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Necessary and sufficient conditions for extremality in certain classes of quasiconformal mappings

Dedicated to Professor Yukio Kusunoki on the occasion of his sixtieth birthday

By

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§ 1. Introduction

Let Γ be a Fuchsian group acting on the upper half-plane U, possibly consisting of only the identity transformation. Let $L_\infty(\Gamma)$ be the closed linear subspace of $L_\infty(U)$ consisting of those $\nu \in L_\infty(U)$ which satisfy

$$\nu(\gamma(w))\bar{\gamma}'(w)/\gamma'(w) = \nu(w) \quad \text{for every} \quad \gamma \in \Gamma. \tag{1.1}$$

We denote by $M(\Gamma)$ the open unit ball of $L_\infty(\Gamma)$. For ν in $M(\Gamma)$, we denote by $z = F_\nu(w)$ the uniquely determined quasiconformal automorphism of U with complex dilatation $\nu = \nu(w)$ such that the homeomorphic extension of F_ν to the closure $\bar{U} = U \cup \hat{R}$, which we call it again F_ν, leaves 0, 1 and ∞ fixed (see Lehto and Virtanen [3, p. 185 and p. 194]).

Let σ be a Γ-invariant closed subset of the extended real line $\hat{R}$, which contains 0, 1 and ∞. Let E be a Γ-invariant measurable, possibly empty, subset of U such that the set $D = U \setminus E$ has a positive measure. Let $b(w)$ be a non-negative bounded measurable function on E, being automorphic for Γ and satisfying

$$\|b\|_\infty = \operatorname*{ess\,sup}_{w \in E} b(w) < 1.$$

Let $h: \hat{R} \to \hat{R}$ be the boundary mapping induced by F_{ν_0} for some $\nu_0 \in M(\Gamma)$. We consider the class $Q \equiv Q(\Gamma, h, \sigma, E, b)$ consisting of those F_ν with $\nu \in M(\Gamma)$, which satisfy the conditions $F_\nu|_\sigma = h|_\sigma$ and $|\nu(w)| \leqq b(w)$ a.e. in E. We suppose that Q is not empty. A mapping F_μ in Q which satisfies $\|\mu|_D\|_\infty = \inf_{F_\nu \in Q} \|\nu|_D\|_\infty$ is said to be extremal within the class Q, where $\|\nu|_D\|_\infty$ means the L_∞ norm of the restriction $\nu|_D$ of ν to D. By [3, pp. 71-74] and Strebel [8, Satz on page 469], we see that there exists at least one extremal quasiconformal mapping within Q.

In this note, as a continuation of the preceding paper [6], we investigate necessary and sufficient conditions for a prescribed mapping $F_\mu \in Q$ to be extremal within Q.

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In §2, we give the summarizing statements of Theorems. Theorem 1 gives a complete characterization for F_μ to be extremal within Q under the hypotheses that $\|\mu|_D\|_\infty>0$ and that either dim $A(\Gamma, \sigma)<\infty$ or E/Γ is relatively compact in $\{U \cup (\hat{R}\setminus\sigma)\}/\Gamma$. This theorem generalizes [6, Theorem 2] where we imposed some additional hypotheses both on E and on b. Theorem 2 gives a necessary condition for extremality under certain hypotheses which are a little weaker than those of Theorem 1. Theorem 3 gives a necessary condition for extremality within a class $\tilde{Q}$ similar to Q. In §3, we give the proofs of Theorems, following the arguments in [6] but with some parts improved.

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§2. Necessary and sufficient conditions for extremality

In order to formulate our theorems, first we collect necessary terminologies and notations. Fix $F_\mu \in Q$ once for all, and we put

$$F=F_\mu, \quad f=F^{-1}, \quad \kappa(z)=f_{\bar{z}}(z)/f_z(z), \tag{2.1}$$

$$G=F\Gamma F^{-1} \quad \text{and} \quad k_0=\|\mu|_D\|_\infty .$$

Let $\delta=h(\sigma)$ and, for $\varepsilon\geqq 0$, let E_ε be the subset of E consisting of those $w\in E$ with $b(w)\leqq\varepsilon$. Put $V_\varepsilon=U\setminus F(E_\varepsilon)$. We denote by $\hat{\kappa}$ the extension of $\kappa|_{F(D\cup E_0)}$ to U which satisfies

$$\hat{\kappa}(z)=k_0\kappa(z)/b(f(z)) \qquad \text{for} \quad z\in F(E\setminus E_0).$$

