J. Math. Kyoto Univ. (JMKYAZ)
26-2 (1986) 233-261

Local structure of analytic transformations of two complex variables, I

By

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Introduction.

Let T be a holomorphic mapping defined on a neighborhood V of the origin O of the space C^2 of two complex variables into C^2 which leaves O invariant. By a local analytic transformation we will mean either such a mapping T or the germ of T at O. The purpose of the present and the forthcoming papers is to investigate the structure of these local analytic transformations. Specifically we will study the following subjects, which are closely related to each other: (i) description of the behavior of the points near the fixed point O under the transformation T and its iterates T^n, $n=1, 2, \cdots$; (ii) intrinsic characterization of the structure of T, i.e., a characterization which does not rely on local coordinates around O.

We treat these problems, especially for semi-attractive transformations of type $(1, b)_1$, i.e., transformations which can be expressed in the form (6.1), in section 6, with the conditions $0<|b|<1$, $a_{20}\neq 0$. The study of transformations of this type is considered as a generalization of the result of Voronin [19] for the case of one complex variable.

In Part A (sections 1-5) of the present paper, we recall some known results and make some additional remarks for general local analytic transformations. These observations will serve to give a proper perspective of our main subject. The reader may begin with Part B and use Part A as reference.

In Part B (sections 6-10), a transformation T of type $(1, b)_1$ is investigated. We study the structure of the set of points P in the vicinity of the fixed point O such that the sequence $T(P)$, $T^2(P)$, $\cdots$ converges to O pointwise or uniformly in a neighborhood of P. The invariant curve of Poincaré lies on the boundary of the domain of uniformly convergent points. On this domain we introduce a system of coordinates so that the transformation T is reduced to a simple form, i.e., translation. As an application we will give, in section 10, an analogous result to the example of Bieberbach [2], which shows that there is a proper subdomain $\mathfrak{D}$ in C^2 which is biholomorphic to C^2. It may be of some interest that by the present method $\mathfrak{D}$ can be so constructed that the boundary $\partial\mathfrak{D}$ contains a complex line.

In the forthcoming paper II, we will give a complete solution for the prob-

Received November 20, 1984.

lem (ii) posed above.

A. Classification and general remarks.

1. Classification.

Let T be a local analytic transformation at $O=(0, 0)\in \boldsymbol{C}^2$. We use the notation $P=(x, y)$, $T(P)=(x_1, y_1)$ to denote any point in the vicinity of O and its image under T. The transformation T is expressed by convergent power series in x, y:

$$(1.1)\qquad \begin{cases} x_1=f(x, y)=\sum_{i,j} a_{ij}x^i y^j, \\ y_1=g(x, y)=\sum_{i,j} b_{ij}x^i y^j, \end{cases}$$

where the constant terms $a_{00}=b_{00}=0$. If the Jacobian determinant $J_T(O)=a_{10}b_{01}-a_{01}b_{10}$ of T at O is not 0, then T has an inverse T^{-1}. All the transformations constitute a semi-group by composition and all the invertible transformations constitute a group.

Let $S:(X, Y)\mapsto(x, y)$ be an invertible transformation and consider $\tilde{T}=S^{-1}\circ T\circ S:(X, Y)\mapsto(X_1, Y_1)$. This new transformation $\tilde{T}$ can be regarded as the expression of T with respect to the coordinate system (X, Y). When $\tilde{T}$ is thus regarded, it will be denoted by the same letter T.

The linear part (differential) of T at O is $dT(O)=\begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01}\end{pmatrix}$. Let a, b denote the eigenvalues of $dT(O)$. We will call T to be of type (a, b). Let us restrict attention to invertible transformations and classify them according to the eigenvalues a, b:

I′. T is called attractive if $|a|, |b|<1$.

I″. T is called repulsive if $|a|, |b|>1$.

II. T is called of saddle type if $|a|<1<|b|$ (or $|b|<1<|a|$).

III′. T is called semi-attractive if $|a|=1$, $|b|<1$ (or $|b|=1$, $|a|<1$).

III″. T is called semi-repulsive if $|a|=1$, $|b|>1$ (or $|b|=1$, $|a|>1$).

IV. T is called neutral if $|a|=|b|=1$.

The fixed point O will be correspondingly called attractive, repulsive, etc.

2. Attractive transformations and repulsive transformations.

Let us first mention the cases of attractive and repulsive transformations. In these cases we can find canonical forms for T, and therefore we have a complete description of T (as far as the local structure is concerned). To fix the ideas we assume that the eigenvalues a, b satisfy the condition $0<|a|\leqq|b|<1$ or $1<|b|\leqq|a|$.

(α) If $b^m\neq a$ for any positive integer m, then by choosing an appropriate coordinate system (x, y), the transformation T is brought to the form

$$
(2.1) \qquad \begin{cases} x_1 = ax, \\ y_1 = by. \end{cases}
$$

(β) If $b^m = a$ for a positive integer m, then T can be brought to the form either (2.1) or (2.2):

$$
(2.2) \qquad \begin{cases} x_1 = ax + y^m, \\ y_1 = by. \end{cases}
$$

We refer to Lattès [10] for the proof. See also Sternberg [17]; and Reich [15], [16] for the case of many variables.

3. The set of fixed points; invariant curves of Poincaré.

3.1. In studying the intrinsic structure of a transformation T it is fundamental to find objects which are invariant under T. The simplest of such objects are the fixed points. We fix a domain of definition V of T with a coordinate system (x, y). T is expressed in the form (1.1). A point P in V is said to be a fixed point of T if $T(P) = P$. The set of all fixed points in V is denoted by J. The set J is the common zero of the holomorphic functions $f(x, y) - x$ and $g(x, y) - y$; therefore J is an analytic set in V containing the origin O. Hence either (i) O is isolated in J, (ii) J is of dimension 1 at O, or (iii) $J = V$.

In the case (iii), T is the identity transformation. In the case (ii), letting $q(x, y)$ be the defining equation of the curve J, the transformation T is written as

$$
(3.1) \qquad \begin{cases} x_1 = x + q(x, y)h(x, y), \\ y_1 = y + q(x, y)k(x, y). \end{cases}
$$

We see from this expression that, when J is singular at O, T is of type (1, 1). When J is non-singular at O, we can choose a coordinate system (x, y) so that $q(x, y) = y$, i.e. J coincides with the x-axis. T takes the form

$$
(3.2) \qquad \begin{cases} x_1 = x + yh(x, y), \\ y_1 = y(1 + k(x, y)). \end{cases}
$$

T is of type $(1, b)$, $b = 1 + k(0, 0)$. Under this situation assume that T is semi-attractive or semi-repulsive: $|b| \neq 0, 1$. Then we can find an appropriate coordinate system (x, y) relative to which T takes the form

$$
(3.3) \qquad \begin{cases} x_1 = x, \\ y_1 = by. \end{cases}
$$

For the proof see Nishimura [11].

3.2. Consider a holomorphic mapping F of a neighborhood of the origin 0 of the complex line $\boldsymbol{C}$ into $\boldsymbol{C}^2$ satisfying the conditions

(i) $F(0)=O$;

(ii) $T(F(\zeta))=F(c\zeta)$, for a constant $c\neq 0$;

(iii) $dF/d\zeta(0)\neq 0$.

By differenciating the equation (ii), we obtain $dT(O)\,dF/d\zeta(0)=c\,dF/d\zeta(0)$. Hence $dF/d\zeta(0)$ is an eigenvector of $dT(0)$ with eigenvalue c. We will call F to be the Poincaré mapping corresponding to the eigenvalue c (relative to the transformation T at O). We denote by C either the image under F of a neighborhood of $\zeta=0$ or its germ. C will be called an invariant curve (of Poincaré). If there exists a Poincaré mapping $\zeta\mapsto F(\zeta)$, then $\zeta\mapsto F(k\zeta)$ is also, for any $k\neq 0$. These are the all Poincaré mappings corresponding to the same eigenvalue. The invariant curve C is uniquely determined.

The following theorem is fundamental in all of our investigations:

Theorem 3.1. (Poincaré [14], see also Lattès [9], Picard [13]). *Let a, b the eigenvalues of $dT(0)$. If $|a|\neq 1$ and $a^m\neq b$ for any integer $m\geqq 0$, then there is a Poincaré mapping F corresponding to the eigenvalue a.*

3.3. Making use of the invariant curves some of the transformations can be brought to a simpler form.

(α) Attractive or repulsive case: When T is of the form (2.1), the x- and y-axes are the invariant curves corresponding to the eigenvalues a and b, respectively. There is no other if $a\neq b$; all lines $x:y=$const. are invariant curves if $a=b$. When T is of the form (2.2), the x-axis is the only invariant curve.

(β) Case of suddle type: We can take a coordinate system (x, y) so that the x- and y-axes are invariant curves corresponding to the eigenvalues a and b respectively, and that T reduces to linear: $(x, 0)\to(ax, 0)$, $(0, y)\to(0, by)$ on these axes, namely,

$$\text{(3.4)}\qquad \begin{cases} x_1=ax+xyh(x, y), \\ y_1=by+xyk(x, y). \end{cases}$$

(γ) Semi-attractive or semi-repulsive case: We can take a coordinate system (x, y) so that the y-axis is the invariant curve and that T is linear:

$$\text{(3.5)}\qquad \begin{cases} x_1=x+xh(x, y), \\ y_1=by+xk(x, y). \end{cases}$$

If, furthermore, O is a non-isolated fixed point, then T reduces to the form (3.3).

4. Simply convergent points and uniformly convergent points.

4.1. Let T be a local analytic transformation at $O\in \boldsymbol{C}^2$. As a domain of definition of T we choose and fix a bounded domain V in $\boldsymbol{C}^2$. Let P be a point in V and $T(P)$ its image (consequent) under T. If $T(P)$ lies in V, then the consequent point $T^2(P)$ of $T(P)$ can be defined. If we can continue this process indefinitely, we will have the sequence

$$P,\ T(P),\ T^2(P),\ \cdots,\ T^n(P),\ \cdots.$$

Such a point P is called *stable*. A stable point P is called *simply convergent* if this sequence $T^n(P)$, $n=0, 1, 2, \cdots$, is convergent. Obviously every fixed point is simply convergent and every consequent point of a simply convergent point is also. A simply convergent point P is called *uniformly convergent* if there is a neighborhood W of P in V consisting of simply convergent points on which the sequence of the holomorphic mappings $T^n|W$, $n=1, 2, \cdots$ is uniformly convergent.

Let S, K and U denote the sets of all stable, simply convergent and uniformly convergent points in V. Then U coincides with the interior K° of K. Indeed, it is clear by definition that every uniformly convergent point is in K°; on the other hand, since the sequence $T^n|K^\circ$, $n=0, 1, 2, \cdots$, constitutes a normal family and is simply convergent, it is uniformly convergent on compact sets in K° by Vitali's theorem.

Let T^∞ denote the limit of the sequence $T^n|K$; $T^\infty(P)=\lim_{n\to\infty} T^n(P)$, $P\in K$. $T^\infty(P)$ is said to be the limit point of P. Obviously T^∞ is holomorphic on U. T^∞ is the identity mapping on the set J of the fixed points. Since $T\circ T^\infty=T^\infty$, every limit point is a fixed point. In particular, if O is the only fixed point of T in V, then $T^\infty(K)=\{O\}$.

4.2. It should be emphasized that the definitions of stable, simply convergent and uniformly convergent points depend upon the choice of the domain of definition V. We cannot even define a concept as "the germ of the set of simply (or uniformly) convergent points".

