$ is a complex manifold. The map $\omega_n\colon V_B^n\to B$ induces a proper holomorphic map $\pi_n\colon S_B^n(V)\to B$. The derivative of π_n is surjective at every point, and $\pi_n^{-1}(t)=S^n(X_t)$ for all t in B.*

Proof. $S_B^n(V)$ is the quotient of the complex manifold V_B^n by a finite group of

2) The claim in [18] that π_{p-1} always has holomorphic sections was too optimistic. Such a result remains unknown.

biholomorphic maps. Such a quotient space is a complex manifold if the stabilizer of every point is generated by transformations whose fixed point set has codimension one [13, Satz 1]. If $x \in V_B^n$, the stabilizer of x in $\sum(n)$ is generated by the set of transpositions in $\sum(n)$ that fix x, and it is easy to see that the fixed point set of a transposition has codimension one in V_B^n.

The holomorphic functions on $S_B^n(V)$ are precisely the $\sum(n)$-invariant holomorphic functions on V_B^n. Since ω_n: $V_B^n \to B$ is holomorphic and $\sum(n)$-invariant, it induces a holomorphic map π_n: $S_B^n(V) \to B$. Since ω_n is proper and has a surjective derivative at every point, the same is true of π_n. Finally, $\pi_n^{-1}(t)$ is the quotient of $\omega_n^{-1}(t) = X_t^n$ by $\sum(n)$, which is $S^n(X_t)$.

2.2. A point of $S_B^n(V)$ in the fiber $\pi_n^{-1}(t) = S^n(X_t)$ determines an (unordered) n-tuple $(x_1, \dots, x_n)$ of points of X_t. This in turn determines an integral divisor $x_1 + \cdots + x_n$ on the Riemann surface X_t, and one can construct in the usual way a line bundle L_t on X_t having a holomorphic section with the given divisor. We shall now describe a similar correspondence between holomorphic sections of π_n: $S_B^n(V) \to B$ and the divisors of certain holomorphic sections of line bundles on V.

Let ω: $L \to V$ be a holomorphic line bundle on V. We call the holomorphic section s: $V \to L$ a *nontrivial relative section* if the restriction of s to each fiber X_t is nontrivial. If ω_1: $L_1 \to V$ and ω_2: $L_2 \to V$ are line bundles with sections s_1 and s_2, we call s_1 and s_2 *equivalent* if and only if there is a bundle isomorphism f: $L_1 \to L_2$ such that $f \circ s_1 = s_2$ (or, equivalently, s_1/s_2 defines a nonvanishing holomorphic section of the line bundle $L_1 \otimes L_2^{-1}$).

Proposition. *Let s be a nontrivial relative section of the line bundle ω: $L \to V$, and let n ($\geqq 0$) be the number of zeros of s in some fiber $\pi^{-1}(t_0)$. Then s has exactly n zeros in each fiber $\pi^{-1}(t)$. If $n > 0$, the divisor of s determines a holomorphic section $\sigma(s)$ of π_n: $S_B^n(V) \to B$. The sections s_1 and s_2 determine the same σ if and only if they are equivalent.*

Proof. Let x be any zero of s on $\pi^{-1}(t_0)$, and let its order be $m > 0$. Choose an open neighborhood U_x of x in V with these properties:

(i) there are local coordinates (t, z) in U_x so that $\pi(t, z) = t$ and x has coordinates $(t_0, 0)$,

(ii) the line bundle L is trivial over U_x, so that s is defined in U_x by a function $f(t, z)$,

(iii) in U_x we can write $f = gh$, where $h(t, z)$ is nonzero and g is a Weierstrass polynomial

$$g(t, z) = z^m + a_1(t) z^{m-1} + \cdots + a_m(t) \tag{2.1}$$

with $a_i(t_0) = 0$ for all $i \geqq 1$,

(iv) if $t \in \pi(U_x)$, then $(t, z) \in U_x$ whenever $z \in \boldsymbol{C}$ and $g(t, z) = 0$.

Since s has only finitely many zeros in the fiber $\pi^{-1}(t_0)$, we can and do require that the sets U_x corresponding to different zeros be disjoint.

Let U be the union of the sets U_x and the set of all points of V where $s \neq 0$.

Since proper maps are closed and U is an open neighborhood of $\pi^{-1}(t_0)$, there is an open neighborhood D of t_0 in B so that $\pi^{-1}(D)\subset U$. If $t\in D$, the zeros of s on $X_t=\pi^{-1}(t)$ all belong to one of the disjoint sets U_x, and (iv) says their total number is independent of t. The number of zeros of s on X_t is thus a locally constant function of t, hence constant.

If s has $n>0$ zeros in each fiber, we define $\sigma\colon B\to S_B^n(V)$ by putting $\sigma(t)$ equal to the (unordered) n-tuple of zeros of s in the fiber X_t. (A zero of order m is of course listed m times.) It is clear that equivalent sections s_1 and s_2 produce the same map σ. Conversely, if s_1 and s_2 determine the same σ, then s_1/s_2 defines a nonvanishing holomorphic section of the line bundle $L_1\otimes L_2^{-1}$, so s_1 and s_2 are equivalent.

It remains to prove that the map $\sigma\colon B\to S_B^n(V)$ is holomorphic. For this we need local coordinates in $S_B^n(V)$. Fix $t_0\in B$, and at each zero of s in the fiber $\pi^{-1}(t_0)$ choose local coordinates (t, z) as above. Local coordinates for $S_B^n(V)$ at $\sigma(t_0)$ are given by t and appropriate combinations of the "fiber coordinates" z. If $(t_0, 0)$ corresponds to a zero x of order m, so that x occurs m times in the unordered n-tuple $\sigma(t_0)$, we need m coordinate functions that are symmetric functions of the variables $z_1,\dots, z_m$. It is convenient to use the symmetric functions $a_1,\dots, a_m$ defined by

$$z^m+a_1z^{m-1}+\cdots+a_m=\prod_{j=1}^{m}(z-z_j). \tag{2.2}$$

It is now clear that σ is a holomorphic function of t, since its description in these coordinates is given by $a_j=a_j(t)$, for all j, where the holomorphic functions $a_j(t)$ are defined by (2.1). The proof is complete.

2.3. Now we shall prove that every holomorphic section $\sigma\colon B\to S_B^n(V)$ can be produced in the above way.

Proposition. *Let σ be a holomorphic section of $\pi_n\colon S_B^n(V)\to B$. There exist a line bundle $\omega\colon L\to V$ and a nontrivial relative section $s\colon V\to L$ such that s determines σ.*

Proof. First we need to obtain a divisor on V from σ. We shall do this by covering V with open sets U_j and defining holomorphic functions f_j in U_j so that f_j/f_k is holomorphic and never zero in $U_j\cap U_k$ and so that the zeros of the functions f_j are exactly at the points of V determined by the image of σ. We observe first that the image of σ determines a closed set C in V. Indeed $\sigma(B)$ is a closed subset of $S_B^n(V)$, and if $q\colon V_B^n\to S_B^n(V)=V_B^n/\sum(n)$ is the quotient map, then $q^{-1}(\sigma(B))$ is closed in V_B^n. Therefore C is closed in V, since it is the image of $q^{-1}(\sigma(B))$ under the closed map $(x_1,\dots, x_n)\mapsto x_1$ of V_B^n onto V.

For $x\in C$ we choose as before an open neighborhood U_x of x in V with local coordinates (t, z) so that $\pi(t, z)=t$ and x has coordinates $(t_0, 0)$. We also choose as before local coordinates for $S_B^n(V)$ at $\sigma(t_0)$ so that if x occurs m times in the unordered n-tuple $\sigma(t_0)$, then the coordinate functions include the functions $a_1,\dots, a_m$ defined by (2.2). The holomorphic section $\sigma(t)$ determines holomorphic functions $a_1(t),\dots, a_m(t)$, and we define the holomorphic function f on U_x by

(2.3) $$f(t, z) = z^m + a_1(t)z^{m-1} + \cdots + a_m(t).$$

That determines our divisor in a neighborhood of every point of C, and to determine it on the rest of V we use the constant function $f=1$, defined in the open set $V \backslash C$.

Now any open cover of V by sets U_j in which holomorphic functions f_j are defined so that $f_{jk} = f_j / f_k$ is holomorphic and never zero in $U_j \cap U_k$ allows us to use the f_{jk} as transition functions to define a holomorphic line bundle $L \to V$ with a holomorphic section $s\colon V \to L$ whose local description in U_j is given by f_j. Our divisor therefore determines a line bundle $L \to V$ with a section s that has no zeros in $V \backslash C$ and that has local description (2.3) in U_x if $x \in C$. Comparison of (2.1) and (2.3) shows that s determines the section $\sigma\colon B \to S_B^n(V)$ with which we started. The proof is complete.

2.4. If we call the holomorphic sections $\sigma\colon B \to S_B^n(V)$ *positive relative divisors*, the results of this section can be summarized by the statement that the positive relative divisors are precisely the divisors of the nontrivial relative sections of line bundles $L \to V$.

3. Line bundles over the Teichmüller curve; restatement of Theorem 1.

3.1. The general considerations of section two can of course be applied to the Teichmüller curve. We start by recalling some of its basic properties, established by Bers [2]. For any integer $p \geqq 2$, let T_p be the Teichmüller space of closed Riemann surfaces of genus p. T_p is a complex manifold of dimension $3p-3$ and can be embedded in $\boldsymbol{C}^{3p-3}$ as a bounded contractible domain of holomorphy.