As is known, the property of (1.1) of μ implies that G is a Fuchsian group. A holomorphic function ϕ on U is called a quadratic differential for G on U if $\phi(g(z))\cdot g'(z)^2=\phi(z)$ for every $g\in G$. We denote by $A(G, \delta)$ the space consisting of all the quadratic differentials ϕ for G on U, which are continuously extensible to $\hat{R}\setminus\delta$ and real on $\hat{R}\setminus\delta$, and satisfy

$$\|\phi\|_{U/G}\equiv\iint_{U/G}|\phi(z)|dxdy<\infty.$$

In this note, further, we require that $A(G, \delta)\neq\{0\}$. This requirement eliminates certain trivial cases (see [6]). We denote by $A(G, \delta)_1$ the set of those $\phi\in A(G, \delta)$ with $\|\phi\|_{U/G}=1$.

For $\varepsilon\geqq 0$ and $\alpha\in L_\infty(G)$, we put

$$L_{V_\varepsilon/G}(\alpha)(\phi)\equiv\mathrm{Re}\iint_{V_\varepsilon/G}\alpha(z)\phi(z)dxdy \qquad \text{for} \quad \phi\in A(G, \delta),$$

$$a_\varepsilon\equiv\sup L_{V_\varepsilon/G}(\hat{\kappa})(\phi),$$

where the supremum is taken over all $\phi\in A(G, \delta)$ with $\|\phi\|_{V_\varepsilon/G}=1$. For every ϕ in $A(G, \delta)\setminus\{0\}$, following Reich [4], we put

$$D[\phi, \kappa] = \iint_{F(D)/G} |\phi(z)|\,|1-\kappa(z)\phi(z)/|\phi(z)|\,|^2(1-|\kappa(z)|^2)^{-1}dxdy,$$

$$E[\phi, \kappa] = \iint_{F(E)/G} |\phi(z)|\,|1-\kappa(z)\phi(z)/|\phi(z)|\,|^2(1-|\kappa(z)|^2)^{-1}B(f(z))dxdy,$$

where $B(w)=(1+b(w))(1-b(w))^{-1}$.

Now we give the summarizing statements of theorems as follows.

Theorem 1. *Suppose that* $k_0>0$ *in* (2.1) *and that either* $\dim A(\Gamma, \sigma)<\infty$ *or* E/Γ *is relatively compact in* $\{U \cup (\hat{R}\smallsetminus\sigma)\}/\Gamma$. *Then each one of the conditions* (I), (II) *and* (III) *below is necessary and sufficient for* F *to be extremal within* Q.

Condition (I): *Either there exists* $\phi_0 \in A(G, \delta)_1$ *such that*

$$\kappa(z)=b(f(z))\,|\phi_0(z)|/\phi_0(z) \quad \text{a.e.} \quad \textit{in} \quad F(E), \quad \textit{and}$$

$$\kappa(z)=k_0|\phi_0(z)|/\phi_0(z) \quad \text{a.e.} \quad \textit{in} \quad F(D)$$

or there exists a sequence $\{\phi_n\}$ *in* $A(G, \delta)_1$ *such that*

$$\lim_{n\to\infty} \phi_n(z)=0 \textit{ locally uniformly in } U \cup (\hat{R}\smallsetminus\delta), \textit{ and}$$

$$\lim_{n\to\infty} L_{U/G}(\kappa)(\phi_n)=k_0.$$

Condition (II):

$$\inf\{K_0 D[\phi, \kappa]+E[\phi, \kappa]\}=1,$$

where $K_0=(1+k_0)/(1-k_0)$ *and the infimum is taken over all* $\phi \in A(G, \delta)_1$.

Condition (III):

$$a_0=k_0.$$

The following theorem is a slightly generalized form of a part of Theorem 1.