However the following properties for T are obviously independent of the choice of V:

(i) O is a uniformly convergent point.

(ii) O is the limit of a uniformly convergent point.

(iii) O is the limit of a simply convergent point other than O.

Let us examine these properties according to the classification given in 1.

(α) Suppose that T is attractive. We assume that T is of the form (2.1) or (2.2). If the domain of definition V is sufficiently small, then we can find a constant k, $0<k<1$, such that $|x_1|\leqq k|x|$, $|y_1|\leqq k|y|$. If follows that O is a uniformly convergent point and the limit of every point in the vicinity of O is O.

(β) Suppose that T is repulsive, and of the form (2.1) or (2.2). Choosing a sufficiently small V, we can find a constant $K>1$ such that $|x_1|>K|x|$, $|y_1|>K|y|$. It follows that O is the only stable point in the vicinity of O.

(γ) Suppose that T is of saddle type. To fix the ideas we choose a coordinate system (x, y) and a domain of definition V of T in such a way that

$$V=\{|x|<\rho,\ |y|<\rho\}$$

and that T takes the form (3.4) with $|a|<1<|b|$.

Obviously the points on the x-axis: $y=0$ are simply convergent. These are the only stable points in V. To show this notice first there is a number $K>1$ such that $|y_1|\geqq K|y|$ for all $P=(x, y)$ in V, by choosing ρ sufficiently small. If P is a stable point in V, then $T^n(P)$ are all in V and $|y_n|\geqq K^n|y|$. Therefore $y=0$, i.e., P is on the x-axis. Consequently there is no uniformly convergent point in V.

(δ) Suppose that T is semi-repulsive. There is no uniformly convergent point in a sufficiently small neighborhood V of O. Indeed, since $|J_T(O)|>1$, there are a neighborhood V and a constant $K>1$ such that $|J_T(P)|\geqq K$ on V. Let P be any stable point in V. Consider the sequence of Jacobian determinants $J_{T^n}(P)$ of T^n, $n=1, 2, \cdots$. We have $|J_{T^n}(P)|=|J_T(T^{n-1}(P))\cdots J_T(T(P))J_T(P)|\geqq K^n$. Therefore $J_{T^n}(P)$ is non-convergent and hence P is not a uniformly convergent point. (P may be a simply convergent point.)

By (β)~(δ), we obtain the following

Proposition 4.1. *If O is the limit of a uniformly convergent point, then T is attractive, semi-attractive or neutral.*

Proposition 4.2. *Suppose that O is a uniformly convergent point. Then T is either* (i) *attractive,* (ii) *semi-attractive and the set J of fixed points is a curve, (i.e., it can be brought to the form* (3.3)), *or* (iii) *the identity.*

Proof. Letting a, b, be the eigenvalues of $dT(O)$, the sequences a^n, b^n, $n=0, 1, 2, \cdots$, are convergent. Consequently ($|a|<1$ or $a=1$) and ($|b|<1$ or $b=1$). Consider the sequence of the Jacobian determinants $J_{T^n}(P)$, $n=0, 1, 2, \cdots$, which converges uniformly to $J_{T^\infty}(P)$ on U. Since $J_{T^n}(P)$ are non-vanishing, the limit $J_{T^\infty}(P)$ is non-vanishing or identically 0. If $a=b=1$, then $J_{T^\infty}(O)=1$ and hence T^∞ is invertible. Therefore a neighborhood of O is covered by the image under T^∞ of U, hence by the set J of the fixed points. Thus T is the identity. In the other cases $J_{T^\infty}(O)=0$, and hence $J_{T^\infty}(P)=0$ identically on U. Therefore T^∞ is a mapping of rank $\leqq 1$. If ($a=1$, $|b|<1$) or ($|a|<1$, $b=1$), then T^∞ is of rank 1 at O and hence on a neighborhood of O. $T^\infty(U)$ is a curve which is regular at O. Since $T^\infty(U)$ is contained in J, we get the case (ii). If $|a|$, $|b|<1$, then T is attractive by definition. q. e. d.

5. Global transformations.

5.1. Let $\mathfrak{M}$ be a complex manifold of dimension m and T a holomorphic mapping of $\mathfrak{M}$ into itself. We are concerned with the behavior of the points in $\mathfrak{M}$ under the iterated mappings T^n, $n=1, 2, \cdots$. A point P in $\mathfrak{M}$ is called simply convergent if the sequence of the points $T^n(P)$, $n=1, 2, \cdots$, is convergent. P is called uniformly convergent if there is a neighborhood W of P such that the sequence of mappings $T^n|W$, $n=1, 2, \cdots$, is convergent in compact-open topology. P is called *normal* if there is a neighborhood W of P such that the set of the mappings $T^n|W$ is relatively compact in compact-open topology. The sets of all fixed, simply convergent, uniformly convergent, normal points are denoted by $\mathfrak{J}=\mathfrak{J}(T)$, $\mathfrak{K}=\mathfrak{K}(T)$, $\mathfrak{U}=\mathfrak{U}(T)$, $\mathfrak{N}=\mathfrak{N}(T)$, respectively. It is obvious that $\mathfrak{U}$, $\mathfrak{N}$ are open sets and that $\mathfrak{J}\subseteq\mathfrak{K}$, $\mathfrak{U}\subseteq\mathfrak{K}$, $\mathfrak{U}\subseteq\mathfrak{N}$.

For any integer $q>0$, we have correspondinng concepts for transformation T^q.

(α) It is clear that

$$\mathfrak{N}(T^q)=\mathfrak{N}(T), \qquad \text{for all } q\ ;$$

$$\mathfrak{K}(T^q)\subseteq\mathfrak{K}(T^{q'}),\ \mathfrak{U}(T^q)\subseteq\mathfrak{U}(T^{q'}),\ \text{if } q' \text{ is a multiple of } q.$$

(β) The sets $\mathfrak{J}(T^q)$, $\mathfrak{K}(T^q)$, $\mathfrak{U}(T^q)$, $\mathfrak{N}$ are invariant in the sense: $T(\mathfrak{J}(T^q))\subseteq\mathfrak{J}(T^q)$, $T(\mathfrak{K}(T^q))\subseteq\mathfrak{K}(T^q)$, $T(\mathfrak{U}(T^q))\subseteq\mathfrak{U}(T^q)$, $T(N)\subseteq N$.

(γ) We decompose $\mathfrak{N}$ into connected components: $\mathfrak{N}=\bigcup_\lambda\mathfrak{D}_\lambda$. If $\mathfrak{D}_\lambda\cap\mathfrak{U}(T^q)\neq 0$, then $\mathfrak{D}_\lambda\subseteq\mathfrak{U}(T^q)$. This an immediate consequence of Vitali's theorem. In other words, for every q, $\mathfrak{U}(T^q)$ is composed of some of the connected components of $\mathfrak{N}$.

Proposition 5.1. *Suppose that $\mathfrak{M}$ is a Stein manifold. Then $\mathfrak{N}$ is a Runge domain in $\mathfrak{M}$. Hence so are all the connected conponents $\mathfrak{D}_\lambda$ of $\mathfrak{N}$. In particular they are all Stein.*

Proof. There are holomorphic functions φ_j, $j=1,\cdots,k$, on $\mathfrak{M}$ such that $(\varphi_j)_{j=1}^k$ is a holomorphic imbedding of $\mathfrak{M}$ into C^k. Denote by $\mathfrak{N}_j$ the region of normality of the family of functions $\{\varphi_j(T^n(P)(,\ n=1, 2,\cdots\}$. Then $\mathfrak{N}$ is the intersection of the regions $\mathfrak{N}_j$. By the theorem of Cartan and Thullen [3], each of $\mathfrak{N}_j$ is a Runge domain; hence $\mathfrak{N}$ is a Runge domain. (In the original theorem of Cartan and Thullen, $\mathfrak{M}$ is supposed to be an unramified domain over the space of complex variables C^m, $m=\dim\mathfrak{M}$. This theorem is generalized to the case $\mathfrak{M}$ is a Stein manifold, using the method of Docquier and Grauert [4].)

(δ) Let $T^\infty:\mathfrak{K}\mapsto\mathfrak{M}$ denote the limit of the sequence T^n. Every limit point $T^\infty(P)$, $P\in\mathfrak{K}$, is a fixed point.

Example. Let $\mathfrak{M}$ be the product of two Riemann spheres with inhomogeneous coordinates x, y:

$$\mathfrak{M}=\{|x|\leqq\infty,\ |y|\leqq\infty\}$$

and let T be an automorphism of $\mathfrak{M}$

$$\begin{cases} x_1=x+1 \\ y_1=by \end{cases}$$

with $0<|b|<1$. There are two fixed points $(\infty, 0)$ and (∞, ∞). The fixed point $(\infty, 0)$ is semi-attractive of type $(1, b)$ and (∞, ∞) is semi-repulsive of type $(1, 1/b)$. Every point in $\mathfrak{M}$ is simply convergent. The points (x, y) with $y\neq\infty$ have $(\infty, 0)$, and the points (x, ∞) have (∞, ∞) as limit points respectively. The points (x, y) with $x\neq\infty$, $y\neq\infty$ are uniformly convergent to $(\infty, 0)$. The points (∞, y), $y\neq\infty$, are interior to the set of simply convergent points with limit $(\infty, 0)$, but they are not uniformly convergent.

5.2. Now we consider the case where $\mathfrak{M}$ is of dimension 2 and T is an automorphism of $\mathfrak{M}$ having a fixed point O.

Suppose that O is an attractive fixed point. Let $\mathfrak{D}$ denote the connected component of $\mathfrak{U}(T)$ containing O. Then $\mathfrak{D}$ is the set of all simply convergent points with limit O. Bieberbach [2] showed that this domain $\mathfrak{D}$ is biholomorphic to $\boldsymbol{C}^2$. (See also Kodaira [8].) We note that $\mathfrak{D}$ is a Runge domain if $\mathfrak{M}$ is Stein by Proposition 5.1. Incidentally, the author does not know whether a domain $\mathfrak{D}$ in $\boldsymbol{C}^2$ which is biholomorphic to $\boldsymbol{C}^2$ is always a Runge domain.

Now suppose that O is a fixed point such that there is an invariant curve of Poincaré with center O corresponding to an eigenvalue a, $|a|\neq 0, 1$. The mapping F is analytically continued to an injective holomorphic mapping of $\boldsymbol{C}$ (ζ-plane) into $\mathfrak{M}$ by means of the functional equation $T(F(\zeta))=F(a\zeta)$. The analytic curve $\mathfrak{C}=F(\boldsymbol{C})$ is not generally a closed set in $\mathfrak{M}$. In view of 4.2. (γ), we infer that, when O is a hyperbolic fixed point of type (a, b), $|a|<1<|b|$, the curve $\mathfrak{C}$ is the set of simply convergent points with limit O.

5.3. (Appendix). In connection with the example of Bieberbach, we will point out a property of the boundary of a domain $\mathfrak{D}$ in a complex manifold of dimension 2 which is biholomorphic to $\boldsymbol{C}^2$.