Let Γ be the fundamental group of a fixed orientable closed surface X of genus p. The *Bers fiber space* over T_p is a subregion F_p of $T_p \times \boldsymbol{C}$ with these properties:

(i) F_p can be embedded in $\boldsymbol{C}^{3p-2}$ as a bounded contractible domain of holomorphy;

(ii) Γ acts freely and properly discontinuously on F_p as a group of biholomorphic maps

(3.1) $$\gamma(t, \zeta) = (t, \gamma^t(\zeta)) \qquad \text{for all} \quad \gamma \in \Gamma \quad \text{and} \quad (t, \zeta) \in F_p,$$

and $\zeta \mapsto \gamma^t(\zeta)$ is a Möbius transformation for every t in T_p:

(iii) $D(t) = \{\zeta \in \boldsymbol{C};\ (t, \zeta) \in F_p\}$ is a Jordan region in $\boldsymbol{C}$ for every t in T_p, and the quotient space $D(t)/\Gamma$ is the closed Riemann surface represented by the point t;

(iv) the projection $(t, \zeta) \mapsto t$ of F_p onto T_p induces a holomorphic map π from the quotient manifold $V_p = F_p/\Gamma$ onto T_p, and $\pi\colon V_p \to T_p$ is a holomorphic family of closed Riemann surfaces of genus p, which we call the *Teichmüller curve* (of genus p).

Properties (i), (ii), and (iii) of F_p were proved by Bers [2], and (iv) follows easily from the results of [2] although it is neither stated nor proved there. For a proof of (iv) in a more general setting see Theorem 1 (a) of [6].

3.2. As in §2, there is a canonical correspondence between nontrivial relative sections s of line bundles over V_p and holomorphic sections of $\pi_n\colon S_T^n(V_p) \to T_p$. The complex manifold $S^n_T(V_p)$ — we write T instead of T_p for obvious typographical reasons — can be described explicitly in terms of the fiber space F_p in the following way.

Since V_p is the quotient of F_p by the action (3.1) of the group Γ, $(V_p)^n_T$ is the quotient of the complex manifold

$$F_n(T_p)=\{(t,\zeta)\in T_p\times \mathbf{C}^n;\ \zeta=(\zeta_1,\dots,\zeta_n)\ \text{ and }\ (t,\zeta_j)\in F_p\ \text{ for }\ j=1,\dots,n\}$$

by the free, properly discontinuous action of the group Γ^n, acting in the obvious way:

$$(\gamma_1,\dots,\gamma_n)(t,\zeta_1,\dots,\zeta_n)=(t,\gamma_1^t(\zeta_1),\dots,\gamma_n^t(\zeta_n)).$$

The action of $\sum(n)$ on $(V_p)^n_T$ lifts to the action

$$\sigma(t,\zeta_1,\dots,\zeta_n)=(t,\zeta_{\sigma^{-1}(1)},\dots,\zeta_{\sigma^{-1}(n)})$$

on $F_n(T_p)$. The groups Γ^n and $\sum(n)$ generate a properly discontinuous group G_n of biholomorphic mappings of $F_n(T_p)$. The quotient space $F_n(T_p)/G_n$ is $S^n_T(V_p)$, and we can identify the set of holomorphic functions on $S^n_T(V_p)$ with the set of G_n-invariant holomorphic functions on $F_n(T_p)$.

3.3. The line bundles over V_p and their relative sections can also be described in terms of F_p. Every line bundle over V_p can be lifted to a line bundle over F_p. The Bers isomorphism theorem [2] asserts that F_p is biholomorphically equivalent to the contractible domain of holomorphy $T_{p,1}$ (the Teichmüller space of closed Riemann surfaces of genus p with one distinguished point). All line bundles over F_p are therefore trivial, and it follows (see Gunning [14]) that all line bundles over V_p are determined by "factors of automorphy" on F_p. Recall that a factor of automorphy is a map $\xi\colon \Gamma\times F_p\to \mathbf{C}$ such that $\xi(\gamma,\cdot)$ is a nowhere vanishing holomorphic function on F_p for each $\gamma\in\Gamma$, and

$$\xi(\gamma_1\gamma_2,z)=\xi(\gamma_1,\gamma_2(z))\xi(\gamma_2,z) \tag{3.2}$$

for all $\gamma_1,\gamma_2\in\Gamma$ and $z=(t,\zeta)\in F_p$. The holomorphic sections of the line bundle determined by ξ are described by the *ξ-automorphic functions* $f\colon F_p\to\mathbf{C}$. These are the holomorphic functions f that satisfy

$$f(\gamma z)=\xi(\gamma,z)f(z) \qquad \text{for all }\ \gamma\in\Gamma\ \text{ and }\ z=(t,\zeta)\in F_p. \tag{3.3}$$

f describes a nontrivial relative section if and only if for each $t\in T_p$ the function $f(t,\cdot)$ on $D(t)$ is not identically zero.

3.4. In terms of line bundles, Theorem 1 takes the following equivalent form.

Theorem 2. *If $2\leqq p\leqq 4$, there are $2^{p-1}(2^p-1)$ line bundles $L\to V_p$ with nontrivial relative sections whose divisor on every fiber is half-canonical. If $p\geqq 5$, there are no such line bundles.*

It is this formulation of the theorem that we will prove.

4. Half-canonical divisors and Riemann's theta function.

4.1. We need to review some classical theorems about Jacobi varieties and theta functions. For details we refer to the books of Farkas and Kra [11, Chapter VI] and Fay [12, Chapter 1]. Let $A_1,\dots, A_p, B_1,\dots, B_p$ be a canonical homology basis for the closed Riemann surface X of genus $p\geqq 2$, and let $\omega_1,\dots,\omega_p$ be a basis for the space $H^{1,0}(X)$ of holomorphic 1-forms on X that satisfies

$$\int_{A_k}\omega_j=\delta_{jk} \quad \text{if} \quad 1\leqq j\leqq p \quad \text{and} \quad 1\leqq k\leqq p.$$

The *Riemann period matrix* $\tau=(\tau_{jk})$ of X (with respect to the given homology basis) is the $p\times p$ matrix with

$$\tau_{jk}=\int_{B_k}\omega_j \quad \text{if} \quad 1\leqq j\leqq p \quad \text{and} \quad 1\leqq k\leqq p. \tag{4.1}$$

The columns of τ and of the $p\times p$ identity matrix I are linearly independent over $\boldsymbol{R}$ and generate a lattice subgroup of $\boldsymbol{C}^p$ whose members are the column vectors

$$(I, \tau)N, \quad N\in Z^{2p}. \tag{4.2}$$

The quotient of $\boldsymbol{C}^p$ by that lattice is a complex torus $J(X)$, the *Jacobi variety* of X. It is both a complex manifold and an additive group. If $\lambda\in \boldsymbol{C}^p$, we denote by $[\lambda]$ its image in $J(X)$.

If $x_0\in X$ is any fixed point, the multivalued "function" from X to $\boldsymbol{C}^p$ whose j-th component at $x\in X$ is

$$\int_{x_0}^{x}\omega_j$$

induces a well defined holomorphic map $\phi\colon X\to J(X)$ with the property that $\phi(x_0)=0$. Using the group structure of $J(X)$ we obtain holomorphic maps $\phi_n\colon S^n(X)\to J(X)$, $n\geq 1$, given by

$$\phi_n(x_1,\dots, x_n)=\phi(x_1)+\cdots+\phi(x_n).$$

The sets $W_n=\phi_n(S^n(X))$, $n\geqq 1$, are analytic subvarieties of $J(X)$.

If $\omega\in H^{1,0}(X)$ is not identically zero, its divisor has degree $2p-2$ and determines a point (ω) in $S^{2p-2}(X)$. Its image $\phi_{2p-2}((\omega))$ is called the *canonical point* of $J(X)$. The canonical point is independent of the choice of ω, but it does depend on the basepoint x_0, so we denote it by $K(x_0)$. The maps ϕ_n and the sets W_n also depend on x_0, but our notation suppresses that dependence.

4.2. Let $\mathscr{H}_p$ be the Siegel half space of symmetric $p\times p$ matrices with positive definite imaginary parts. The *Riemann theta function* $\theta\colon \mathscr{H}_p\times \boldsymbol{C}^p\to \boldsymbol{C}$ is defined by

$$\theta(\tau, z)=\sum_{N\in Z^p}\exp\left[\pi i({}^tN\tau N+2{}^tNz)\right], \quad (\tau, z)\in\mathscr{H}_p\times \boldsymbol{C}^p.$$

(As usual, ${}^tN$ denotes the transpose of the matrix N.) The Riemann period matrix $\tau \in \mathscr{H}_p$, so we can consider the set

$$\{z \in \mathbf{C}^p;\ \theta(\tau, z) = 0\}.$$

That set is invariant under the lattice (4.2), so it projects to a closed analytic subvariety $\Theta \subset J(X)$.