Theorem 2. *Suppose that either* $\dim A(\Gamma, \sigma)<\infty$ *or the condition* (A) *below is satisfied for a positive number* ε_0. *Let* ε *be a non-negative real number which is arbitrarily chosen* (*resp. less than or equal to* ε_0) *if* (*resp. unless*) $\dim A(\Gamma, \sigma)<\infty$. *If* F *is extremal within* Q, *then* $a_\varepsilon=k_0$.

Condition (A): $\dim A(\Gamma, \sigma)=\infty$ *and* E_{ε_0}/Γ *is relatively compact in* $\{U \cup (\hat{R}\smallsetminus\sigma)\}/\Gamma$.

Now we define a class $\tilde{Q}$ similar to Q. Let $\tilde{b}$ be a measurable function on E, being automorphic for Γ and satisfying $\|\tilde{b}\|_\infty<1$. We note that $\tilde{b}$ is not necessarily required to be non-negative. We consider the class $\tilde{Q}=\tilde{Q}(\Gamma, h, \sigma, E, \tilde{b})$ consisting of those F_ν with $\nu \in M(\Gamma)$, which satisfy $F_\nu|_\sigma=h|_\sigma$ and $\nu(w)=\tilde{b}(w)$ a.e. in E. We suppose that $\tilde{Q}$ is not empty. We say that $F \in \tilde{Q}$ is extremal if $F|_D$ has minimal maximal dilatation within the class $\tilde{Q}$. For a prescribed mapping $F \in \tilde{Q}$, we put

$$a^* \equiv \sup L_{F(D)/G}(\kappa)(\phi),$$

where κ is defined by (2.1) and where the supremum is taken over all $\phi \in A(G, \delta)$

with $\|\phi\|_{F(D)/G}=1$. Then, by the same method as that of the proof of Theorem 2, we can show the following theorem.

Theorem 3. *Suppose that either* $\dim A(\Gamma, \sigma)<\infty$ *or* E/Γ *is relatively compact in* $\{U \cup (\hat{R}\setminus\sigma)\}/\Gamma$. *If* F *is extremal within* $\tilde{Q}$, *then* $a^*=k_0$. *In other words, either there exists some* $\phi_0 \in A(G, \delta)\setminus\{0\}$ *such that* $\kappa=k_0|\phi_0|/\phi_0$ *on* $F(D)$ *or there exists a sequence* $\{\phi_n\}$ *in* $A(G, \delta)_1$ *such that*

$$\lim_{n\to\infty} L_{F(D)/G}(\kappa)(\phi_n)=k_0.$$

§3. Proofs of Theorems

In this section, we give the proofs of Theorems. For this purpose, first we recall some result from [7] and prove some lemmas.

Let $M_0(G, \delta)$ be the subset of $M(G)$ consisting of those $\tau \in M(G)$ such that $F_\tau|_\delta$ is identical with the identity automorphism of δ. We denote by $N(G, \delta)$ the space consisting of all the elements $\alpha \in L_\infty(G)$ which satisfy $L_{U/G}(\alpha)(\phi)=0$ for every $\phi \in A(G, \delta)$.

The following Lemma 1 follows from Gardiner [1, Theorem 1] and [7, Corollary].

Lemma 1. *Let* S *be a* G*-invariant measurable subset of* U *such that the set* $U\setminus S$ *has a positive measure and such that* S/G *is relatively compact in* $\{U \cup (\hat{R}\setminus\delta)\}/G$ *if* $\dim A(\Gamma, \sigma)=\infty$. *Suppose that* $\alpha \in N(G, \delta)$ *vanishes on* S. *Then there exists a curve* $\tau(t) \in M_0(G, \delta)$, *defined in an interval* $(0, t_0)$, *which vanishes on* S, *and which satisfies*