Proposition 5.2. *Let $\mathfrak{M}$ be a complex manifold of dimension 2 and $\mathfrak{E}$ a subset of $\mathfrak{M}$. Assume that, for every point P in $\mathfrak{E}$, there exist a parabolic Riemann surface $\mathfrak{R}$ and a non-constant holomorphic mapping f of $\mathfrak{R}$ into $\mathfrak{M}$ such that $P\in f(\mathfrak{R})\subseteq\bar{\mathfrak{E}}$. Then the set $\mathfrak{M}\setminus\bar{\mathfrak{E}}$, namely the set of all exterior points of $\mathfrak{E}$, is a pseudoconvex region (if it is not empty). In particular, if $\mathfrak{M}$ is Stein, then $\mathfrak{M}\setminus\bar{\mathfrak{E}}$ is also Stein.*

Corollary. *If $\mathfrak{D}$ is a domain in $\boldsymbol{C}^2$ biholomorphic to $\boldsymbol{C}^2$. Then $\boldsymbol{C}^2\setminus\bar{\mathfrak{D}}$ is a region of holomorphy. If F is a non-constant holomorphic mapping of $\boldsymbol{C}$ into $\boldsymbol{C}^2$ then $\boldsymbol{C}^2\setminus\overline{F(\boldsymbol{C})}$ is a region of holomorphy.*

Proposition 5.2 is a variant of theorems of Suzuki [18, Corollaire au lemme 8] and Nishino-Suzuki [12, Théorème 1]. The proof is done in the same way: Consider an open set W in $\mathfrak{M}$ biholomorphic to the dicylinder $\{|x|, |y|<1\}$ and a subset W^* of W defined by

$$W^*=\{|x|<r,\ |y|<1\}\cup\{|x|<1,\ s<|y|<1\}\,, \qquad (0<r,\ s<1)\,,$$

such that $W^*\cap\bar{\mathfrak{E}}=\emptyset$. The proposition is proved if we show that $W\cap\bar{\mathfrak{E}}=\emptyset$ under this situation. Suppose that $W\cap\bar{\mathfrak{E}}\neq\emptyset$. Then there are a point $P\in W\cap\mathfrak{E}$ and a holomorphic mapping f of a parabolic Riemann surface $\mathfrak{R}$ into $\mathfrak{M}$ such that $f^{-1}(W)$ is a non-empty region in $\mathfrak{R}$. Consider the subharmonic function $1/|x\circ f|$ on $f^{-1}(W)$. This function takes the boundary value 1 at every boundary point of $f^{-1}(W)$. By the maximum principle for a parabolic Riemann surface (see, for example, Ahlfors and Sario [1, Chap. IV, §1, 6]), we have $1/|x\circ f|<1$ on $f^{-1}(W)$. This contradiction proves the proposition.

B. Semi-attractive transformations and semi-repulsive transformations of type $(1, b)_1$.

6. Reduction of local coordinates.

Let us consider a local analytic transformation of type $(1, b)$, with $|b|\neq 0, 1$. With respect to an appropriate local coordinate system (x, y), the linear part $dT(O)$ of T is diagonalized and $T:(x, y)\mapsto(x_1, y_1)$ takes the form

$$(6.1)\qquad \begin{cases} x_1=f(x, y)=x+\sum_{i+j\geq 2} a_{ij}x^i y^j, \\ y_1=g(x, y)=by+\sum_{i+j\geq 2} b_{ij}x^i y^j. \end{cases}$$

We will first reduce the expression of T as simple as possible by means of successive changes of local coordinates. If $|b|>1$, then we consider the inverse T^{-1} in place of T; and we assume T to be semi-attractive: $0<|b|<1$.

(α) We take a coordinate system (x, y) with respect to which the Poincaré mapping F takes the form $\zeta\to(x, y)=(0, \zeta)$. Then T is expressed by the power series in x:

$$(6.2)\qquad \begin{cases} x_1=a_1(y)x+a_2(y)x^2+\cdots, \qquad (a_1(0)=1), \\ y_1=by+b_1(y)x+\cdots, \end{cases}$$

where $a_i(y)$ and $b_j(y)$ are holomorphic functions of the variable y on a neighborhood of $y=0$. (See 3.3, (γ).)

(β) In the expression (6.2) we can assume that $a_1(y)\equiv 1$, by choosing an appropriate coordinate system. To see this we consider a new coordinate system (X, Y) defined by

$$\begin{cases} x=P(Y)X, \\ y=Y, \end{cases} \qquad \begin{cases} X=x/P(y), \\ Y=y, \end{cases}$$

where $P(Y)$ is holomorphic on a neighborhood of $Y=0$ and $P(0)=1$. The transformation T of the form (6.2) is expressed with respect to (X, Y) as

$$\begin{aligned} X_1&=\frac{1}{P(y_1)}x_1=\frac{1}{P(by+\cdots)}\{a_1(y)x+\cdots\} \\ &=\frac{1}{P(bY+\cdots)}\{a_1(Y)P(Y)X+\cdots\} \\ &=a_1(Y)\frac{P(Y)}{P(bY)}X+(\text{terms containing } X^2), \end{aligned}$$

$$Y_1=bY+(\text{terms containing } X).$$

In order that the coefficient of $X\equiv 1$, the function $P(Y)$ should satisfy the equation

$$P(Y)/P(bY)=1/a_1(Y).$$

This has a unique solution

$$P(Y)=1/\prod_{n=0}^{\infty} a_1(b^n Y).$$

The infinite product is convergent since $a_1(0)=1$ and $|b|<1$. This shows our assertion.

(γ) For any integer $i \geqq 2$, we can choose a coordinate system with respect to which $a_2(y), \cdots, a_i(y)$ are all constants. We prove this by induction. Supopse that a coordinate system (x, y) is already chosen so that T has the form

$$\begin{cases} x_1=x+a_2x^2+\cdots+a_{i-1}x^{i-1}+a_i(y)x^i+\cdots, & (i \geqq 2), \\ y_1=by+\cdots. \end{cases}$$

We introduce a new coodinate system (X, Y) by

$$\begin{cases} x=X+P_i(Y)X^i, \\ y=Y, \end{cases} \qquad \begin{cases} X=x-P_i(y)x^i+\cdots, \\ Y=y, \end{cases}$$

with respect to which T takes the form

$$\begin{aligned} X_1 &= x_1-P_i(y_1)x_1^i+\cdots \\ &= x+a_2x^2+\cdots+a_{i-1}x^{i-1}+\{a_i(y)-P_i(by)\}x^i+\cdots \\ &= X+a_2X^2+\cdots+a_{i-1}X^{i-1}+\{P_i(Y)+a_1(Y)-P_i(bY)\}X^i+\cdots, \\ Y_1 &= bY+\cdots. \end{aligned}$$

We should determine $P_i(Y)$ satisfying the equation

$$P_i(Y)-P_i(bY)=a_i(0)-a_i(Y).$$

The solution is given by the convergent series

$$P_i(Y)=\sum_{n=0}^{\infty}\{a_i(0)-a_i(b^nY)\}+(\text{arbitrary constant}).$$

(δ) Furthermore, for any integer $j \geqq 1$ we can choose a coordinate system with respect to which $b_1(y), \cdots, b_j(y)$ are all linear monomials: $b_1(y)=b_1y, \cdots, b_j(y)=b_jy$. We prove this by induction. Suppose that a coordinate system (x, y) is already chosen so that T has the form

$$\begin{cases} x_1=x+a_2x^2+\cdots+a_ix^i+\cdots, \\ y_1=by+b_1yx+\cdots+b_{j-1}yx^{j-1}+b_j(y)x^j+\cdots, & (j \geqq 1). \end{cases}$$

We introduce a new coordinate system (X, Y) by

$$\begin{cases} x=X, \\ y=Y+Q_j(Y)X^j, \end{cases} \qquad \begin{cases} X=x, \\ Y=y-Q_j(y)x^j+\cdots, \end{cases}$$

with respect to which T takes the form

$$\begin{aligned} X_1 &= X + a_2X^2 + \cdots + a_iX^i + \cdots, \\ Y_1 &= y_1 - Q_j(y_1)x_1{}^j + \cdots \\ &= by + b_1yx + \cdots + b_{j-1}yx^{j-1} + \{b_j(y) - Q_j(by)\}x^j + \cdots \\ &= bY + b_1YX + \cdots + b_{j-1}YX^{j-1} + \{bQ_j(Y) + b_j(Y) - Q_j(bY)\}X^j + \cdots. \end{aligned}$$

We should determine $Q_j(Y)$ satisfying the equation

$$bQ_j(Y) - Q_j(bY) = b_j'(0) - b_j(Y).$$

To solve this equation we put

$$\begin{aligned} b_j(Y) &= \beta_0 + \beta_1Y + \beta_2Y^2 + \cdots, \qquad (\beta_1 = b_j'(0)), \\ Q_j(Y) &= q_0 + q_1Y + q_2Y^2 + \cdots. \end{aligned}$$

Then the above equation reduces to

$$\sum_{n=0}^{\infty}(b-b^n)q_nY^n = -\beta_0 - \sum_{n=0}^{\infty}\beta_nY^n.$$

Hence $q_n = \beta_n/(b^n - b)$ for $n \neq 1$, while q_1 is arbitray. The power series thus determined is convergent since

$$|q_n| = |\beta_n|/|b^n - b| \leqq |\beta_n|/|b^2 - b|.$$

(ε) *Remark and definition.* Thus we can obtain, for any i, j, a coordinate system (x, y) with respect to which T takes the form

$$(6.3)_{ij} \qquad \begin{cases} x_1 = x + a_2x^2 + \cdots + a_ix^i + a_{i+1}(y)x^{i+1} + \cdots, \\ y_1 = by + b_1yx + \cdots + b_jyx^i + b_{j+1}(y)x^{j+1} + \cdots. \end{cases}$$

When T is of the form $(6.3)_{ij}$, the inverse T^{-1} has a similar form with respect to the same coordinate system. So our result is valid for any transformation of type $(1, b)$ with $|b| \neq 0, 1$.

If we admit a change of coordinates by means of formal power series, it is possible to reduce all $a_2(y)$, $a_3(y)$, $\cdots$ to constants and all $b_1(y)$, $b_2(y)$, $\cdots$ to linear monomials. However these formal power series are not generally convergent, as we shall see later.

We will divide the class of all transformations of type $(1, b)$, $|b| \neq 0, 1$, into subclasses. Suppose that T is expressed, relative to a certain coordinate system (x, y), in the form $(6.3)_{ij}$ in such a way that $a_2 = \cdots = a_k = 0$ and $a_{k+1} \neq 0$ $(k+1 \leqq i)$. Then, for any coordinate system (x, y) which expresses T in the form $(6.3)_{ij}$, this condition for coefficients is invariant. Such a transformation T is said to be of type $(1, b)_k$. If, for any coordinate system with arbitrarily large i, we have $a_2 = \cdots = a_i = 0$, then T is said to be of type $(1, b)_\infty$.

In the sequel we treat the case of type $(1, b)_1$, i. e., $a_2 \neq 0$. This condition is equivalent to say that $a_{20} \neq 0$ in the original expression (6.1).

(ζ) Finally we make the change of coordinate system

$$\begin{cases} x=1/z, \\ y=w, \end{cases} \quad \begin{cases} z=1/x, \\ w=y, \end{cases}$$

and regard T as a local analytic transformation defined on a neighborhood of the point $(\infty, 0)$ in the product space

$$\hat{\boldsymbol{C}}\times\boldsymbol{C}=\{|z|\leqq\infty,\ |w|<\infty\}.$$

The expression $(6.3)_{ij}$ is brought to the form

$$(6.4)\qquad \begin{cases} z_1=z+A_0+A_1/z+\cdots, \\ w_1=bw+b_1w/z+\cdots, \end{cases}$$

where $A_0=-a_2$, $A_1=a_2{}^2-a_3$, $\cdots$.