According to a remarkable theorem of Riemann, there is a unique point $k(x_0)$ in $J(X)$ such that

$$W_{p-1} = \Theta + k(x_0). \tag{4.3}$$

We shall need two additional properties of $k(x_0)$. First, $2k(x_0) = K(x_0)$. Second, if $\lambda \in \mathbf{C}^p$ and $[\lambda] + k(x_0) \in W_{p-1}$, then the set $\phi_{p-1}^{-1}([\lambda] + k(x_0))$ contains more than one point in $S^{p-1}(X)$ if and only if the holomorphic 1-form

$$\omega = \sum_{j=1}^{p} \frac{\partial \theta}{\partial z_j}(\tau, \lambda)\omega_j \tag{4.4}$$

is identically zero. Further, if ω is not identically zero, $D \in S^{p-1}(X)$, and $\phi_{p-1}(D) = [\lambda] + k(x_0)$, then ω generates the space of holomorphic 1-forms on X whose divisors are $\geqq D$. (See Lewittes [19] and Farkas [10] for details.)

4.3. It is important to understand the dependence of $k(x_0)$ on x_0. If ϕ and W_{p-1} are defined using the basepoint x_0, the effect of replacing x_0 by x_1 is to replace ϕ by $x \mapsto \phi(x) - \phi(x_1)$ and W_{p-1} by $W_{p-1} - (p-1)\phi(x_1)$. Using (4.3) twice we get

$$\Theta + k(x_1) = W_{p-1} - (p-1)\phi(x_1) = \Theta + k(x_0) - (p-1)\phi(x_1),$$

so $k(x_1) = k(x_0) - (p-1)\phi(x_1)$. Hence the map

$$x \mapsto k(x) = k(x_0) - (p-1)\phi(x)$$

is holomorphic, and there is a holomorphic map $\psi\colon X \to J(X)$ such that

$$k(x) = (1-p)\psi(x) \qquad \text{for all} \quad x \in X. \tag{4.5}$$

Now $(1-p)(\psi(x) - \phi(x)) = k(x_0)$ is constant, so $\psi(x) - \phi(x)$ is also constant, and $\phi(x) = \psi(x) - \psi(x_0)$.

As in §4.1, there are holomorphic maps $\psi_n\colon S^n(X) \to J(X)$, $n \geqq 1$, given by

$$\psi_n(x_1, \ldots, x_n) = \psi(x_1) + \cdots + \psi(x_n),$$

and $\psi_n(D) = \phi_n(D) + n\psi(x_0)$ for all $D \in S^n(X)$.

Proposition. *If $\psi\colon X \to J(X)$ is holomorphic and satisfies* (4.5), *then*

$$\psi_{p-1}(S^{p-1}(X)) = \Theta, \tag{4.6}$$

and $D \in S^{2p-2}(X)$ is a canonical divisor if and only if $\psi_{2p-2}(D) = 0$. Moreover, if $\lambda \in \mathbf{C}^p$ and $[\lambda] \in \Theta$, then $\psi_p^{-1}{}_{-1}([\lambda])$ contains more than one point of $S^{p-1}(X)$ if and only if the 1*-form ω defined by* (4.4) *is identically zero. Finally, if ω is not identi-*

cally zero, $D \in S^{p-1}(X)$, and $\psi_{p-1}(D)=[\lambda]$, then ω generates the space of holomorphic 1-forms on X whose divisors are $\geqq D$.

Proof. This is merely a restatement of properties of $k(x_0)$ from §4.2. Indeed, by (4.5), $\psi_{p-1}(D)=\phi_{p-1}(D)-k(x_0)$, so

$$\psi_{p-1}(S^{p-1}(X))=W_{p-1}-k(x_0)=\Theta,$$

by (4.3). Similarly, the divisor $D \in S^{2p-2}(X)$ is canonical if and only if

$$K(x_0)=\phi_{2p-2}(D)=\psi_{2p-2}(D)-(2p-2)\psi(x_0)$$
$$=\psi_{2p-2}(D)+2k(x_0)=\psi_{2p-2}(D)+K(x_0),$$

and that happens if and only if $\psi_{2p-2}(D)=0$. Finally, the sets $\psi_{p-1}^{-1}([\lambda])$ and $\phi_{p-1}^{-1}([\lambda])+k(x_0))$ are equal, so they bear the same relationship to the 1-form ω defined by (4.4). This proves the Proposition.

4.4. Now we turn our attention to half-canonical divisors. By the Proposition, if $D \in S^{p-1}(X)$ is half-canonical, then $2\psi_{p-1}(D)=\psi_{2p-2}(2D)=0$, so $\psi_{p-1}(D)$ belongs to the *group of half-periods*, which by definition consists of the points $t \in J(X)$ with $2t=0$. That group is isomorphic to $(Z/2Z)^{2p}$. In fact any half-period t can be written as $t=[\lambda]$, with

$$\lambda=\frac{1}{2}(I, \tau)N \qquad \text{for some} \quad N \in Z^{2p}, \tag{4.7}$$

and $[\lambda]=0 \in J(X)$ if and only if $N \in 2Z^{2p}$. We call $t=[\lambda]$ the half-period determined by $N \in Z^{2p}$ and we identify t with the corresponding element $\bar{N} \in (Z/2Z)^{2p}$. Write ${}^tN=({}^tU, {}^tV)$ with $U, V \in Z^p$. We call the half-period t or $\bar{N}$ *odd* or *even* according as the inner product $U \cdot V$ is odd or even.

Every odd half-period $t=[\lambda]$ belongs to the set Θ (see for instance Farkas-Kra [11, p. 286]), so the Proposition implies that $\psi_{p-1}^{-1}([\lambda])$ is a nonempty set of half-canonical divisors. That set contains only one divisor exactly when (4.4) defines a nontrivial 1-form on X.

4.5. We emphasize that all the above results are classical and go back to Riemann. To apply them to our problem we need to know how τ and ψ depend on moduli. For that purpose we shall use an explicit formula (due to Riemann) for ψ (see also [5]).

We recall from §3 that the Teichmüller curve V_p is the quotient of the Bers fiber space $F_p \subset T_p \times \boldsymbol{C}$ by the group Γ, and that each fiber X_t, $t \in T_p$, is the quotient by Γ of the Jordan region

$$D(t)=\{\zeta \in \boldsymbol{C};\ (t, \zeta) \in F_p\}.$$

The quotient map $D(t) \to X_t = D(t)/\Gamma$ is a univeral covering, and we choose generators $A_1, \dots, A_p, B_1, \dots, B_p$ for Γ that satisfy the defining relation

$$\prod_{j=1}^{p} A_j B_j A_j^{-1} B_j^{-1}=1$$

and determine a canonical homology basis on each X_t. The corresponding basis $\omega_1, \ldots, \omega_p$ for $H^{1,0}(X_t)$ lifts to a set of 1-forms $\alpha_j^t(\zeta)d\zeta$ in $D(t)$. The embedding ψ: $X_t \to J(X_t)$ lifts to a holomorphic map η: $D(t) \to \boldsymbol{C}^p$. Classical formulas ensure that ψ satisfies (4.5) if

$$(1-p)\eta_j(\zeta) = -\frac{1}{2}\tau_{jj} + \sum_{k=1}^{p} \int_{s=\zeta}^{A_k^t(\zeta)} ds \int_{u=\zeta}^{s} \alpha_j^t(u)\alpha_k^t(s)du, \quad 1 \leqq j \leqq p.$$

(Here τ_{jj} is defined by (4.1).)

By a theorem of Bers [1], there are holomorphic functions $\alpha_j(t, \zeta)$ on F_p, $1 \leqq j \leqq p$, such that $\alpha_j(t, \zeta) = \alpha_j^t(\zeta)$ if $\zeta \in D(t)$. These functions satisfy

$$\alpha_j(t, \zeta) = \alpha_j(\gamma(t, \zeta))\frac{\partial \gamma}{\partial \zeta}(t, \zeta) \qquad \text{for all} \quad \gamma \in \Gamma \tag{4.8}$$

and

$$\int_{\zeta}^{A_k^t(\zeta)} \alpha_j(t, u)du = \delta_{jk}$$

whenever $(t, \zeta) \in F_p$, $1 \leqq j \leqq p$, and $1 \leqq k \leqq p$. The 1-form $\alpha_j(t, \zeta)d\zeta$ is not closed in F_p, so the above integral and all similar integrals must be computed using paths in F_p along which t is constant. With that understanding, the functions τ_{jk}: $T_p \to \boldsymbol{C}$ and η: $F_p \to \boldsymbol{C}^p$ defined by

$$\tau_{jk}(t) = \int_{\zeta}^{B_k^t(\zeta)} \alpha_j(t, u)du, \; 1 \leqq j \leqq p \quad \text{and} \quad 1 \leqq k \leqq p, \tag{4.9}$$

and

$$(1-p)\eta_j(t, \zeta) = -\frac{1}{2}\tau_{jj}(t) + \sum_{k=1}^{p} \int_{s=\zeta}^{A_k^t(\zeta)} ds \int_{u=\zeta}^{s} \alpha_j(t, u)\alpha_k(t, s)du, \quad 1 \leqq j \leqq p, \tag{4.10}$$

are holomorphic.