$$\lim_{t\to 0} \|\tau(t)/t-\alpha\|_\infty=0. \tag{3.1}$$

Lemma 2. *Suppose that either* $\dim A(\Gamma, \sigma)<\infty$ *or the condition* (A) *is satisfied for some* $\varepsilon_0 \geqq 0$. *Let* ε *be a non-negative real number which is arbitrarily chosen (resp. less than or equal to* ε_0*) if (resp. unless)* $\dim A(\Gamma, \sigma)<\infty$. *Let* $\{\phi_n\}$ *be a sequence in* $A(G, \delta)$ *with* $\|\phi_n\|_{V_\varepsilon/G}=1$ *such that* $\{\phi_n|_{V_\varepsilon \cup (\hat{R}\setminus\delta)}\}$ *converges to the limit function* ϕ_0 *locally uniformly in* $V_\varepsilon \cup (\hat{R}\setminus\delta)$. *Then* $\{\|\phi_n\|_{U/G}\}$ *is a bounded sequence and there exists some* $\phi \in A(G, \delta)$ *which satisfies* $\phi|_{V_\varepsilon \cup (\hat{R}\setminus\delta)}=\phi_0$ *and such that* $\{\phi_n\}$ *converges to* ϕ *locally uniformly in* $U \cup (\hat{R}\setminus\delta)$. *Moreover, the following equality holds*:

$$\lim_{n\to\infty} \|\phi_n-\phi\|_{V_\varepsilon/G}=1-\|\phi\|_{V_\varepsilon/G}. \tag{3.2}$$

Proof. Put $K_n=\|\phi_n\|_{U/G}$. Assume that there exists a subsequence of $\{\phi_n\}$, which we call it again $\{\phi_n\}$, such that $\{K_n\}$ converges to ∞ as n tends to ∞. Put $\psi_n=\phi_n/K_n$. Then we have $\psi_n \in A(G, \delta)_1$ and

$$\lim_{n\to\infty} \|\psi_n\|_{F(E_\varepsilon)/G}=1. \tag{3.3}$$

Thus $\{\psi_n\}$ forms a normal family in $U \cup (\hat{R} \setminus \delta)$. We choose a convergent subsequence of $\{\psi_n\}$, which we call it again $\{\psi_n\}$. Since, by our hypotheses, $\{\psi_n|_{V_\varepsilon \cup (\hat{R} \setminus \delta)}\}$ converges to 0 locally uniformly in $V_\varepsilon \cup (\hat{R} \setminus \delta)$, so does $\{\psi_n\}$ locally uniformly in $U \cup (\hat{R} \setminus \delta)$. But this is impossible if $\dim A(\Gamma, \sigma) < \infty$ and this contradicts (3.3) if the condition (A) is satisfied. Therefore we see that $\{K_n\}$ is a bounded sequence.

By the boundedness of $\{K_n\}$, $\{\phi_n\}$ forms a normal family in $U \cup (\hat{R} \setminus \delta)$. Then, by our hypotheses and Vitali's theorem, we have the former part of the lemma.

Now we show that (3.2) holds. By a routine work, we see that the following inequality holds:

$$\limsup_{n\to\infty} \|\phi_n - \phi\|_{U/G} \leqq \limsup_{n\to\infty} \|\phi_n\|_{U/G} - \|\phi\|_{U/G} \tag{3.4}$$

(see Harrington and Ortel [2, Proposition 1.1] and [5, Lemma 3]). But, under the hypotheses of the lemma, we easily check that the right hand side of (3.4) is identical with $1 - \|\phi\|_{V_\varepsilon/G}$ and that

$$\begin{aligned}\limsup_{n\to\infty} \|\phi_n - \phi\|_{U/G} &= \limsup_{n\to\infty} \|\phi_n - \phi\|_{V_\varepsilon/G} \\ &\geqq \liminf_{n\to\infty} \|\phi_n - \phi\|_{V_\varepsilon/G} \\ &\geqq 1 - \|\phi\|_{V_\varepsilon/G}.\end{aligned}$$

Thus we have (3.2). q.e.d.

By making use of (3.2), we have Lemma 3 below (see [2, Proposition 1.2] and [5, Lemma 4] for the procedure).