7. Base of uniform convergence.

7.1. Let us first introduce the concept of bases of convergence in a somewhat general situation. Let T be an invertible local analytic transformation at O and suppose that O is an isolated fixed point. At first we fix a domain V of T. An open subset D of V is said to be a base of uniform convergence of T if it satisfies the conditions:

(i) Every point in D is uniformly convergent to O.

(ii) For every uniformly convergent point P in V, there is a sufficintly large number n_0 such that $T^{n_0}(P)$ is in D.

(iii) $T(D)\subseteq D$.

This definition does not depend on the choice of V in the sense of (α) of the following lemma.

Lemma 7.1. (α) *Let D be a base of uniform convergence relative to a domain V, and let V' another domain of T containing D. Then D is a base of convergence relative to V' also.*

(β) *If D is a base of uniform convergence, then $T(D)$ is also.*

(γ) *If D_1 and D_2 are bases of uniform convergence, then $D_1\cup D_2$ and $D_1\cap D_2$ are also.*

Proof. To prove (α), it suffices to verify the condition (ii) relative to V'. Let P be a uniformly convergent point in V'. Then by difinition there is a number n_1 such that $T^n(P)\in V\cap V'$ for all $n\geqq n_1$. Then $T^{n_1}(P)$ is a uniformly convergent point in V, and consequently $T^{n_1+n_0}(P)\in D$ for some n_0. Thus (α) is shown. Assertions (β) and (γ) are obvious.

We have the corresponding concept for simple convergence. A subset (which need not be open) E of V is said to be a base of simple convergence of T if it satisfies the conditions:

(i) Every point in E is simply convergent to O.

(ii) For every simply convergent point P in V, there is a sufficiently large number n_0 such that $T^{n_0}(P)$ is in E.

(iii) $T(E)\subseteq E$.

The properties corresponding to the above lemma hold for E.

7.2. Now we return to the semi-attractive transformation T of type $(1, b)_1$. We use the expression (6.4) with changed notation: T is a transformation

$$\left\{\begin{array}{l} z_1=z+a_0+a_1/z+a_2(w)/z^2+\cdots, \\ w_1=bw+b_1w/z+b_2(w)/z^2+\cdots \end{array}\right. \tag{7.1}$$

of a neighborhood of $O=(\infty, 0)$ satisfying the conditions $0<|b|<1$, $a_0\neq 0$. As the domain of definition of T we specify a neighborhood V of O of the form

$$V=\{R'<|z|\leqq\infty,\ |w|<\rho\}.$$

The domain V is assumed to be so chosen that T is analytically continued to a neighborhood of the closure of V and injective there, and that there is no fixed point other than O. By C we denote the portion of the invariant curve in V, i. e.,

$$C=\{z=\infty,\ |w|<\rho\}.$$

Our purpose is to construct bases of uniform and simple convergence for T.

Let K_1, K_2 be positive numbers such that

$$|z_1-z-a_0|=|a_1/z+a_2(w)/z^2+\cdots|<K_1/|z|,$$

and that

$$|w_1-bw|=|b_1w/z+b_2(w)/z^2+\cdots|<K_2/|z|$$

on V. Let α denote the argument of $a_0=|a_0|e^{i\alpha}$. We take and fix a real number θ with $|\theta-\alpha|<\pi/2$. Let $R=R(\theta)$ be a sufficiently large number such that

$$\delta:=|a_0|\cos(\theta-\alpha)-K_1/R>0,$$

and

$$|b|\rho+K_2/R<\rho.$$

We define the domain D in V by

$$D=D_{\theta,R}=\{\mathrm{Re}\,(e^{-i\theta}z)>R,\ |w|<\rho\}, \tag{7.2}$$

namely, the product of a half plane in the z-plane and a disk in the w-plane. The invariant curve C lies in the boundary of D.

Proposition 7.2. (i) *The domain D is a base of uniform convergence.* (ii) *The set $D\cup C$ is a base of simple convergence.*

We prove this proposition in the steps $(\alpha)\sim(\varepsilon)$.

(α) First we notice the following: If $|z(P)|>R$, $|w(P)|<\rho$, then

$$\text{(7.3)}\qquad \begin{aligned}\operatorname{Re}(e^{-i\theta}z_1)&=\operatorname{Re}(e^{-i\theta}z)+\operatorname{Re}(e^{-i\theta}a_0)+\operatorname{Re}\{e^{-i\theta}(a_1/z+\cdots)\}\\ &\geqq \operatorname{Re}(e^{-i\theta}z)+|a_0|\cos(\theta-\alpha)-K_1/R\\ &=\operatorname{Re}(e^{i\theta}z)+\delta,\end{aligned}$$

and

$$|w_1|\leqq|bw|+K_2/R\leqq|b|\rho+K_2/R<\rho.$$

Hence, in particular, if $P\in D$, then $T(P)\in D$; i.e., $T(D)\subseteq D$. Since $T(C)\subseteq C$, we have $T(D\cup C)\subseteq D\cup C$.

(β) Next we show that the sequence $T^n(P)=(z_n, w_n)$, $n=1, 2, \cdots$, is uniformly convergent to $O=(\infty, 0)$ on D. Suppose that $P\in D$. Using the inequality (7.3) n times, we obtain

$$\operatorname{Re}(e^{-i\theta}z_n)\geqq\operatorname{Re}(e^{-i\theta}z)+n\delta\geqq R+n\delta.$$

This proves that $z_n\to\infty$ as $n\to\infty$ uniformly on D. To prove that $w_n\to 0$, we choose $\varepsilon>0$ such that $|b|+\varepsilon<1$. We claim that, for any integer $\nu>0$, there exists a number $n(\nu)$ such that if $n>n(\nu)$ then $|w_n|<(|b|+\varepsilon)^\nu\rho$. This is shown by induction on ν. For $\nu=0$, it suffices to put $n(0)=0$. Suppose that $n(\nu)$ is already determined. We take $n(\nu+1)$ so large that $n(\nu+1)>n(\nu)$ and $K_2/((R+(n(\nu+1)-1)\delta)<\varepsilon(|b|+\varepsilon)^\nu\rho$. Then, for $n>n(\nu+1)$, we have

$$\begin{aligned}|w_n|&\leqq|b|\,|w_{n-1}|+K_2/|z_{n-1}|\\ &\leqq|b|(|b|+\varepsilon)^\nu\rho+K_2/(R+(n-1)\delta)\\ &\leqq(|b|+\varepsilon)^{\nu+1}\rho.\end{aligned}$$

The above assertion is thus shown.

(γ) If $P\in V\setminus C$ is a simply convergent point, then there exists an integer n_0 such that $T^{n_0}(P)\in D$. Indeed, since $z_n\to\infty$ by hypothesis, there is an n' such that if $n\geqq n'$ then $|z_n|>R$. For $|z_n|>R$, we have by (7.3)

$$\operatorname{Re}(e^{-i\theta}z_{n+1})\geqq\operatorname{Re}(e^{-i\theta}z_n)+\delta.$$

Hence

$$\operatorname{Re}(e^{-i\theta}z_n)\geqq\operatorname{Re}(e^{-i\theta}z_{n'})+(n-n')\delta.$$

Therefore, for sufficiently large n, we have $\operatorname{Re}(e^{-i\theta}z_n)>R$ and hence $T^n(P)=(z_n, w_n)\in D$.

(δ) It follows from (γ) that every simply convergent point in $V\setminus C$ is also uniformly convergent.

(ε) Every point on C is a simply convergent point but not uniformly convergent. To see this let us return to the coordinate system (x, y) and the expression $(6.3)_{2j}$ of T:

$$\begin{cases} x_1=x+a_2x^2+a_3(y)x^3+\cdots, & a_2\neq 0,\\ y_1=by+b_1(y)x+b_2(y)x^2+\cdots. \end{cases}$$

The iterates T^n, $n=1, 2, \cdots$, have the expressions

$$\begin{cases} x_n = x + n a_2 x^2 + \cdots, \\ y_n = b^n y + \cdots. \end{cases}$$

Every point $P=(0, y)\in C$ is simply convergent since $T^n(P)=(0, b^n y)$. If P were uniformly convergent, then the sequence of functions x_n, $n=1, 2, \cdots$, would be uniformly convergent on some neighborhood of P and the sequence $\partial^2 x_n/\partial x^2$, $n=1, 2, \cdots$, would be also. But this is not the case because $\partial^2 x_n/\partial x^2(P)=2na_2$.

Proposition 7.2 now follows from the observations $(\alpha)\sim(\varepsilon)$.

It is impossible to obtain practically the set of all uniformly convergent points. But the following remarks are important and useful: For every θ satisfying $|\theta-\alpha|<\pi/2$, there exists a base of uniform convergence of the form $D_{\theta,R}$. If D_{θ_1,R_1} and D_{θ_2,R_2} are bases of uniform convergence, then their intersection and union are also.

8. Abel's functional equation.

Now we shall construct a holomorphic function $\varphi(P)$ on a base of uniform convergence $D=D_{\theta,R}$ satisfying Abel's functional equation

$$\varphi(T(P))=\varphi(P)+a_0. \tag{8.1}$$

The coordinate function z satisfies approximately this equation. Among the solutions of this equation we shall single out the class of solutions which are approximated by z (cf. Fatou [5, Chapitre II]). Such a function φ together with w will form a coordinate system on D.

8.1. We choose and fix a branch of $\log z$ on D. Then

$$\log z_1 - \log z = \log\left(1+\frac{a_0}{z}+\cdots\right)=\frac{a_0}{z}+\cdots$$

is a holomorphic function on V. We put

$$z_1-\frac{a_1}{a_0}\log z_1 - a_0 = z-\frac{a_1}{a_0}\log z + A(P). \tag{8.2}$$

Here $A(P)$ is a holomorphic function on V such that $|A(P)|\leqq K/|z|^2$ with a sufficiently large number K. If we replace $P=(z, w)$ by $T^\nu(P)=(z_\nu, w_\nu)$ in (8.2), then we have

$$z_{\nu+1}-\frac{a_1}{a_0}\log z_{\nu+1}-a_0=z_\nu-\frac{a_1}{a_0}\log z_\nu + A(T^\nu(P)), \qquad \nu=0, 1, 2, \cdots.$$

Summation for $\nu=0, 1, \cdots, n-1$ yields

$$z_n-\frac{a_1}{a_0}\log z_n - na_0 = z-\frac{a_1}{a_0}\log z+\sum_{\nu=0}^{n-1}A(T^\nu(P)). \tag{8.3}$$

This expression converges uniformly on D as $n\to\infty$. Indeed, we have, by (7.3),

$$|z_\nu|\geqq \operatorname{Re}(e^{-i\theta}z_\nu)\geqq \operatorname{Re}(e^{-i\theta}z)+\nu\delta; \tag{8.4}$$

and hence

$$(8.5)\qquad \sum_{\nu=0}^{\infty} |A(T^{\nu}(P))| \leqq K\sum_{\nu=0}^{\infty} 1/|z_{\nu}|^2$$

$$\leqq K\sum_{\nu=0}^{\infty} 1/(\mathrm{Re}\,(e^{-i\theta}z)+\nu\delta)^2$$

$$\leqq \frac{K}{\delta}\int_{\mathrm{Re}\,(e^{i\theta}z)-\delta}^{\infty} \frac{dx}{x^2} = K/\{\delta(\mathrm{Re}\,(e^{-i\theta}z)-\delta)\}.$$

Thus we obtain as the limit of (8.3) a holomorphic function on D

$$(8.6)\qquad \varphi_0(P) = \lim_{n\to\infty}\left(z_n - \frac{a_1}{a_0}\log z_n - na_0\right)$$

$$= z - \frac{a_1}{a_0}\log z + \sum_{\nu=0}^{\infty} A(T^{\nu}(P)) .$$

Definition. A function of the form $\varphi_0(P)$+constant is said to be an Abel-Fatou function (A-F function, for short).