4.6. As is well known (see for example §3 of Mayer [20]), the fact that $\tau_{jk}(t)$ is holomorphic implies that the group Z^{2p} acts on $T_p \times \boldsymbol{C}^p$ by

$$N \cdot (t, z) = (t, z + (I, \tau(t))N) \tag{4.11}$$

as a group of biholomorphic maps, producing a quotient manifold $J(V_p)$. The map $(t, z) \mapsto t$ induces a holomorphic projection ρ: $J(V_p) \to T_p$ so that $\rho^{-1}(t) = J(X_t)$ for all $t \in T_p$. It is easy to prove (see [5]) that the map $(t, \zeta) \mapsto (t, \eta(t, \zeta))$ from F_p to $T_p \times \boldsymbol{C}^p$ induces a holomorphic embedding ψ: $V_p \to J(V_p)$; the restriction to each fiber is an embedding ψ: $X_t \to J(X_t)$ that satisfies (4.5). Similarly, we can define holomorphic maps ψ_n: $S_T^n(V_p) \to J(V_p)$, $n \geqq 1$, whose restrictions to fibers are the maps ψ_n: $S^n(X_t) \to J(X_t)$ of §4.3. Recall that $S_T^n(V_p)$ is a quotient space of the complex manifold $F_n(T_p)$ defined in §3.2, and define η_n: $F_n(T_p) \to \boldsymbol{C}^p$ by

$$\eta_n(t, \zeta_1, \ldots, \zeta_n) = \sum_{j=1}^{n} \eta(t, \zeta_j), \quad (t, \zeta_1, \ldots, \zeta_n) \in F_n(T_p).$$

The map $(t, \zeta_1, \ldots, \zeta_n) \mapsto (t, \eta_n(t, \zeta_1, \ldots, \zeta_n))$ from $F_n(T_p)$ to $T_p \times C^p$ covers a holomorphic map ψ_n: $S_T^n(V_p) \to J(V_p)$ with the required property.

5. Proof of Theorem 2: Existence and Uniqueness.

5.1. The considerations of §4 lead quickly to a proof of the existence part of Theorem 2. They also lead to a uniqueness statement that will be useful both for counting relative sections when $2 \leqq p \leqq 4$ and for proving their nonexistence when $p \geqq 5$. We shall begin with the uniqueness statement. We need some notation. For $N \in Z^{2p}$, put

$$\lambda_N(t) = \frac{1}{2}(I, \tau(t))N \qquad \text{for all} \quad t \in T_p, \tag{5.1}$$

so that for each $t \in T_p$ the half-period in $J(X_t)$ determined by N is $[\lambda_N(t)]$. In addition, let H_N: $F_p \to C$ be the function

$$H_N(t, \zeta) = \sum_{j=1}^{p} \frac{\partial \theta}{\partial z_j}(\tau(t), \lambda_N(t))\alpha_j(t, \zeta). \tag{5.2}$$

Recall that the relative sections of the line bundle $L \to V_p$ determined by a factor of automorphy ξ are described by the ξ-automorphic functions on F_p.

Lemma. *Suppose the ξ-automorphic function f: $F_p \to C$ defines a relative section whose divisor on every fiber is half-canonical. Then there exist $N \in Z^{2p}$ and a holomorphic function ϕ: $F_p \to C$ such that*

(*a*) *N determines an odd half-period $\lambda_N(t)$ on each $J(X_t)$,*

(*b*) *for fixed $t \in T_p$, the function $\phi(t, \cdot)$ either has no zeros in $D(t)$ or else vanishes identically,*

(*c*) $H_N = \phi f^2$.

Proof. First we find N. Let s: $V_p \to L$ be the relative section determined by f. By Proposition 2.2, the divisor of s determines a holomorphic section σ: $T_p \to S_T^{p-1}(V_p)$. The composition $\psi_{p-1} \circ \sigma$: $T_p \to J(V_p)$ is holomorphic, and $\psi_{p-1}(\sigma(t))$ is a half-period in $J(X_t)$ for each t in T_p, since $\sigma(t)$ is half-canonical. The map $\psi_{p-1} \circ \sigma$ lifts to a holomorphic map $t \mapsto (t, g(t))$ from T_p to $T_p \times C^p$. Since $\psi_{p-1}(\sigma(t))$ is a half-period, for each t there is $N \in Z^{2p}$ such that

$$g(t) = \frac{1}{2}(I, \tau(t))N.$$

Since g: $T_p \to C^p$ is holomorphic, N is independent of t, and $g = \lambda_N$. Since $\psi_{p-1}(\sigma(t)) \in \Theta \subset J(X_t)$,

$$\theta(\tau(t), \lambda_N(t)) = \theta(\tau(t), g(t)) = 0$$

for all $t \in T_p$. It follows (see Farkas [9] or §7.2) that N determines an odd half-period on each $J(X_t)$. That proves (a).

5.2. It remains to produce the function ϕ. Let

$$A=\{t\in T_p;\ H_N(t,\zeta)=0 \quad \text{for all} \quad \zeta\in D(t)\},$$

and let $B=\{(t,\zeta)\in F_p;\ t\in A\}$. For each $t\in T_p\backslash A$, the functions $H_N(t,\cdot)$ and $f(t,\cdot)^2$ have exactly the same zeros (counting multiplicities) in $D(t)$, so

$$\phi(t,\zeta)=\frac{H_N(t,\zeta)}{f(t,\zeta)^2}$$

is a nowhere vanishing holomorphic function in $F_p\backslash B$. Since H_N vanishes identically on B, ϕ extends to a holomorphic function on F_p that vanishes identically on B and satisfies (b) and (c). That proves Lemma 5.1.

5.3. The existence proof requires an additional

Lemma. *If $H_N(t,\cdot)$ does not vanish identically in $D(t)$ for any fixed $t\in T_p$, there is a holomorphic function $f: F_p\to C$ such that $f^2=H_N$.*

Proof. Since T_p is contractible, the existence of f is a local question. The existence of f is obvious near any point (t_0,ζ_0) where $H_N(t_0,\zeta_0)\neq 0$. If $H_N(t_0,\zeta_0)=0$, then (t_0,ζ_0) has an open neighborhood in which H_N is the product of a nonvanishing function h and a Weierstrass polynomial

$$W(t,\zeta)=(\zeta-\zeta_0)^m+a_1(t)(\zeta-\zeta_0)^{m-1}+\cdots+a_m(t)$$

with $a_i(t_0)=0$ for all $i\geqq 1$. It is obvious that h has a holomorphic square root in some open neighborhood of (t_0,ζ_0). Also, since every zero of $H_N(t,\cdot)$ has even order, for each t near t_0 there is a unique polynomial

$$F(t,\zeta)=(\zeta-\zeta_0)^n+b_1(t)(\zeta-\zeta_0)^{n-1}+\cdots+b_n(t)$$

such that $F(t,\zeta)^2=W(t,\zeta)$. It suffices to prove that $F(t,\zeta)$ is a holomorphic function in a neighborhood of (t_0,ζ_0), and that follows from the formulas

$$2b_1(t)=a_1(t),$$

$$2b_k(t)=a_k(t)-\sum_{j=1}^{k-1} b_j(t)b_{k-j}(t),\ 2\leqq k\leqq n.$$

The lemma is proved.

5.4. The hypothesis of Lemma 5.3 is satisfied if $2\leqq p\leqq 4$ and $N\in Z^{2p}$ defines an odd half-period. Indeed, $\theta(\tau(t),\cdot)$ vanishes to odd order at $\lambda_N(t)$ if $[\lambda_N(t)]$ is odd, so if $H_N(t,\cdot)$ vanishes identically for some t, then $\theta(\tau(t),\cdot)$ vanishes to order at least 3 at $\lambda_N(t)$. Let $D\in S^{p-1}(X_t)$ be a divisor with $\psi_{p-1}(D)=[\lambda_N(t)]$. The Riemann vanishing theorem (see [11, p. 298]) would then imply that D has index of speciality

$$i(D)\geqq 3. \tag{5.3}$$

But Clifford's theorem (see [11, pp. 306 or 107]) gives

$$i(D)\leqq\left[\frac{p+1}{2}\right],$$

where as usual $[k]$ is the greatest integer $\leqq k$. That inequality contradicts (5.3) if $p<5$, and we conclude that $H_N(t,\cdot)$ cannot vanish identically if $2\leqq p\leqq 4$.

5.5. Now we can prove Theorem 2 for $2\leqq p\leqq 4$. Let $N\in Z^{2p}$ define an odd half-period. By Lemma 5.3, there is a holomorphic function $f\colon F_p\to \boldsymbol{C}$ such that $f^2=H_N$. By (4.8) and (5.2), H_N satisfies

$$H_N(t,\zeta)=H_N(\gamma(t,\zeta))\frac{\partial\gamma}{\partial\zeta}(t,\zeta) \qquad \text{for all} \quad \gamma\in\Gamma,$$

so there is a factor of automorphy $\xi(\gamma, z)$ for Γ on F_p such that

$$f(\gamma z)=\xi(\gamma,z)f(z) \qquad \text{for all} \quad \gamma\in\Gamma \quad \text{and} \quad z=(t,\zeta)\in F_p,$$

and $\xi(\gamma,\cdot)^{-2}=\dfrac{\partial\gamma}{\partial\zeta}$. Thus f determines a nontrivial relative section s of the line bundle $L\to V_p$ determined by ξ. It is clear that the divisor of s on every fiber is half-canonical.