Lemma 3. *Suppose that the condition* (A) *is satisfied for some* $\varepsilon_0 \geqq 0$. *Let* ε *be a non-negative real number which is less than or equal to* ε_0. *Suppose, further, that* $a_\varepsilon = k_0 > 0$ *and that there exists no* $\phi \in A(G, \delta) \setminus \{0\}$ *satisfying* $\hat{\kappa} = k_0|\phi|/\phi$ *on* V_ε. *Then every sequence* $\{\phi_n\}$ *in* $A(G, \delta)$, *with* $\|\phi_n\|_{V_\varepsilon/G} = 1$, *which satisfies* $\lim_{n\to\infty} L_{V_\varepsilon/G}(\hat{\kappa})(\phi_n) = k_0$ *converges to* 0 *locally uniformly in* $U \cup (\hat{R} \setminus \delta)$.

Lemma 4. *Suppose that either* $\dim A(\Gamma, \sigma) < \infty$ *or the condition* (A) *is satisfied for a positive number* ε_0. *Then* $a_0 = k_0$ *if and only if* $a_\varepsilon = k_0$ *for every positive number* ε (*resp. for every* ε *satisfying* $\varepsilon_0 \geqq \varepsilon > 0$) *if* (*resp. unless*) $\dim A(\Gamma, \sigma) < \infty$.

Proof. If $k_0 = 0$, then nothing has to be shown. Thus we may assume that $k_0 > 0$.

Suppose that $a_0 = k_0$. If $\hat{\kappa} = k_0|\phi|/\phi$ on V_0 for some $\phi \in A(G, \delta) \setminus \{0\}$, then clearly we have $a_\varepsilon = k_0$ for every $\varepsilon > 0$. Next we consider the case where there exists no $\phi \in A(G, \delta) \setminus \{0\}$ such that $\hat{\kappa} = k_0|\phi|/\phi$ on V_0. Under the hypothesis of the lemma, this occurs only in the case where the condition (A) is satisfied for ε_0. Thus, by Lemma 3, every sequence $\{\phi_n\}$ in $A(G, \delta)$ with $\|\phi_n\|_{V_0/G} = 1$, satisfying $\lim_{n\to\infty} L_{V_0/G}(\hat{\kappa})(\phi_n) = k_0$, converges to 0 locally uniformly in $U \cup (\hat{R} \setminus \delta)$. Let ε be a number satisfying $\varepsilon_0 \geqq \varepsilon > 0$. Then, by the condition (A) for ε_0, we see that $\{\|\phi_n\|_{V_\varepsilon/G}\}$ converges to 1 as n tends to ∞. Put $\psi_n=\phi_n/\|\phi_n\|_{V_\varepsilon/G}$. Then we have $\lim_{n\to\infty} L_{V_\varepsilon/G}(\hat{\kappa})(\psi_n)dxdy=k_0$. This implies that $a_\varepsilon=k_0$.

Conversely suppose that $a_\varepsilon=k_0$ for every ε mentioned in this lemma. Suppose that, for every such ε, there exists some $\phi_\varepsilon\in A(G, \delta)\smallsetminus\{0\}$ with $\hat{\kappa}=k_0|\phi_\varepsilon|/\phi_\varepsilon$ on V_ε. Then, for such ε and ε' with $\varepsilon>\varepsilon'$, we easily see that $\phi_{\varepsilon'}$ is a positive constant times ϕ_ε on V_ε. We may assume that $\phi_\varepsilon=\phi_{\varepsilon'}\in A(G, \delta)_1$ and put $\phi_0=\phi_\varepsilon$ on V_ε. Since it follows from definition that $V_0=\bigcup_{\varepsilon>0} V_\varepsilon$, we have $\hat{\kappa}=k_0|\phi_0|/\phi_0$ on V_0. Thus we see that $a_0=k_0$. Next we consider the case where, for some $\varepsilon>0$ with $\varepsilon_0\geqq\varepsilon$ if $\dim A(\Gamma, \sigma)=\infty$, there exists no $\phi_\varepsilon\in A(G, \delta)\smallsetminus\{0\}$ satisfying $\hat{\kappa}=k_0|\phi_\varepsilon|/\phi_\varepsilon$ on V_ε. Under the hypothesis of the lemma, this occurs only in the case where the condition (A) is satisfied for ε_0. Thus, by Lemma 3, we can prove that $a_0=k_0$ in the same manner as in the former part of the proof of the lemma. q. e. d.