There is an arbitrariness of an additive constant in the definition of φ_0 caused by the choice of the branch of $\log z$. This arbitrariness is absorbed into the constant in the definition of the class of A-F functions.

Proposition 8.1. *A-F functions satisfy Abel's functional equation.*

Indeed, we have

$$\varphi_0(T(P)) = \lim_{n\to\infty}\left(z_{n+1} - \frac{a_1}{a_0}\log z_{n+1} - na_0\right)$$

$$= \lim_{n\to\infty}\left(z_{n+1} - \frac{a_1}{a_0}\log z_{n+1} - (n+1)a_0\right) + a_0$$

$$= \varphi_0(P) + a_0 ;$$

hence the proposition is true for every A-F function. q. e. d.

Let $\varphi(P)$ be any solution of Abel's equation (8.1) defined on a base of uniform convergence of any form. Then $\varphi(P)$ is analytically continued to a single-valued function on the set U of all uniformly convergent points, by putting

$$\varphi(P) = \varphi(T^n(P)) - na_0 ,$$

where n is a sufficiently large integer such that $T^n(P) \in D$. In particular, the A-F functions are defined on U independently of the choice of $D = D_{\theta,R}$. (It may however happen that A-F functions are continued analytically beyond the boundary of U and yield multiple-valued functions.)

By (8.5) and (8.6), every A-F function φ has the form

$$(8.7)\qquad \varphi(P) = z - \frac{a_1}{a_0}\log z + B(P) , \qquad P = (z, w) \in U ,$$

where $B(P)$ is a holomorphic function which tends to a constant uniformly as $\mathrm{Re}\,(e^{-i\theta}z)\to\infty$ for any θ with $|\theta-\alpha|<\pi/2$. Since

$$\frac{\partial\varphi}{\partial z}=1-\frac{a_1}{a_0}\frac{1}{z}+\frac{\partial B}{\partial z}, \qquad \frac{\partial\varphi}{\partial w}=\frac{\partial B}{\partial w},$$

we have, estimating Cauchy's integral representation of $B(P)$,

$$\frac{\partial\varphi}{\partial z}\longrightarrow 1, \quad \frac{\partial\varphi}{\partial w}\longrightarrow 0, \tag{8.8}$$

uniformly on the same condition. Further from (8.7) we obtain rough estimates

$$|\varphi(P)-z(P)|=O(|z(P)|^{\varepsilon}), \tag{8.9}$$

$$|\varphi(P)|=O(|z(P)|^{1+\varepsilon}),$$

$$|z(P)|=O(|\varphi(P)|^{1+\varepsilon}), \qquad \text{for any } \varepsilon>0.$$

8.2. Let θ_1, θ_2 be real numbers with $\alpha-\pi/2<\theta_1<\theta_2<\alpha+\pi/2$, and ε a positive number $<\pi/2$. We take a base of uniform convergence of the form

$$D=D_{\theta_1,R}\cup D_{\theta_2,R}.$$

Denoting by z^* the intersection point of the lines $\mathrm{Re}\,(e^{-i\theta_1}z)=R$ and $\mathrm{Re}\,(e^{-i\theta_2}z)=R$ in the z-plane, D is the product of the angular domain

$$\mathcal{A}=\{\theta_1-\pi/2<\arg(z-z^*)<\theta_2+\pi/2\}$$

and the disk

$$\{|w|<\rho\}.$$

We choose and fix an A-F function φ. Taking R sufficiently large, we assume that φ is extended to a continuous mapping of the closure $\bar{D}$ of D into the Riemann sphere $\hat{\boldsymbol{C}}=\boldsymbol{C}\cup\{\infty\}$ and that $|\arg\partial\varphi/\partial z|<\varepsilon$ on D. This is possible by the estimate (8.8).

We want to show that, for every w ($|w|<\rho$), the mapping φ_w of $\mathcal{A}$ into $\boldsymbol{C}$ (s-plane) defined by $z\mapsto s=\varphi_w(z)=\varphi(z, w)$ is injective. To see this, we examine the image under φ_w of the contour $\partial\mathcal{A}$ of the angular domain $\mathcal{A}$. When z traces the ray $\arg(z-z^*)=\theta_1-\pi/2$ from z^* to ∞, its image $\varphi_w(z)$ in the s-plane traces a simple curve from $\varphi_w(z^*)$ to ∞ lying in the angular domain

$$|\arg(s-\varphi_w(z^*))-(\theta_1-\pi/2)|<\varepsilon.$$

This is because the deviation of the direction of tangent to the image curve from the direction of the ray $\arg(z-z^*)=\theta_1-\pi/2$ does not exceed ε at every point. In the same manner, when z traces the ray $\arg(z-z^*)=\theta_2+\pi/2$ from ∞ to z^*, its image in the s-plane traces a simple curve from ∞ to $\varphi_w(z^*)$ lying in the angular domain

$$|\arg(s-\varphi_w(z^*))-(\theta_2+\pi/2)|<\varepsilon.$$

Thus the contour $\partial\mathcal{A}$ traced in the positive sense is mapped to a simple closed curve $\varphi_w(\partial\mathcal{A})$ on the s-sphere and bounds a domain Δ_w lying on the left of

$\varphi_w(\partial\mathcal{A})$. This domain Δ_w contains the angular domain

$$\theta_1-\pi/2+\varepsilon<\arg(s-\varphi_w(z^*))<\theta_2+\pi/2-\varepsilon$$

and is contained in the angular domain

$$\theta_1-\pi/2-\varepsilon<\arg(s-\varphi_w(z^*))<\theta_2+\pi/2+\varepsilon\,. \tag{8.10}$$

To show that φ_w maps $\mathcal{A}$ injectively onto Δ_w we rely on the argument principle. Let L be a sufficiently large number and consider the image of the contour of $\mathcal{A}\cap\{|z|<L\}$. When z traces the portion of the circle $|z|=L$ lying in $\mathcal{A}$ in the positive sense, its image traces with increasing argument a simple curve lying in the angular domain (8.10). This image curve tends to ∞ as $L\to\infty$ by (8.9). From this we see that the winding number of the image of the contour of $\mathcal{A}\cap\{|z|<L\}$ relative to a point s is 1 or 0 according as $s\in\Delta_w$ or $s\notin\Delta_w$, for sufficiently large L. This shows that φ_w is an injective mapping of $\mathcal{A}$ onto Δ_w.

Let us consider the mapping (φ, w) of $D_{\theta_1,R}\cup D_{\theta_2,R}$ into C^2.

$$P\mapsto(s, v)=(\varphi(P), w(P))\,.$$

We know by the above observation that this mapping is injective. Let $\theta'=\theta_1-\pi/2+\varepsilon$ and $\theta''=\theta_2+\pi/2-\varepsilon$. Then we have

$$\alpha-\pi<\theta'<\alpha<\theta''<\alpha+\pi\,.$$

We put

$$\mathcal{B}=\bigcap_{|w|<\rho}\{s\in C|\,\theta''<\arg(s-\varphi_w(z^*))<\theta''\},$$

which has the form of an angular domain

$$\mathcal{B}=\{s\in C|\,\theta'<\arg(s-s^*)<\theta''\}. \tag{8.11}$$

Since for every w, $|w|<\rho$, the image of $\mathcal{A}$ under φ_w contains $\mathcal{B}$, the image of $D_{\theta\cdot,R}\cup D_{\theta_2,R}$, under (φ, w) contains $\mathcal{B}\times\{|v|<\rho\}$. Thus by putting

$$D[\mathcal{B}]=\{P\in U\,|\,\varphi(P)\in\mathcal{B}\}, \tag{8.12}$$

(φ, w) is a biholomorphic mapping of $D[\mathcal{B}]$ onto $\mathcal{B}\times\{|v|<\rho\}$. We can regard (φ, w) as a coordinate system on $D[\mathcal{B}]$. When we do so, the letters s, v will be used in place of φ, w. It is clear that $D[\mathcal{B}]$ is a base of uniform convergence. The transformation T is expressed with respect to the coordinate system (s, v) in the form

$$\begin{cases} s_1=s+a_0, \\ v_1=g'(s, v)\,, \end{cases}$$

where g' is the function determined by $g'(s(z, w), w)=g(z, w)$.

8.3. The following proposition characterizes the class of A-F functions:

Proposition 8.2. *Let φ be a holomorphic function defined on the set U of all uniformly convergent points and satisfying Abel's equation* (8.1). *The conditions* (i), (ii), (iii) *are equivalent to one another*:

(i) *φ is an A-F fnnction.*

(ii) $\dfrac{\partial\varphi}{\partial z}(P)\to 1$, $\dfrac{\partial\varphi}{\partial w}(P)\to 0$ *uniformly as* $\mathrm{Re}\,(e^{-i\theta}z(P))\to\infty$.

(iii) *For every uniformly convergent point P,*

$$\frac{\partial\varphi}{\partial z}(T^n(P))\longrightarrow 1\,,\qquad \frac{\partial\varphi}{\partial w}(T^n(P))\longrightarrow 0\,,\qquad \text{as } n\to\infty\,.$$

Proof. We have seen at the end of 8.1, that (i) implies (ii). Condition (ii) implies (iii) because $\mathrm{Re}\,(e^{-i\theta}z_n(P))\to\infty$ as $n\to\infty$. It remains to show that (iii) implies (i).

Denoting by $s(P)$ an A-F function and setting $\pi(P)=\varphi(P)-s(P)$, the condition (iii) is equivalent to

$$\text{(iv)}\qquad \frac{\partial\pi}{\partial z}(T^n(P)),\qquad \frac{\partial\pi}{\partial w}(T^n(P))\longrightarrow 0\qquad \text{as } n\to\infty\,.$$

Note that the Jacobian matrix for change of coordinate systems from (z, w) to (s, v) has the form

$$\text{(8.13)}\qquad \begin{pmatrix}\partial s/\partial z & \partial s/\partial w\\ 0 & 1\end{pmatrix}$$

and tends to the identity matrix as $\mathrm{Re}\,(e^{-i\theta}z(P))\to\infty$. Hence we may replace $\partial\pi/\partial z$, $\partial\pi/\partial w$ by $\partial\pi/\partial s$, $\partial\pi/\partial v$ respectively in the condition (iv).

(α) If for any $P\in U$, the sequence $\partial\pi/\partial v\,(T^n(P))$, $n=0, 1, 2, \cdots$ is bounded, then $\pi(P)$ depends only on $s(P)$. Indeed, since $\pi(P)$ is invariant relative to T, we have

$$\frac{\partial(\pi\circ T^n)}{\partial v}(P)=\frac{\partial\pi}{\partial v}(P)\,,\qquad n=0, 1, 2, \cdots.$$

On the other hand, since $\partial s_1/\partial v\,(P)=0$, we have

$$\frac{\partial(\pi\circ T^n)}{\partial v}(P)=\frac{\partial(\pi\circ T^{n-1})}{\partial v}(T(P))\frac{\partial v_1}{\partial v}(P)\,.$$

Hence, by induction,

$$\frac{\partial\pi}{\partial v}(P)=\frac{\partial(\pi\circ T^n)}{\partial v}(P)=\frac{\partial\pi}{\partial v}(T^n(P))\frac{\partial v_1}{\partial v}(T^{n-1}(P))\cdots\frac{\partial v_1}{\partial v}(T(P))\frac{\partial v_1}{\partial v}(P).$$

The right-hand side tends to 0 as $n\to\infty$, because $\partial\pi/\partial v(T^n(P))$ is bounded and $|\partial v_1/\partial v(T^n(P))|<|b|+\varepsilon<1$ for sufficiently large n. This shows that $\partial\pi/\partial v(P)=0$ and that $\pi(P)$ depends only on $s(P)$.