We show next that N_1 and N_2 produce equivalent sections s_1 and s_2 if and only if they define the same half-period for some (hence all) $t\in T_p$. First suppose $[\lambda_{N_1}(t)]=[\lambda_{N_2}(t)]\in J(X_t)$ for all $t\in T_p$. Proposition 4.3 implies that $H_{N_1}(t,\cdot)$ is a nonzero multiple of $H_{N_2}(t,\cdot)$, so there is a nonvanishing holomorphic function $\phi(t)$ on T_p with

$$H_{N_1}(t,\zeta)=\phi(t)H_{N_2}(t,\zeta) \qquad \text{for all} \quad (t,\zeta)\in F_p.$$

Therefore the square roots satisfy

$$f_1(t,\zeta)=F(t)f_2(t,\zeta) \qquad \text{for all} \quad (t,\zeta)\in F_p,$$

where $F(t)$ is a nonvanishing holomorphic function on T_p. That means f_1 and f_2 determine relative sections s_1 and s_2 of the same line bundle, and s_1/s_2 is a nonvanishing section of the trivial bundle. Therefore s_1 and s_2 are equivalent.

Conversely, if s_1 and s_2 are equivalent, then the line bundles $L_1\to V_p$ and $L_2\to V_p$ determined by N_1 and N_2 are equivalent. The same is true of their restrictions to any fiber X_t, so $[\lambda_1(t)]=[\lambda_2(t)]\in J(X_t)$ as required.

5.6. Finally, we shall prove that if s is a relative section of a line bundle $L\to V_p$, and the divisor of s on every fiber is half-canonical, then s is equivalent to one of the sections constructed in §5.5. Since there are exactly $2^{p-1}(2^p-1)$ odd half-periods (see [11, p. 285]), this will complete the proof of Theorem 2 for $2\leqq p\leqq 4$. Let the ξ-automorphic function f determine s, and choose N and ϕ as in Lemma 5.1. Using Lemma 5.3, choose $f_1\colon F_p\to\boldsymbol{C}$ so that $f_1^2=H_N$. Then $f_1^2=\phi f^2$, so there is a nonvanishing holomorphic function F on F_p such that $f_1=Ff$. Clearly the relative section s_1 determined by f_1 is equivalent to s. The proof is complete.

6. The action of the modular group on half-periods.

6.1. We begin by reviewing the action of the modular group on F_p and T_p.

As in §3.1, let Γ be the fundamental group of the closed orientable surface X of genus $p \geqq 2$. Every homeomorphism $f: X \to X$ which fixes the base point x_0 induces an automorphism of Γ, and f induces an inner automorphism if and only if f is homotopic to the identity. Further, every automorphism of Γ is induced by some homeomorphism of X.

Using the generators of Γ described in §4.5, we define a homomorphism ν: $\Gamma \to Z^{2p}$ by

$$\nu(A_k) = \begin{pmatrix} e_k \\ 0 \end{pmatrix} \quad \text{and} \quad \nu(B_k) = \begin{pmatrix} 0 \\ e_k \end{pmatrix}, \quad 1 \leqq k \leqq p, \tag{6.1}$$

where e_k = the kth column of the $p \times p$ identity matrix. The kernel of ν is the commutator subgroup of Γ, so every automorphism g of Γ induces a unique automorphism $\beta(g)$ of Z^{2p} satisfying

$$\beta(g)(\nu(\gamma)) = \nu(g(\gamma)) \qquad \text{for all} \quad \gamma \in \Gamma. \tag{6.2}$$

Usually one interprets $\beta(g)$ as a $2p \times 2p$ unimodular matrix (with integer entries). If g is induced by the homeomorphism f, then $\beta(g)$ describes the effect of f on $H_1(X, Z)$, the first homology group of X. In particular, g is induced by an orientation-preserving homeomorphism if and only if the matrix $\beta(g)$ preserves the intersection matrix of the homology basis $A_1, \ldots, B_p$; that is, if and only if

$${}^t\beta(g) J \beta(g) = J, \; J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{6.3}$$

The group of automorphisms g of Γ such that $\beta(g)$ satisfies (6.3) is the modular group $\operatorname{mod}(\Gamma)$. The map β is a homomorphism of $\operatorname{mod}(\Gamma)$ onto the symplectic modular group

$$\operatorname{Sp}(p, Z) = \{Q \in \operatorname{SL}(2p, Z);\ {}^tQJQ = J\}.$$

6.2. The modular group acts on T_p as a group of biholomorphic maps so that for each $t \in T_p$ and $g \in \operatorname{mod}(\Gamma)$, the Riemann surfaces X_t and $X_{g(t)}$ are equivalent. In addition, the transformation theory of the θ-function (see Igusa [17, pp. 50 and 85] or Rauch-Farkas [21, p. 87]) leads to an action of $\operatorname{mod}(\Gamma)$ on the group $(Z/2Z)^{2p}$ of half-periods, which we shall need in §7. To describe that action we need some notation.

If Q is any square matrix, $\operatorname{Diag}(Q)$ is the column vector whose components are the entries on the main diagonal of Q. The $2p \times 2p$ matrix L is defined by

$$L = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix},$$

where I is the $p \times p$ identity matrix. If $g \in \operatorname{mod}(\Gamma)$ and $N \in Z^{2p}$, the vectors $\chi(g)$ and $g \cdot N$ in Z^{2p} are defined by

$$\chi(g) = -\operatorname{Diag}(\beta(g) L\, {}^t\beta(g)) \tag{6.4}$$

and

$$g\cdot N=\beta(g)N+\chi(g)\,. \tag{6.5}$$

Formula (6.5) does not define an action of $\mathrm{mod}(\Gamma)$ on Z^{2p}, since $gh\cdot N\neq g\cdot(h\cdot N)$ in general. However, by reducing modulo two, we obtain the desired action on the group $(Z/2Z)^{2p}$ of half-periods.

6.3. We shall need the following classical fact.

Proposition. *The sets of odd and even half-periods are invariant under the action* (6.5) *and* $\mathrm{mod}(\Gamma)$ *acts transitively on each of them.*

For completeness we indicate the proof. To see that $g\cdot\bar{N}$ is odd if and only if $\bar{N}\in(Z/2Z)^{2p}$ is odd, one can verify it directly from (6.5) on a convenient set of generators of $\mathrm{mod}(\Gamma)$. Alternatively one can use the transformation theory of the θ-function (see Rauch-Farkas [21, pp. 16 and 87]).

For the proof of transitivity we follow Igusa [17, pp. 211–213]. First let $N\in Z^{2p}$ represent an even half-period. Write ${}^tN=({}^ta, {}^tb)$ with $a, b\in Z^p$. Then ${}^tab\equiv 0 \pmod 2$. It is easy to verify that the matrix of integers

$$P=\begin{pmatrix} I-a{}^tb & a{}^ta \\ -b{}^tb & I+b{}^ta\end{pmatrix}$$

satisfies ${}^tPJP=J$, and it follows (see, for example, Siegel [22, p. 115]) that $P\in\mathrm{Sp}(p, Z)$. An easy computation gives $g\cdot 0=\chi(g)\equiv N \pmod 2$ if $\beta(g)=P$.

If $N\in Z^{2p}$ represents an odd half-period, we write ${}^tN=({}^tc, {}^td)$ with $c, d\in Z^p$. Since ${}^tcd\equiv 1 \pmod 2$, there is some j, $1\leqq j\leqq p$, with $c_j\equiv d_j\equiv 1 \pmod 2$. We may assume that $c_1=d_1=1$. Put $a=c-e_1$, $b=d-e_1$, and define P as above. If $\beta(g)=P$, then

$$g\cdot\begin{pmatrix} e_1 \\ e_1\end{pmatrix}=P\begin{pmatrix} e_1 \\ e_1\end{pmatrix}+\chi(g)\equiv\begin{pmatrix} e_1 \\ e_1\end{pmatrix}+\begin{pmatrix} a \\ b\end{pmatrix}\equiv\begin{pmatrix} c \\ d\end{pmatrix} \pmod 2,$$

so the orbit of $\begin{pmatrix} 0 \\ 0\end{pmatrix}$ contains all even half-periods and the orbit of $\begin{pmatrix} e_1 \\ e_1\end{pmatrix}$ contains all odd half-periods.

7. The zeros of H_N.

7.1. We shall complete the proof of Theorem 2 by showing that for $p\geqq 5$ there are no $N\in Z^{2p}$ and holomorphic functions $\phi\colon F_p\to C$ satisfying the conditions of Lemma 5.1. Toward that end we study the set

$$A(N)=\{t\in T_p\colon H_N(t, \zeta)=0 \quad \text{for all} \quad \zeta\in D(t)\}\,,$$

where $N\in Z^{2p}$ defines an odd half-period. Since the functions $\alpha_j(t, \cdot)$ on $D(t)$, $1\leqq j\leqq p$, are linearly independent, the definition (5.2) of H_N implies that $A(N)$ is the analytic variety

(7.1) $$A(N)=\left\{t\in T_p;\ \frac{\partial\theta}{\partial z_j}(\tau(t),\ \lambda_N(t))=0\quad\text{if}\quad 1\leqq j\leqq p\right\}.$$

We shall need some basic properties of the varieties $A(N)$.

Lemma. *If N and $N'\in Z^{2p}$ define odd half-periods, then*
 (*a*) $A(N)=A(N')$ *if* $N\equiv N'\pmod 2$,
 (*b*) $A(g\cdot N)=g(A(N))$ *if* $g\in\mathrm{mod}\,(\Gamma)$.