Now we give the proofs of Theorems.

Proof of Theorem 1. Under the hypotheses of the theorem, the condition (I) is clearly sufficient for the condition (III) to hold. We have the reverse implication as an easy corollary to Lemma 3. If E/Γ is relatively compact in $\{U\cup(\hat{R}\smallsetminus\sigma)\}/\Gamma$, then, by [6, Theorem 1], we know that the condition (I) is sufficient for the condition (II) to hold, and that the condition (II) is sufficient for F to be extremal within Q. Examination of the proof of [6, Theorem 1], however, shows that if $\dim A(\Gamma, \sigma)<\infty$, then these assertions still hold even if E/Γ is not necessarily relatively compact in $\{U\cup(\hat{R}\smallsetminus\sigma)\}/\Gamma$. Thus it suffices to prove that, under the hypotheses of our theorem, the condition (III) holds provided that F is extremal within Q. Since Theorem 2 is a generalized form of this part of Theorem 1, the proof of Theorem 1 is accomplished once we verify Theorem 2.

Proof of Theorem 2. We may assume that $k_0>0$. Suppose the theorem does not hold. Then, by Lemma 4, it holds that $a_\varepsilon<k_0$ with some $\varepsilon>0$ (resp. $\varepsilon_0\geqq\varepsilon>0$) if (resp. unless) $\dim A(\Gamma, \sigma)<\infty$. By the Hahn-Banach and Riesz representation theorems, there exists $\beta\in L_\infty(G)$ which vanishes on $F(E_\varepsilon)$ and which satisfies the conditions $\|\beta\|_\infty=a_\varepsilon$ and

$$L_{V_\varepsilon/G}(\hat{\kappa})(\phi)=L_{V_\varepsilon/G}(\beta)(\phi)\quad\text{for every }\phi\in A(G, \delta).$$

Put $\alpha=\hat{\kappa}-\beta$ on V_ε and $\alpha=0$ on $F(E_\varepsilon)$. Then α is an element in $N(G, \delta)$ which vanishes on $F(E_\varepsilon)$ and which satisfies

$$\|(\hat{\kappa}-\alpha)|_{V_\varepsilon}\|_\infty=a_\varepsilon<k_0\,. \tag{3.5}$$

By our hypotheses and Lemma 1, there exists a curve $\tau(t)\in M_0(G, \delta)$, defined in an interval $(0, t_0)$, which vanishes on $F(E_\varepsilon)$ and which satisfies (3.1) for the above α. Let $\nu(t)$ be the complex dilatation of $F_{\tau(t)}\circ F$. Since $\tau(t)$ belongs to $M_0(G, \delta)$, we have $F_{\nu(t)}|_\sigma=F|_\sigma$. Thus, in the same way as in the proof of [6, Lemma 4], we see by (3.5) that $F_{\nu(t)}\in Q$ and that $\|\nu(t)|_D\|_\infty<k_0$ for a sufficiently small t. Therefore F is not extremal within Q. This contradiction proves Theorem 2.

Proof of Theorem 3. Assume that $a^* < k_0$. Then, by the Hahn-Banach and Riesz representation theorems, there exists $\beta \in L_\infty(G)$ which vanishes on $F(E)$ and which satisfies the conditions $\|\beta\|_\infty = a^*$ and

$$L_{F(D)/G}(\kappa)(\phi) = L_{F(D)/G}(\beta)(\phi) \quad \text{for every} \quad \phi \in A(G, \delta).$$

Put $\alpha = \kappa - \beta$ on $F(D)$ and $\alpha = 0$ on $F(E)$. Then α is an element in $N(G, \delta)$ which vanishes on $F(E)$ and which satisfies

$$\|(\kappa - \alpha)|_{F(D)}\|_\infty = a^* < k_0.$$

If we note that $E = E_\varepsilon$ for a sufficiently large $\varepsilon > 0$, then, by the hypothesis of the theorem, we can repeat the proof of Theorem 2 verbatim. Then we see that there exists some $F_\nu \in \tilde{Q}$ which satisfies $\|\nu|_D\|_\infty < k_0$. This contradiction proves Theorem 3.

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