(β) If moreover $\partial\pi/\partial z(T^n(P))\to 0$ as $n\to\infty$, then $\pi(P)$ is a constant. This is because

$$\frac{\partial\pi}{\partial s}(P)=\frac{\partial(\pi\circ T^n)}{\partial s}(P)=\frac{\partial\pi}{\partial s}(T^n(P))\,.$$

Combining (α) and (β), we know that (iv) implies that $\pi(P)$ is a constant and that $\varphi(P)$ is an A-F function. Thus Proposition 8.2 is proved.

9. Coordinates on the fibers of Abel-Fatou function.

To reduce the expression of T on a base of uniform convergence D to a simpler form, we want to construct an invariant holomorphic function which, together with the A-F function, will form a coordinate system on D.

9.1. We begin with constructing, on the set U of all uniformly convergent points, a non-vanishing holomorphic 2-form Ω which is invariant relative to T, i.e., $T^{*}\Omega=\Omega$. In view of this invariance relation, it suffices to construct Ω on a base of uniform convergence. To fix the ideas we take a base of uniform convergence $D=D[\mathscr{B}]$ equipped with the coordinate system (s, v) as in 8.2. We use the notations

$$P=(s(P), v(P))=(s, v),$$

$$T^{n}(P)=(s(T^{n}(P)), v(T^{n}(P))=(s_{n}(P), v_{n}(P)), \qquad n=0, 1, 2, \cdots.$$

By (8.13) the Jacobian determinant of T relative to the coordinate system (s, v) is equal to $\partial v_{1}/\partial v$. If we write $\Omega(P)=\xi(P)ds\wedge dv$ with a non-vanishing holomorphic function $\xi(P)=(s, v)$, then the invariance condition is expressed by

$$\xi(T(P))\frac{\partial v_{1}}{\partial v}(P)=\xi(P). \tag{9.1}$$

We define $\Omega=\xi(P)ds\wedge dv$ by

$$\xi(P)=c\lim_{n\to\infty}b^{-s(T^{n}(P))/a_{0}}s(T^{n}(P))^{-b_{1}/a_{0}b}\frac{\partial v_{n}}{\partial v}(P) \tag{9.2}$$

$$=c\,b^{-s(P)/a_{0}}s(P)^{-b_{1}/a_{0}b}\prod_{n=0}^{\infty}\left\{\frac{1}{b}\left(1+\frac{a_{0}}{s(T^{n}(P))}\right)^{-b_{1}/a_{0}b}\frac{\partial v_{1}}{\partial v}(T^{n}(P))\right\}.$$

Here, a_{0}, b, b_{1} are the coefficients in (7.1); c is an arbitrary non-zero constant; $b^{-s(P)/a_{0}}=\exp\{-(s(P)/a_{0})\log b\}$ and $s(P)^{-b_{1}/a_{0}b}=\exp\{-(b_{1}/a_{0}b)\log s(P)\}$ with a fixed choice of $\log b$ and a branch of $\log s(P)$. The two expressions in (9.2) are equivalent, since $s(T^{n}(P))=s(P)+na_{0}$ and

$$\frac{\partial v_{n}}{\partial v}(P)=\frac{\partial v_{1}}{\partial v}(T^{n-1}(P))\cdots\frac{\partial v_{1}}{\partial v}(T(P))\frac{\partial v_{1}}{\partial v}(P).$$

Besides Ω we define a holomorphic 2-form $\tilde{\Omega}=\tilde{\xi}(P)ds\wedge dv$ by

$$\tilde{\xi}(P)=\prod_{n=0}^{\infty}\left\{\frac{1}{b}\left(1+\frac{a}{s(T^{n}(P))}\right)^{-b_{1}/a_{0}b}\frac{\partial v_{1}}{\partial v}(T^{n}(P))\right\}, \tag{9.3}$$

so that the following relation holds:

$$\Omega=c\,b^{-s(P)/a_{0}}s(P)^{-b_{1}/a_{0}b}\tilde{\Omega}.$$

Let us assume the convegence of (9.2) and (9.3) for a while. We have obviously the following

Proposition 9.1. *The 2-form Ω is invariant relative to T. $\tilde{\Omega}$ satisfies the* *equation*

$$(9.4)\qquad T^{*}\tilde{\Omega}(P)=b\Big(1+\frac{a_0}{s(P)}\Big)^{b_1/a_0 b}\tilde{\Omega}(P)\,.$$

There is some arbitrariness in the definition of Ω. If we replace $\log s(P)$ by another choice, then the new 2-form differs from the original one by a constant factor, which can be absorbed into the constant c. If we replace the value of $\log b$ by $\log b+2\pi i\nu$ with an integer ν, the new form differs from the original by the factor $e^{-2\pi i\nu s(P)/a_0}$. We will call any holomorphic 2-form Ω which can be expressed by (9.2) a *canonical invariant 2-form.* Ω is determined up to a constant factor when $\log b$ is specified.

On the other hand, $\tilde{\Omega}$ is determined if the A-F function $s(P)$ is determined. We will call $\tilde{\Omega}$ *normalized 2-form.*

9.2. Now we show the convergence of the infinite product (9.3). The convergence of (9.2) follows from this. We put

$$\eta(P)=\frac{1}{b}\Big(1+\frac{a_0}{s(P)}\Big)^{-b_1/a_0 b}\frac{\partial v_1}{\partial v}(P)\,,$$

and make an estimate of $\log \eta(P)$. Firstly we have

$$\log\Big(1+\frac{a_0}{s(P)}\Big)^{-b_1/a_0 b}=-\frac{b_1}{b}\frac{1}{s(P)}+O\Big(\frac{1}{|s|^2}\Big)\,.$$

Next we notice that

$$\frac{\partial v_1}{\partial v}=\frac{\partial w_1}{\partial v}=\frac{\partial w_1}{\partial z}\frac{\partial z}{\partial v}+\frac{\partial w_1}{\partial w}\frac{\partial w}{\partial v}$$

and, in view of (7.1), that

$$\frac{\partial w_1}{\partial z}=-\frac{b_1 w}{z^2}+\cdots=O\Big(\frac{1}{|z|^2}\Big)\,,\qquad \frac{\partial z}{\partial v}=O(1)\,,$$

$$\frac{\partial w_1}{\partial w}=b+\frac{b_1}{z}+O\Big(\frac{1}{|z|^2}\Big)\,,\qquad \frac{\partial w}{\partial v}=1\,.$$

It follows that

$$\log\Big(\frac{1}{b}\frac{\partial v_1}{\partial v}\Big)=\frac{b_1}{bz}+O\Big(\frac{1}{|z|^2}\Big)\,.$$

Therefore, in view of the estimate (8.9), we obtain

$$\log \eta(P)=O\Big(\frac{1}{|s|^{\lambda}}\Big)\qquad \text{for } \lambda>1\,.$$

This, combined with (8.4), shows the convergence of

$$\tilde{\xi}(P)=\prod_{n=0}^{\infty}\eta(T^{n}(P))\,.$$

Further, this estimate implies that $\tilde{\xi}(P)\to 1$ uniformly as $\mathrm{Re}\,(e^{-i\theta}z)\to\infty$.

Proposition 9.2. *Let $\tilde{\Omega}=\tilde{\xi}(P)ds\wedge dv$ be a holomorphic 2-form on U satisfying the equation* (9.4). *The following conditions are equivalent to one another*:

(i) *$\tilde{\Omega}$ is the normalized 2-form.*

(ii) *$\tilde{\xi}(P)\to 1$ uniformly as* $\mathrm{Re}(e^{-i\theta}z)\to\infty$.

(iii) *For any $P\in U$, $\tilde{\xi}(T^n(P))\to 1$ as $n\to\infty$.*

Proof. We have already shown that (i) implies (ii). (ii) implies (iii) because $\mathrm{Re}(e^{-i\theta}z_n)\to\infty$ as $n\to\infty$. It remains to prove that (iii) implies (i). By iterating the equation (9.4) we have

$$\tilde{\xi}(P)=\tilde{\xi}(T^n(P))\prod_{\nu=0}^{n-1}\left\{\frac{1}{b}\left(1+\frac{a_0}{s(T^\nu(P))}\right)^{-b_1/a_0 b}\frac{\partial v_1}{\partial v}(T^\nu(P))\right\}.$$

If (iii) is satisfied, letting $n\to\infty$ we see that $\tilde{\Omega}=\tilde{\xi}(P)ds\wedge dv$ is the normalized 1-form. q. e. d.

9.3. Consider the expressions

$$\omega=\Omega/ds=\xi(P)dv\,,$$

$$\tilde{\omega}=\tilde{\Omega}/ds=\tilde{\xi}(P)dv\,.$$

They are regarded as families of holomorphic 1-forms on the fibers of the A-F function $s(P)$ depending holomorphically on s. Correspondingly to Ω and $\tilde{\Omega}$, these 1-forms have the properties $T^*\omega=\omega$, $T^*\tilde{\omega}=b(1+a_0/s(P))^{b_1/a_0 b}\tilde{\omega}$, $\omega=c\,b^{-s(P)/a_0}s(P)^{-b_1/a_0 b}\tilde{\omega}$.

We define a holomorphic function $\phi(P)=\phi(s, v)$ on $D[\mathcal{B}]$ in the following manner: For a point $P=(s, v)$ in $D[\mathcal{B}]$, we denote by $\underline{P}$ the point $(s, 0)$. We put

$$\phi(P)=\int_{\underline{P}}^{P}\omega=\int_0^v\xi(s, v)dv,\qquad \tilde{\phi}(P)=\int_{\underline{P}}^{P}\tilde{\omega}=\int_0^v\tilde{\xi}(s, v)dv\,,$$

where the path of the integrals is a curve joining $\underline{P}$ to P on the fiber of the A-F function s. ϕ and $\tilde{\phi}$ satisfy the relation

$$\phi(P)=c\,b^{-s/a_0}s^{-b_1/a_0 b}\tilde{\phi}(P)\,. \tag{9.5}$$

They behave under T as

$$\begin{aligned}&\phi(T(P))=\phi(P)+\kappa(s(P))\,,\\ &\tilde{\phi}(T(P))=b\left(1+\frac{a_0}{s(P)}\right)^{b_1/a_0 b}\tilde{\phi}(P)+\tilde{\kappa}(s(P))\,,\end{aligned}\tag{9.6}$$

where $\kappa(s)$, $\tilde{\kappa}(s)$ are holomorphic functions on $\mathcal{B}$

$$\kappa(s)=\int_0^{v_1}\xi(s+a_0, v)dv\,,\qquad \tilde{\kappa}(s)=\int_0^{v_1}\tilde{\xi}(s+a_0, v)dv$$

with $v_1=v(T(s, 0))$.