Proof. To prove (a), fix $t\in T_p$, write $\tau=\tau(t)$, and write $z=\lambda_N(t)\in \boldsymbol{C}^p$. If $N\equiv N'\pmod 2$, then

$$z'=\lambda_{N'}(t)=z+(I,\ \tau)M,\ M\in Z^{2p},$$

so the functional equation of the θ-function [11, p. 282] gives

$$\theta(\tau,\ z')=\sigma(\tau,\ z)\theta(\tau,\ z)$$

for some nonvanishing holomorphic function σ. Since $\theta(\tau,\ z)=0$, we have

$$\frac{\partial\theta}{\partial z_j}(\tau,\ z')=\sigma(\tau,\ z)\frac{\partial\theta}{\partial z_j}(\tau,\ z),\quad 1\leqq j\leqq p.$$

That proves (a).

The proof of (b) is similar. Let $g\in\mathrm{mod}\,(\Gamma)$ and let

$$\beta(g)=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{Sp}\,(p,\ Z),$$

where a, b, c, d are $p\times p$ matrices. Then (6.3) implies

$$\beta(g)^{-1}=\begin{pmatrix} {}^t d & -{}^t b\\ -{}^t c & {}^t a\end{pmatrix}.$$

Fix $t\in T_p$ and put $\tau=\tau(t)$ and $\tau^\#=\tau(g(t))$. Formula (6.8) of [5] says

(7.2) $${}^t(-c\tau+d)^{-1}(I,\ \tau)=(I,\ \tau^\#)\beta(g),$$

so $(I,\ \tau^\#)={}^t(-c\tau+d)^{-1}(I,\ \tau)\beta(g)^{-1}$, and

$$\tau^\#={}^t(-c\tau+d)^{-1}(\tau\,{}^t a-{}^t b)=(a\tau-b)(-c\tau+d)^{-1}.$$

Put $z^\#={}^t(-c\tau+d)^{-1}z$ if $z\in\boldsymbol{C}^p$. The transformation theory of the θ-function (see Igusa [17, pp. 50 and 85] or Rauch-Farkas [21, p. 87]) gives

(7.3) $$\theta\left(\tau^\#,\ z^\#+\frac{1}{2}(I,\ \tau^\#)\chi(g)\right)=\theta(\tau,\ z)\phi_g(\tau,\ z)$$

for some nonvanishing holomorphic function ϕ_g on $\mathscr{H}_p\times\boldsymbol{C}^p$. If $\theta(\tau,\ z)=0$, differentiation of (7.3) gives

(7.4) $$\frac{\partial\theta}{\partial z_k}(\tau,\ z)\phi_g(\tau,\ z)=\sum_{j=1}^{p}u_{jk}\frac{\partial\theta}{\partial z_j}\left(\tau^\#,\ z^\#+\frac{1}{2}(I,\ \tau^\#)\chi(g)\right),$$

if (u_{jk}) is the matrix ${}^t(-c\tau+d)^{-1}$. In particular, if $z=\lambda_N(t)$, then (7.4) holds, and (5.1), (7.2), and (6.5) give

$$z^\# + \frac{1}{2}(I, \tau^\#)\chi(g) = \frac{1}{2}[{}^t(-c\tau+d)^{-1}(I, \tau)N + (I, \tau^\#)\chi(g)]$$

$$= \frac{1}{2}(I, \tau^\#)(\beta(g)N + \chi(g))$$

$$= \frac{1}{2}(I, \tau^\#)(g\cdot N) = \lambda_{g\cdot N}(g(t)).$$

Substitution in (7.4) gives

$$\text{(7.5)} \qquad \frac{\partial\theta}{\partial z_k}(\tau(t), \lambda_N(t))\,\sigma_{g,N}(t) = \sum_{j=1}^{p} u_{jk}\frac{\partial\theta}{\partial z_j}(\tau(g(t)), \lambda_{g\cdot N}(g(t))),$$

with $\sigma_{g,N}(t)=\phi_g(\tau(t), \lambda_N(t))$. Since the matrix (u_{jk}) is invertible, that proves (b).

Remark. Lemma 7.1(a) says that the set $A(N)$ is determined by the half-period defined by N.

7.2. The crucial issue is the size of the sets $A(N)$. As a first result in that direction we prove this

Lemma. *If $N\in Z^{2p}$ defines an odd half-period and $p\geqq 5$, the set $A(N)$ is neither empty nor all of T_p.*

Proof Fix $t\in T_p$ so that X_t is hyperelliptic. It is shown in Chapter VII of Farkas-Kra [11] that there are half-canonical integral divisors D_1 and D_2 on X_t with index of speciality $i(D_1)=1$ and $i(D_2)=3$. Choose N_1 and N_2 in Z^{2p} so that $\psi_{p-1}(D_j)=[\lambda_{N_j}(t)]$, $j=1, 2$. The Riemann vanishing theorem implies that $\theta(\tau(t), \cdot)$ vanishes at $\lambda_{N_1}(t)$ and $\lambda_{N_2}(t)$ to orders 1 and 3 respectively. Therefore N_1 and N_2 define odd half-periods such that $t\notin A(N_1)$ and $t\in A(N_2)$.

Now let $N\in Z^{2p}$ define an odd half-period. By Proposition 6.3, there are g_1 and g_2 in $\mathrm{mod}(\Gamma)$ such that

$$g_1\cdot N_1 \equiv g_2\cdot N_2 \equiv N \pmod 2.$$

By Lemma 7.1, $g_1(t)\notin A(N)$ and $g_2(t)\in A(N)$. That proves the lemma.

Remark. Lemma 7.2 shows that $A(N)$ is always a proper subvariety of T_p, so we have proved the following result of Farkas.

Theorem (*Farkas* [9]). *The vanishing of the gradient of the θ-function at an odd half-period is special in the sense of moduli.*

A similar argument proves the companion

Theorem (*Farkas* [9]). *The vanishing of the θ-function at an even half-period is special in the sense of moduli.*

Proof. Just as (7.4) leads to (7.5), (7.3) leads to the formula

$$\theta(\tau(g(t)), \lambda_{g\cdot N}(g(t))) = \theta(\tau(t), \lambda_N(t))\sigma_{g,N}(t),$$

for any $g \in \mathrm{mod}(\Gamma)$ and $N \in Z^{2p}$. It is shown in [11, Chapter VII] that $\theta(\tau(t), 0) \neq 0$ for some fixed $t \in T_p$, and the above formula (with $N=0$) implies that

$$\theta(\tau(g(t)), \lambda_{g\cdot 0}(g(t))) \neq 0$$

for all $g \in \mathrm{mod}(\Gamma)$.

If $N \in Z^{2p}$ defines an even half-period, Proposition 6.3 gives $g \in \mathrm{mod}(\Gamma)$ with $g \cdot 0 \equiv N \pmod 2$. Then

$$\theta(\tau(g(t)), \lambda_N(g(t))) \neq 0.$$

That proves the theorem.

7.3. The connection between Lemma 5.1 and the sets $A(N)$ is given by the following

Lemma. *Suppose $N \in Z^{2p}$ defines an odd half-period and $p \geqq 5$. If there is a holomorphic function ϕ satisfying the conditions of Lemma 5.1, then the variety $A(N) \subset T_p$ has pure dimension $3p-4$.*

Proof. Given $t_0 \in A(N)$, choose $\zeta_0 \in D(t_0)$ and an open neighborhood U of t_0 in T_p so that $\zeta_0 \in D(t)$ if $t \in U$. Then

$$A(N) \cap U = \{t \in U;\ \phi(t, \zeta_0) = 0\},$$

so $A(N)$ is a hypersurface in T_p. That proves the lemma.

7.4. The nonexistence of $\phi: F_p \to C$ when $p=5$ is a consequence of Lemma 7.3 and the following

Lemma. *If $p=5$ and $N \in Z^{2p}$ defines an odd half-period, then $A(N)$ is an open-closed subset of the hyperelliptic submanifold of T_p.*

Proof. If $t \in A(N)$, there is a half-canonical integral divisor D on X_t such that $\psi_{p-1}(D) = [\lambda_N(t)]$ and the index of speciality of D satisfies

$$i(D) \geqq 3 = \frac{p+1}{2}.$$

Clifford's theorem [11, p. 106] implies that $i(D)=3$ and X_t is hyperelliptic. The set of $t \in T_p$ such that X_t is hyperelliptic forms a closed subset S of T_p, each of whose (countably many) connected components is a closed submanifold of dimension $2p-1$. The computations in Chapter VII of Farkas-Kra [11] show that if $t \in A(N)$, then the connected component of t in S is contained in $A(N)$. The conclusion of the lemma follows.

Corollary. *If $p=5$ and $N \in Z^{2p}$ defines an odd half-period, there is no $\phi: F_p \to C$ satisfying the conditions of Lemma 5.1.*

Proof. By Lemmas 7.2 and 7.4, $A(N)$ has dimension $2p-1$ ($\neq 3p-4$) if $p=5$.