Let ε be a sufficiently small number >0 and assume that $\mathcal{B}$ is so chosen that $|\tilde{\xi}(P)-1|<\varepsilon$ on $D[\mathscr{B}]$. Consider, for every s in $\mathscr{B}$, the holomorphic mapping $\tilde{\psi}_s$ of $\{|v|<\rho\}$ into $\boldsymbol{C}$ ($\tilde{u}$-plane) defined by $\tilde{\psi}_s(v)=\tilde{\psi}(s, v)$. This mapping $\tilde{\psi}_s$ is injective and its image contains the disk $\{|u|<\rho_0\}$, $\rho_0=(1-\varepsilon)\rho$.

Consequently the holomorphic mapping $(\varphi, \tilde{\psi})$ of $D[\mathscr{B}]$ to $\boldsymbol{C}^2$

$$P \longmapsto (s, \tilde{u})=(\varphi(P), \tilde{\psi}(P))$$

is injective and its image contains the product domain $\mathscr{B}\times\{|\tilde{u}|<\rho_0\}$. In view of the relation (9.5), the holomorphic mapping (φ, ψ) of $D[\mathscr{B}]$ is also injective and its image contains the domain

$$\{(s, u)\,|\,s\in\mathscr{B},\ |u|<\rho_0|c\, b^{-s/a_0}s^{-b_1/a_0 b}|\}.$$

Thus we can regard (φ, ψ) and $(\varphi, \tilde{\psi})$ coordinate systems on $D[\mathscr{B}]$. When we do so, the notations (s, u), $(s, \tilde{u})$ will be used. With respect to these coordinate systems, the transformation T is expressed as

$$\begin{cases} s_1=s+a_0, \\ u_1=u+\kappa(s); \end{cases} \qquad \begin{cases} s_1=s+a_0, \\ \tilde{u}_1=b\left(1+\dfrac{a_0}{s}\right)^{b_1/a_0 b}\tilde{u}+\tilde{\kappa}(s). \end{cases}$$

We notice that $\kappa(s)$, $\tilde{\kappa}(s)$ have no canonical meanings, since they depend on the coordinate $w=v$. The coordinate u is so far canonically determined except for a factor of the form const. $\exp(-2\pi i\nu s/a_0)$ and translations on the fibers of $s=\varphi$. The coordinate u is determined except for translations on the fibers (when s is fixed). We will later exclude this arbitrariness.

10. Application to global transformations.

10.1. Let $\mathfrak{M}$ be a complex manifold of dimension 2 and T a holomorphic automorphism of $\mathfrak{M}$. Suppose that there is a fixed point O in $\mathfrak{M}$ of T and that T is semi-attractive of type $(1, b)_1$ at O. Let $\mathfrak{D}$ (resp. $\mathfrak{E}$) denote the set of all uniformly (resp. simply) convergent points whose limit is O. Let $\mathfrak{C}$ denote the invariant curve of Poincaré with center O and corresponding to the eigenvalue b. Under this situation we show a result analogous to the example of Bieberbach: The domain $\mathfrak{D}$ is biholomorphic to $\boldsymbol{C}^2$. More precisely, we prove the following

Theorem 10.1. *There is a holomorphic function $\varphi:\mathfrak{D}\to\boldsymbol{C}$ which satisfies the equation $\varphi(T(P))=\varphi(P)+1$ and induces a structure of fiber bundle over $\boldsymbol{C}$ with fibers $\cong\boldsymbol{C}$. Further there is a holomorphic function ψ^* on $\mathfrak{D}$ which is invariant under T and induces on every fiber of φ a biholomorphic mapping onto $\boldsymbol{C}$.*

The proof is done in the steps $(\alpha)\sim(\varepsilon)$.

(α) We choose a neighborhood V of O together with a local coordinate system (z, w) as in 7.2. Here in the expression (7.1) we suppose $a_0=1$, which is possible by replacing z by $a_0 z$.

If D is a base of uniform convergence to O, then $T^{-n}(D)$, $n=0, 1, 2, \cdots$, is an increasing sequence of domains in $\mathfrak{M}$ which exhausts $\mathfrak{D}$, i. e., $\mathfrak{D}=\bigcup_{n=0}^{\infty} T^{-n}(D)$. Similarly, if E is a base of simple convergence to O, then $\mathfrak{E}=\bigcup_{n=0}^{\infty} T^{-n}(E)$.

We take D and E as in 8.2: $D=D[\mathscr{B}]$, $E=D\cup C$. We notice that, since C is in the boundary of D and T^{-n} is an automorphism for every n, $T^{-n}(C)$ is in the boundary of $T^{-n}(D)$. Therefore $\mathfrak{E}$ is in the boundary of $\mathfrak{D}$.

(β) The A-F function $\varphi(P)$ on D satisfies the equation

$$\varphi(T(P))=\varphi(P)+1\,, \tag{10.1}$$

for $P\in D$. By putting

$$\varphi(P)=\varphi(T^{n}(P))-n\,, \qquad P\in T^{-n}(D)\,,$$

we can extend φ to all of $\mathfrak{D}$. The equation (10.1) is then valid for all $P\in\mathfrak{D}$.

The differential $d\varphi$ of φ is an invariant holomorphic 1-form on $\mathfrak{D}$: $T^{*}d\varphi(P)=d(\varphi\circ T)(P)=d\varphi(P)$. Since $d\varphi$ is non-vanishing on D, it is non-vanishing everywhere on $\mathfrak{D}$. In other words $\varphi:\mathfrak{D}\to\boldsymbol{C}$ is a mapping of rank 1.

This mapping $\varphi:\mathfrak{D}\to\boldsymbol{C}$ is surjective; and for every $s\in\boldsymbol{C}$ the fiber $\varphi^{-1}(s)=\{P\in\mathfrak{D}\,|\,\varphi(P)=s\}$ is connected and simply connected. In fact, the fiber $\varphi^{-1}(s)$ is exhausted by the increasing sequence

$$\varphi^{-1}(s)\cap T^{-n}(D)=\{P\in T^{-n}(D)\,|\,\varphi(P)=s\}, \qquad n=0,\ 1,\ 2,\ \cdots,$$

each of which is biholomorphic to

$$\varphi^{-1}(s+n)\cap D=\{P\in D\,|\,\varphi(P)=s+n\}, \qquad n=0,\ 1,\ 2,\ \cdots,$$

via the automorphism T^{n}. When n is sufficiently large so that $s+n\in\mathscr{B}$, this set is non-empty and biholomorphic to the disk $|v|<\rho$; hence connected and simply connected. Therefore $\varphi^{-1}(s)$ is also.

(γ) We choose and fix a canonical invariant 2-form Ω and the 1-form $\omega=\Omega/d\varphi$ on D. They can be extended to the whole $\mathfrak{D}$ by means of the invariance relation.

The holomorphic function ψ on D defined in 9.3. can be extended to $\varphi^{-1}(\mathscr{B})$ in the following manner: For a point P with $\varphi(P)=s\in\mathscr{B}$, let $\underline{P}$ denote the point in D with $\varphi(\underline{P})=s$, $w(\underline{P})=0$. We define $\psi(P)$ by the integral $\psi(P)=\int_{\underline{P}}^{P}\omega$ over a path which joins $\underline{P}$ to P on the fiber $\varphi^{-1}(s)$. The relation

$$\psi(T(P))=\psi(P)+\kappa(s)\,, \qquad s=\varphi(P)\in\mathscr{B} \tag{10.2}$$

remains true for this extended function.

We claim that, for every $s\in\mathscr{B}$, the restriction of ψ to the fiber $\varphi^{-1}(s)$ is a biholomorphic mapping of $\varphi^{-1}(s)$ onto $\boldsymbol{C}$. For the proof we consider first the restriction of ψ to $\varphi^{-1}(s)\cap T^{-n}(D)$. We use the relation derived from (10.2):

$$\psi(P)=\psi(T^{n}(P))-\kappa(s)-\kappa(s+1)-\cdots-\kappa(s+n-1)\,.$$

Here $\varphi^{-1}(s)\cap T^{-n}(D)$ is mapped by T^{n} biholomorphically onto $\varphi^{-1}(s+n)\cap D$, which is mapped by ψ into $\boldsymbol{C}$ injectively and its image contains the disk $\{|u|<\rho_0|b^{-(s+n)}(s+n)^{-b_1/b}|\}$. Consequently $\varphi^{-1}(s)\cap T^{-n}(D)$ is mapped by ψ into $\boldsymbol{C}$ injectively and its image contains the disk

$$\{|u+\kappa(s)+\cdots+\kappa(s+n-1)|<\rho_0|b^{-(s+n)}(s+n)^{-b_1/b}|\}.$$

The radius tends to infinity as $n\to\infty$. The assertion is proved.

It follows that $\varphi^{-1}(\mathscr{B})$ is biholomorphic to $\mathscr{B}\times\boldsymbol{C}$ via the mapping (φ, ψ).

(δ) We denote by $\mathscr{B}-n$ the angular domain in $\boldsymbol{C}$ (s-plane) obtained by the translation of $\mathscr{B}$ by $-n$:

$$\mathscr{B}-n=\{s-n\,|\,s\in\mathscr{B}\}=\{\theta'<\arg(s-s^*+n)<\theta''\}.$$

We consider on the domain $\varphi^{-1}(\mathscr{B}-n)=T^{-n}(\varphi^{-1}(\mathscr{B}))$ the holomorphic function $\psi(T^n(P))$. The pair of functions $(\varphi, \psi\circ T^n)$ is a biholomorphic mapping of $\varphi^{-1}(\mathscr{B}-n)$ onto $(\mathscr{B}-n)\times\boldsymbol{C}$. This is so because every fiber $\varphi^{-1}(s)$, $s\in\mathscr{B}-n$, is biholomorphic to $\varphi^{-1}(s+n)$ via T^n and $\varphi^{-1}(s+n)$ is biholomorphic to $\boldsymbol{C}$ via ψ.

The collection of the mapping $\varphi:\mathfrak{D}\to\boldsymbol{C}$, the open covering $\{\mathscr{B}-n\}_{n=0}^{\infty}$ of $\boldsymbol{C}$, and the fiber coordinates $\psi\circ T^n$ on $\varphi^{-1}(\mathscr{B}-n)$ defines on $\mathfrak{D}$ a structure of complex analytic fiber bundle, with base $\boldsymbol{C}$ and fibers $\cong\boldsymbol{C}$. In view of the relation

$$\psi(T^n(P))=\psi(T^m(P))+\kappa(\varphi(P)+m)+\cdots+\kappa(\varphi(P)+n-1), \qquad n\geqq m,$$

the structure group is the translations of $\boldsymbol{C}$.

(ε) This fiber bundle is trivial, i.e., there is a holomorphic function ψ^* on $\mathfrak{D}$, which induces on each fiber a biholomorphic mapping onto $\boldsymbol{C}$. Further we can choose as such ψ^* an invariant function under T.

We remark that this assertion is equivalent to say that the difference equation

$$\lambda(s+1)-\lambda(s)=\kappa(s), \qquad s\in\mathscr{B}, \tag{10.3}$$

has a solution. Indeed, if $\lambda(s)$ is a solution to (10i3), then we put

$$\psi^*(P)=\psi(P)-\lambda(\varphi(P)), \qquad P\in\varphi^{-1}(\mathscr{B}).$$

by (10.2) and (10.3) we find that $\psi^*(P)$ is invariant under T; and hence extended to a holomorphic function on all of $\mathfrak{D}$ with the desired properties.

Instead of solving (10.3) directly, we deduce as follows: Let $\langle T\rangle=\{T\,|\,n\in\boldsymbol{Z}\}$ denote the group of automorphisms of $\mathfrak{D}$ generated by T, and consider the quotient manifold $\mathfrak{D}/\langle T\rangle$ with the projection $\tilde{\omega}:\mathfrak{D}\to\mathfrak{D}/\langle T\rangle$. The function $\exp 2\pi i\varphi(P)$ on $\mathfrak{D}$ is invariant relative to T. Hence there is a holomorphic function $\check{\varphi}$ of $\mathfrak{D}/\langle T\rangle$ onto $\boldsymbol{C}^*$ such that $\check{\varphi}\circ\tilde{\omega}(P)=\exp 2\pi i\varphi(P)$. This mapping $\check{\varphi}$ defines on $\mathfrak{D}/\langle T\rangle$ a structure of fiber bundle with base $\boldsymbol{C}^*$ and fibers $\cong\boldsymbol{C}$. This fiber bundle is trivial since the first cohomology group of $\boldsymbol{C}^*$ with coefficients in holomorphic functions $H^1(\boldsymbol{C}^*, \mathcal{O})=0$. Therefore there is a holomorphic function $\check{\psi}^*$ on $\mathfrak{D}/\langle T\rangle$ which induces on every fiber a biholomorphic mapping onto $\boldsymbol{C}$. Then $\psi^*=\check{\psi}^*\circ\tilde{\omega}$ has the desired property.

Thus the proof of Theorem 10.1 is completed.

10.2. We will show that, roughly speaking. the fiber $\varphi^{-1}(s)$ converges to the Poincaré invariant curve $\mathfrak{C}$ as Re $s\to+\infty$.

We extend the A-F function φ to $\mathfrak{D}\cup\mathfrak{C}$ by letting $\varphi(P)=\infty$ for $P\in\mathfrak{C}$. φ is a mapping onto $\hat{C}$ (s-sphere). Let us write $\dot{\mathscr{B}}=\mathscr{B}\cup\{\infty\}$ and restrict our considerations to $\varphi^{-1}(\dot{\mathscr{B}})=\varphi^{-1}(\mathscr{B})\cup\mathfrak{C}$. We take the holomorphic function $\tilde{\phi}$ on $D[\mathscr{B}]$ (introduced in 9.3). $\tilde{\phi}$ can be extended to a holomorphic function on $\varphi^{-1}(\mathscr{B})$ by the relation (9.5). We extend $\tilde{\phi}$ to $\varphi^{-1}(\infty)=\mathfrak{C}$ as follows: For $P\in C=\mathfrak{C}\cap U$, we let $\tilde{\phi}(P)=w(P)$. Then we have, in view of (7.1), $\tilde{\phi}(T(P))=b\tilde{\phi}(P)$ for $P\in C$. Using this relation, $\tilde{\phi}$ is extended to all of $\varphi^{-1}(\infty)$. We notice that $(\tilde{\phi}|\mathfrak{C})^{-1}$ is the Poincaré mapping $F: \boldsymbol{C}\to\mathfrak{C}$. We have thus a bijective mapping $(\varphi, \tilde{\phi})$ of $\varphi^{-1}(\dot{\mathscr{B}})$ onto $\dot{\mathscr{B}}\times\boldsymbol{C}$, which maps $\varphi^{-1}(\mathscr{B})$ biholomorphically onto $\mathscr{B}\times\boldsymbol{C}$.

Theorem 10.2. (i) *For any positive number M, the restriction of $(\varphi, \tilde{\phi})$ to $\varphi^{-1}(\dot{\mathscr{B}})\cap\{|\tilde{\phi}|<M\}$ is a homeomorphism onto $\dot{\mathscr{B}}\times\{|\tilde{u}|<M\}$.* (ii) *The inverse of $(\varphi, \tilde{\phi})$ is a continuous bijective mapping of $\dot{\mathscr{B}}\times\boldsymbol{C}$ onto $\varphi^{-1}(\dot{\mathscr{B}})$.*

Proof. First we consider the restriction of $(\varphi, \tilde{\phi})$ to $D[\dot{\mathscr{B}}]=D[\dot{\mathscr{B}}]\cup C$. φ is continuous on $D[\dot{\mathscr{B}}]$ as we have seen in section 8. As for $\tilde{\phi}$, when P approaches C in $D[\mathscr{B}]$, we have $\tilde{\xi}(P)\to1$, and hence $\tilde{\phi}(P)=\int_0^{v_1}\tilde{\xi}(s, v)dv\to v(P)=w(P)$. Hence $\tilde{\phi}$ is continous on $D[\dot{\mathscr{B}}]$.

In the functional equation (9.6) we let $\tilde{\kappa}(\infty)=0$. Then κ is a continuous function on $\dot{\mathscr{B}}$; and the equation (9.6) remains valid for all P in $D[\dot{\mathscr{B}}]$. For any M, the set $\varphi^{-1}(\dot{\mathscr{B}})\cap\{|\tilde{\phi}|<M\}$ is mapped homeomorphically into $D[\dot{\mathscr{B}}]$ by with sufficiently large n. Therefore, in view of (9.6), the assertion (i) is true.

The assertion (ii) is true, since for every $M>0$, the inverse of $(\varphi, \tilde{\phi})$ is continuous on $\dot{\mathscr{B}}\times\{|\tilde{u}|<M\}$. q. e. d.

Remark. This argument does not imply that $(\varphi, \tilde{\phi})$ is continuous on $\varphi^{-1}(\dot{\mathscr{B}})$. For, the sets $\varphi^{-1}(\dot{\mathscr{B}})\cap\{|\tilde{\phi}|<M\}$ may be non-open in $\varphi^{-1}(\dot{\mathscr{B}})$.

10.3. Now we want to show that not every local analytic transformation appears as the germ at a fixed point of an automorphism of a complex manifold.

Let us first prove a lemma in a slightly general situation. Let $\mathfrak{M}$ be a complex manifold of dimension 2 and $\mathfrak{D}$ a (schlicht) domain in $\mathfrak{M}$. Let (φ, ϕ) be a pair of holomorphic functions on $\mathfrak{D}$ which yields a biholomorphic mapping of $\mathfrak{D}$ onto $\boldsymbol{C}^2$. By analytic continuation of φ we get a holomorphic function $\hat{\varphi}$ in its domain of existence $\hat{\mathfrak{D}}$. By definition, $\hat{\mathfrak{D}}$ is an unramified domain over $\mathfrak{M}$ containing $\mathfrak{D}$ in a canonical manner so that $\hat{\varphi}|\mathfrak{D}=\varphi$. Under this situation we have

Lemma 10.3. *$\hat{\mathfrak{D}}$ is a schlicht domain in $\mathfrak{M}$ and $\hat{\mathfrak{D}}\setminus\mathfrak{D}$ is either empty or an analytic set of pure dimension* 1. *Further, if $\mathfrak{M}$ is a Stein manifold and $\mathfrak{D}$ is a Runge domain in M, then $\hat{\mathfrak{D}}=\mathfrak{D}$,* *Proof.* Suppose that $\hat{\mathfrak{D}}\setminus\mathfrak{D}$ is non-empty. When $P\in\mathfrak{D}$ approaches any boundary point of $\mathfrak{D}$ in $\hat{\mathfrak{D}}$ then $\varphi(P)$ tends to a finite value and $|\varphi(P)|^2+|\psi(P)|^2\to\infty$, hence $|\psi(P)|\to\infty$. Therefore, if we set $\hat{\psi}(P)=\psi(P)$ for $P\in D$ and $\hat{\psi}(P)=\infty$ for $P\in\hat{\mathfrak{D}}\setminus\mathfrak{D}$, we get a continuous mapping $\hat{\psi}(P)$ of $\mathfrak{D}$ into the Riemann sphere $\hat{\boldsymbol{C}}$. By Radó's theorem $\hat{\psi}$ is meromorphic. $\hat{\mathfrak{D}}\setminus\mathfrak{D}$ is the set of poles of $\hat{\psi}$, therefore an analytic set of dimension 1. Consequently $\hat{\mathfrak{D}}$ is schlicht. The first assertion is shown. Since any domain of the form (domain)\(non-empty analytic set) is not a Runge domain, the second assertion follows. q. e. d.

Proposition 10.4. *Let T be a semi-attractive local analytic transformation of type* $(1, b)_1$. *If the analytic continuation of an A-F function $\varphi(P)$ yields a multiple-valeud function, then T is not a germ at a fixed point of a automorphism of a complex manifold.*

This follows immediately from Lemma 10.3. Examples of such local analytic transformations will be given later.

Proposition 10.5. *Let T be a semi-attractive tvansformation of type* $(1, b)_1$ *which has an expression* (7.1) *with $a_i(w)=0$ for all $i\geqq 1$. Then T is not a germ at a fixed point of an automorphism of a Stein manifold.*

Proof. Suppose that T is the germ of an automorphism of a Stein manifold $\mathfrak{M}$ at a fixed point O. Let $\mathfrak{D}$, $\mathfrak{C}$ and V have the same meaning as in 10.1. The coordinate z is an A-F function. We can continue z analytically to the whole $\mathfrak{D}$. Since $\mathfrak{D}$ is a Runge domain by Proposition 5.1, $\mathfrak{D}$ is the domain of existence of z. Consequently $\mathfrak{D}\cap V=V\setminus\mathfrak{C}$. This contradicts the fact that $\mathfrak{D}$ is a Runge domain. q. e. d.

We remark that, in the example in Section 5.1, T satisfies the condition of the proposition but $M=\hat{\boldsymbol{C}}^2$ is not Stein.

10.4. Now let us give some examples of automorphisms of $\boldsymbol{C}^2$:

Example 1.

$$T:\begin{cases} x_1=x+f(x+by), \\ y_1=by-f(x+by), \end{cases} \qquad T^{-1}:\begin{cases} x=x_1-f(x_1+y_1), \\ y=\dfrac{1}{b}y_1+\dfrac{1}{b}f(x_1+y_1), \end{cases}$$

where $0<|b|<1$, and $f(z)$ is an entire function of a complex variable such that $f(0)=f'(0)=0$, $f''(0)\neq 0$. The simplest of such f is z^2. The origin O is a semi-attractive fixed point of type $(1, b)_1$.

Example 2.

$$T:\begin{cases} x_1=e^{g(x+by)}x, \\ y_1=by+(1-e^{g(x+by)})x, \end{cases} \qquad T^{-1}:\begin{cases} x=e^{-g(x_1+y_1)}x_1, \\ y=\dfrac{1}{b}y_1+\dfrac{1}{b}(1-e^{-g(x_1+y_1)})x_1, \end{cases}$$

where $g(z)$ is an entire function of a complex variable z with $g(0)=0$, $g'(0)\neq 0$. The orign O is a semi-attractive fixed point of type $(1, b)_1$. The invariant curve $\mathfrak{C}$ with center O is the y-axis. Therefore, denoting by $\mathfrak{D}$ the set of all uniformly convergent points with limit O, the y-axis is contained in the boundary of the domain $\mathfrak{D}$. We can consider $\mathfrak{D}$ also as a subdomain of $\boldsymbol{C}^*\times\boldsymbol{C}$. (Cf. Kodaira [8], Nishimura [11].)

Incidentally, the author does not know whether there is an analytic automorphism of $(\boldsymbol{C}^*)^2$ which has an attractive or a semi-attractive fixed point. It seems even unknown whether there is a domain in $(\boldsymbol{C}^*)^2$ which is biholomorphic to $\boldsymbol{C}^2$.

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