7.5. The above corollary completes the proof of Theorem 2 for genus $p=5$. The proof for $p>5$ will be completed by an induction argument. To set up the induction we refer to Figure 1, which shows a closed surface X of genus $p \geqq 5$, with a canonical set of generators for the fundamental group Γ. Let $g \in \operatorname{mod}(\Gamma)$ be the element of order two determined by the 180° rotation r in the horizontal axis, as indicated in Figure 1, and let

$$H=\{t \in T_p;\ g(t)=t\}. \tag{7.6}$$

H is a connected component of the hyperelliptic locus. Finally, let $N \in Z^{2p}$ be the vector

$$N=e_1+e_2+\cdots+e_p+e_{p+1}+e_{p+3}+e_{p+5}. \tag{7.7}$$

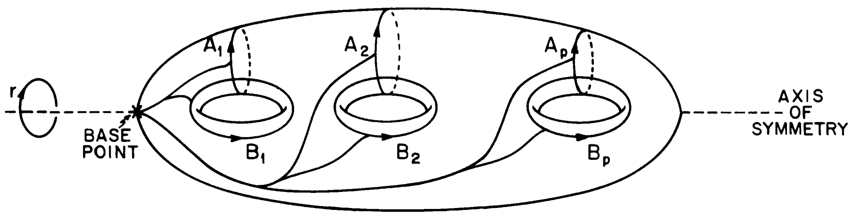

Figure 1. Canonical homotopy basis.

We shall need the following lemma, whose proof we postpone until Section 8.

Lemma. *If $H \subset T_p$ and $N \in Z^{2p}$ are defined by* (7.6) *and* (7.7) (*and* $p \geqq 5$), *then* $H \subset A(N)$.

7.6. Since H is an irreducible analytic subvariety of T_p, at least one irreducible component of $A(N)$ contains H (see Hervé [15]). We shall complete the proof of Theorem 2 by proving the following

Lemma. *Let $H \subset T_p$ and $N \in Z^{2p}$ be defined by* (7.6) *and* (7.7). *If* $p \geqq 5$, *then any irreducible component of $A(N)$ that contains H has dimension* $\leqq 3p-6$.

First we shall derive Theorem 2 from this result. Lemmas 7.5 and 7.6 imply that some irreducible component M of $A(N)$ has dimension $\leqq 3p-6<3p-4$. Proposition 6.3 and Lemma 7.1(b) imply that $A(N')$ has such a component whenever $N' \in Z^{2p}$ defines an odd half-period. Lemmas 5.1 and 7.3 therefore imply that for no factor of automorphy ξ can we find a ξ-automorphic function $f: F_p \to \boldsymbol{C}$ that defines a relative section whose divisor on every fiber is half-canonical, if $p \geqq 5$. That proves Theorem 2.

7.7. The proof of Lemma 7.6 is by induction on $p \geqq 5$. If $p=5$, the only irreducible component of $A(N)$ that contains H is H itself, and H has dimension $2p-1=3p-6$, so the lemma is true when $p=5$. For the induction step, we assume that $p \geqq 6$ and that the lemma holds in genus $p-1$. We shall reduce the problem for genus p to that for genus $p-1$ by pinching the curve γ shown in Figure 2 to a point. For that purpose we must introduce the Bers deformation space (see [3] and [4]) determined by the curve γ.

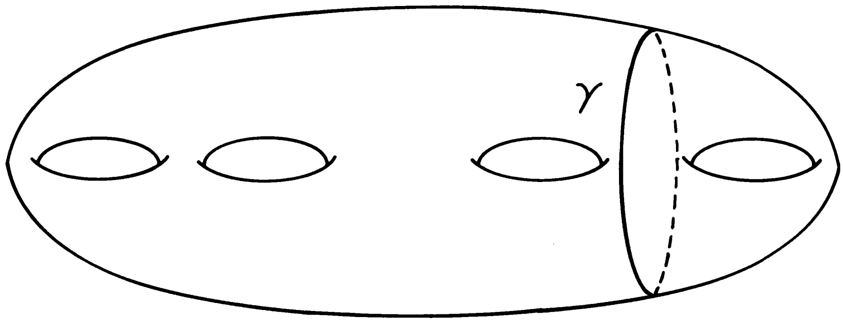

Figure 2. Pinching curve γ is homologous to $A_pB_pA_p^{-1}B_p^{-1}=0$.

That deformation space is a bounded domain $D\subset C^{3p-3}$. The closed analytic hypersurface

$$G=\{w=(w_1,\ldots, w_{3p-3})\in D;\ w_1=0\}$$

parametrizes the singular Riemann surfaces obtained as in Figure 3 by pinching γ to a point. Each such surface has two nonsingular pieces, of genus $p-1$ and 1 respectively, and G is biholomorphically equivalent to the product $T_{p-1,1}\times T_{1,1}$. (As in §3.3 for any $k\geqq 1$, $T_{k,1}$ is the Teichmüller space of closed Riemann surfaces of genus k with 1 distinguished point.)

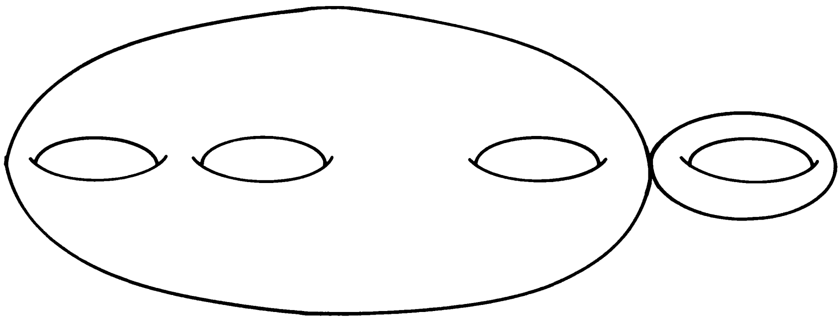

Figure 3. Riemann surface with node obtained by pinching the curve γ.

The points of the open set $D_0=D\backslash G$ represent nonsingular closed Riemann surfaces of genus p. Let f be the element of mod (Γ) determined by the Dehn twist about the curve γ, and let $\langle f\rangle$ be the cyclic subgroup generated by f. Then D_0 is biholomorphically equivalent to the quotient space $T_p/\langle f\rangle$, and there is a surjective holomorphic map $\pi\colon T_p\to D_0$ such that $\pi(t)=\pi(t')$ if and only if $t'=f^n(t)$ for some $n\in Z$.

The period matrix map $t\mapsto\tau(t)$ factors through π. Indeed, there is a holomorphic map $\sigma\colon D\to\mathscr{H}_p$ such that

$$\tau(t) = \sigma(\pi(t)) \quad \text{for all} \quad t \in T_p. \tag{7.8}$$

7.8. Now let $M \subset T_p$ be an irreducible component of $A(N)$ that contains H, and let t be a regular point of M. Since $\pi\colon T_p \to D_0$ is a covering map, (7.1) and (7.8) imply that $\pi(t)$ is a regular point of the analytic variety

$$V = \left\{ w \in D;\ \frac{\partial \theta}{\partial z_j}\left(\sigma(w), \frac{1}{2}(I, \sigma(w))N\right) = 0 \quad \text{if} \quad 1 \leqq j \leqq p \right\}. \tag{7.9}$$

Let W be the unique irreducible component of V that contains $\pi(t)$. (W is the closure in D of the connected component of $\pi(t)$ in the set of regular points of V.) We note that $\pi(M) \subset W$.

Now the elements f and g in $\mathrm{mod}\,(\Gamma)$ commute, so g acts on D_0 as a biholomorphic map in such a way that

$$g(\pi(t)) = \pi(g(t)) \quad \text{for all} \quad t \in T_p.$$

The fixed point set of g in D_0 is precisely $\pi(H)$. Since D is bounded and $G = D \backslash D_0$ is an analytic hypersurface, g extends to a biholomorphic map $g\colon D \to D$ whose fixed point set is the closure of $\pi(H)$. Since $\pi(H) \subset \pi(M) \subset W$ and W is closed in D, every fixed point of $g\colon D \to D$ belongs to W.

7.9. Let d be the codimension of M in T_p. Our goal is to prove that $d \geqq 3$. We know that d equals the codimension of W in D, which in turn equals the codimension of $W \cap G$ in G, by Corollary 1 on p. 105 of [15]. In fact, by that same corollary, any irreducible component Y of $W \cap G$ in G has codimension d.

We fix our attention on G and its subvarieties. Since G is equivalent to $T_{p-1,1} \times T_{1,1}$, all biholomorphic maps of G onto itself have connected fixed point sets. In particular the fixed point set of $g\colon G \to G$ is a connected subset of $W \cap G$, so it is contained in an irreducible component Y of $W \cap G$. We know Y has codimension d. Since $Y \subset W \cap G \subset V \cap G$, some irreducible component X of $V \cap G$ contains Y. We claim that the induction hypothesis implies that X has codimension at least three. The inequalities

$$3 \leqq \mathrm{cod}\,(X) \leqq \mathrm{cod}\,(Y) = d$$

will then complete the proof.

7.10. To show that $\mathrm{cod}\,(X) \geqq 3$ we examine the variety $V \cap G$ in G. For $w \in G$, the period matrix $\sigma(w)$ of the corresponding singular Riemann surface S splits along the main diagonal into two square blocks τ' and τ'' of dimension $p-1$ and 1 respectively. These are the period matrices of the two pieces of S. If $z \in \boldsymbol{C}^p$ is written

$$z = \begin{pmatrix} z' \\ z'' \end{pmatrix}, \quad z' \in \boldsymbol{C}^{p-1}, \quad z'' \in \boldsymbol{C},$$

then the θ-function is a product

$$\theta(\sigma(w), z) = \theta_{p-1}(\tau', z') \times \theta_1(\tau'', z'').$$

Here θ_{p-1} and θ_1 are θ-functions of genus $p-1$ and 1 respectively. In particular, if N is defined by (7.7) and $z = \frac{1}{2}(I, \sigma(w))N$, then

$$z' = \frac{1}{2}(I, \tau')N' \quad \text{and} \quad z'' = \frac{1}{2}(I, \tau'')N''.$$

Here $N' \in Z^{2(p-1)}$ is the odd half-period defined by (7.7), with p replaced by $p-1$, and $N'' = e_1 \in Z^2$ is an even half-period. Therefore $\theta_1(\tau'', z'') \neq 0$ and

$$\theta_{p-1}(\tau', z') = \frac{\partial \theta_1}{\partial z}(\tau'', z'') = 0.$$

Comparing with (7.9) we see that $w \in V \cap G$ if and only if

$$\frac{\partial \theta_{p-1}}{\partial z_j}\left(\tau', \frac{1}{2}(I, \tau')N'\right) = 0 \quad \text{if} \quad 1 \leqq j \leqq p-1.$$

Therefore $V \cap G = \rho^{-1}(A(N'))$, where $\rho\colon G \to T_{p-1}$ is the holomorphic map defined by first projecting $G\,(= T_{p-1,1} \times T_{1,1})$ onto $T_{p-1,1}$, then mapping $T_{p-1,1}$ onto T_{p-1} by the "forgetful map".

Since X is an irreducible component of $V \cap G$ and $V \cap G = \rho^{-1}(A(N'))$, $\rho(X)$ is an irreducible component of $A(N')$ and $\rho^{-1}(\rho(X)) = X$. Further, the codimension of X in G equals the codimension of $\rho(X)$ in T_{p-1}. Since X contains the fixed point set of g in G, $\rho(X)$ contains its image in T_{p-1}. That image is precisely the set $H \subset T_{p-1}$, so the induction hypothesis implies that $\operatorname{cod}(\rho(X)) \geqq 3$. That completes the proof of Lemma 7.6 and of Theorem 2, modulo the proof of Lemma 7.5.

8. Half-periods in $J(X)$ for hyperelliptic X.

8.1. We proceed to prove Lemma 7.5. By the Riemann vanishing theorem, it suffices to prove that for X hyperelliptic and N defined by (7.7) we can find a divisor $D \in S^{p-1}(X)$ with $i(D) \geqq 3$ and

$$\psi_{p-1}(D) = \left[\frac{1}{2}(I, \tau)N\right].$$

(Here the hyperelliptic involution of X and the canonical homology basis used to compute τ are related as specified in §7.5.) We will actually find a divisor (of degree $p-1$ and) of index of speciality precisely 3.

8.2. We adjust the arguments of [11, Chapter VII] to the canonical homology basis determined by the homotopy basis shown in Figure 1. That homology basis is shown in more detail in Figure 4. The Riemann surface X in Figure 4 should be viewed as a hyperelliptic surface conformally embedded in $\boldsymbol{R}^3$. It is of genus $p \geqq 5$ and its Weierstrass points $y_1, y_2, \ldots, y_{2p+2}$ are located on the x-axis. The 180° rotation r about the x-axis preserves X and is identified with the hyperelliptic involution on X. The canonical homology basis $A_1, \ldots, A_p, B_1, \ldots, B_p$ is chosen as indicated in Figure 4. Note that the curves B_j are invariant under r. In fact for $j=1,\dots,p$, let β_j be a curve from y_{2j+1} to y_{2j} as shown by the solid part of B_j. Then $B_j=\beta_j-r(\beta_j)$.

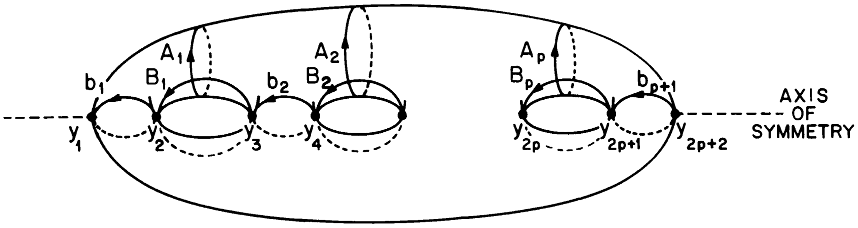

Figure 4. The canonical homology basis corresponding to the homotopy basis of Figure 1 and some additional curves.

Similarly, for $j=1,\dots,p+1$, we define $\hat{\beta}_j$ to be a curve from y_{2j} to y_{2j-1} (as shown by the solid part of b_j in Figure 4) and we set $b_j=\hat{\beta}_j-r(\hat{\beta}_j)$. To determine the homology class of b_k we compute the intersection numbers of b_k with the curves A_j and B_j. All the intersection numbers are zero with the following exceptions:

$$b_k \times B_k = 1 \qquad \text{for} \quad k=1,\dots,p,$$

$$b_k \times B_{k-1} = -1 \qquad \text{for} \quad k=2,\dots,p+1.$$

Therefore $b_1=A_1$, $b_k=A_k-A_{k-1}$ for $k=2,\dots,p$, and $b_{p+1}=-A_p$ as homology classes.

Since r acts as multiplication by -1 on both the homology classes and the abelian differentials of the first kind, we can compute easily the images in $J(X)$ of the Weierstrass points. It is convenient to use y_1 as the basepoint for the map $\phi\colon X \to J(X)$. As in [11, p. 303], we have:

$$\phi(y_1)=[0],$$

$$\phi(y_2)=\left[\frac{1}{2}e_1\right],$$

$$\phi(y_3)=\left[\frac{1}{2}(e_1+\tau e_{p+1})\right],$$

$$\phi(y_4)=\left[\frac{1}{2}(e_2+\tau e_{p+1})\right],$$

$$\phi(y_5)=\left[\frac{1}{2}(e_2+\tau(e_{p+1}+e_{p+2}))\right],$$

$$\vdots$$

$$\phi(y_{2k+1})=\left[\frac{1}{2}(e_k+\tau(e_{p+1}+\cdots+e_{p+k}))\right], \quad k=1,\dots,p-1,$$

$$\phi(y_{2k+2})=\left[\frac{1}{2}(e_{k+1}+\tau(e_{p+1}+\cdots+e_{p+k}))\right], \quad k=1,\dots,p-1,$$

$$\vdots$$

$$\phi(y_{2p+1}) = \left[\frac{1}{2}(e_p + \tau(e_{p+1} + \cdots + e_{2p}))\right],$$

$$\phi(y_{2p+2}) = \left[\frac{1}{2}\tau(e_{p+1} + \cdots + e_{2p})\right].$$

The points $\phi(y_j)$, $1 \leqq j \leqq 2p+2$, are all half-periods. The half-period $\phi(y_j)$ is even if $j=1, 2, 4, 6, \ldots, 2p+2$ and odd if $j=3, 5, \ldots, 2p+1$. The calculations of [11, pp. 305 and 309–311] show that

$$\begin{aligned} k = k(y_1) &= \sum_{j=1}^{p} \phi(y_{2j+1}) \\ &= \left[\frac{1}{2}(e_1 + \cdots + e_p + \tau(pe_{p+1} + (p-1)e_{p+2} + \cdots + e_{2p}))\right]. \end{aligned}$$

Note that $k(y_1)$ is a half-period.

8.3. With these preliminaries out of the way, we are ready to produce the divisor $D \in S^{p-1}(X)$. Let

$$D = 2(y_{10} + y_{11}) + \sum_{k=1}^{n} (y_{4k+10} + y_{4k+11}) \quad \text{if} \quad p = 2n+5,$$

and

$$D = 2(y_{10} + y_{11}) + \sum_{k=1}^{n} (y_{4k+10} + y_{4k+11}) + y_{2p+2} \quad \text{if} \quad p = 2n+6.$$

Then $\deg D = p-1$ and (by the remarks of [11, pp. 306–307]), $i(D) = 3$. The calculations of §8.2 show that

$$\phi_{p-1}(D) = \left[\frac{1}{2}\tau(e_{p+7} + e_{p+9} + e_{p+11} + \cdots + e_{2p})\right] \quad \text{if } p \text{ is odd,}$$

and

$$\phi_{p-1}(D) = \left[\frac{1}{2}\tau(e_{p+1} + e_{p+2} + \cdots + e_{p+6} + e_{p+8} + e_{p+10} + \cdots + e_{2p})\right] \quad \text{if } p \text{ is even.}$$

It follows that

$$\begin{aligned} \psi_{p-1}(D) &= \phi_{p-1}(D) - k(y_1) = \phi_{p-1}(D) + k(y_1) \\ &= \left[\frac{1}{2}(e_1 + \cdots + e_p + \tau(e_{p+1} + e_{p+3} + e_{p+5}))\right] \end{aligned}$$

as required.

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Note added in proof:

J. Harris has brought to our attention his paper: Theta-characteristics on algebraic curves, Trans. Amer. Math. Soc., **271** (1982), 611–638.

In that paper, Harris proves that each variety in the moduli space of surfaces of genus p of Riemann surfaces with a half-canonical divisor D of index of speciality $\geqq r+1$ is either empty or has codimension $\leqq r\,(r+1)/2$. It follows that d of § 7.9 must be equal to 3 (the case $r=2$).