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A characterization of the finely harmonic morphism in R^n

Dedicated to Professor Yukio Kusunoki on his 60th birthday

By

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Introduction.

B. Fuglede [10] gave a characterization of the harmonic morphism in $\boldsymbol{R}^n$ as follows:

Theorem A (Fuglede). *For a continuous mapping φ from a domain U ($\subset \boldsymbol{R}^n$, $n \geqq 2$) into $\boldsymbol{R}^m$ ($m \geqq 2$), the followings are equivalent:*

(i) *φ is a harmonic morphism on U.*

(ii) *The components φ_j of φ ($1 \leqq j \leqq m$), $\varphi_i\varphi_j$ ($i \neq j$) and $\varphi_i^2 - \varphi_j^2$ are harmonic in U.*

(iii) *The components φ_j of φ ($1 \leqq j \leqq m$) are harmonic in U, and $\nabla\varphi_i \cdot \nabla\varphi_j = \delta_{ij}|\nabla\varphi_1|^2$ on U.*

Recently Fuglede introduced the notion of finely harmonic functions in the potential theory on harmonic spaces and he studied finely harmonic morphisms (cf. [7], [8] and [9]).

In this paper we treat a problem of the same type as Theorem A for finely harmonic morphisms in $\boldsymbol{R}^n$. And we obtain the following theorem which is an extension of Theorem A.

Theorem 1. *For a finely continuous mapping φ from a finely open set U ($\subset \boldsymbol{R}^n$, $n \geqq 2$) into $\boldsymbol{R}^m$ ($m \geqq 2$), the followings are equivalent:*

(i) *φ is a finely harmonic morphism on U.*

(ii) *The components φ_j of φ ($1 \leqq j \leqq m$), $\varphi_i\varphi_j$ ($i \neq j$) and $\varphi_i^2 - \varphi_j^2$ are finely harmonic in U.*

(iii) *The components φ_j of φ ($1 \leqq j \leqq m$) are finely harmonic in U, and*

$$\nabla\varphi_i \cdot \nabla\varphi_j = \delta_{ij}|\nabla\varphi_1|^2 \quad \text{a.e. on } U,$$

where $\nabla\varphi_i$ is the gradient defined in Proposition 1 *of* §1.

This theorem will be proved by a probabilistic method. For that purpose we give a probabilistic characterization of the finely harmonic morphism in $\boldsymbol{R}^n$.

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For $n=2$, B. Øksendal [15] gave the following characterization.

Theorem B (Øksendal). *For a finely continuous mapping φ from a finely open set U ($\subset \boldsymbol{R}^2$) into $\boldsymbol{R}^2$, the followings are equivalent:*

(i) *φ is a finely harmonic morphism on U.*

(ii) *For each $x \in U$ and for each Brownian motion $B(t, \omega)$ issued from x, $\{\varphi(B(t))\}_{0 \leq t < \tau}$ is a conformal martingale, where τ is the first exit time of $B(t)$ from U.*

(iii) *φ preserves the paths of Brownian motion.*

Furthermore he remarked that for a finely harmonic morphism in $\boldsymbol{R}^n$, his characterization in $\boldsymbol{R}^2$ will remain valid under an appropriate modification. We introduce the notion of diagonal martingales and extend Theorem B to the case of higher dimensions as follows:

Theorem 2. *For a finely continuous mapping φ from a finely open set U ($\subset \boldsymbol{R}^n$) into $\boldsymbol{R}^m$, the followings are equivalent:*

(i) *φ is a finely harmonic morphism on U.*

(ii) *For each $x \in U$ and for each Brownian motion $B(t, \omega)$ issued from x, $\{\varphi(B(t))\}_{0 \leq t < \tau}$ is a diagonal martingale, where τ is the first exit time of $B(t)$ from U.*

(iii) *φ preserves the paths of Brownian motion.*

Øksendal proved Theorem B by Dynkin's formula. We prove Theorem 2 by the martingale method.

In §1 we provide some definitions and results from potential and probability theories which are used in the next section. In §2 we give the proofs of Theorems 1 and 2.

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§1. Preliminaries.

1. Let $\boldsymbol{R}^n$ ($n \geqq 2$) be the n-dimensional Euclidean space. H. Cartan introduced a topology on $\boldsymbol{R}^n$ which is finer than the Euclidean topology and the coarsest of all topologies for which all the positive superharmonic functions are continuous. This topology is called the *fine topology* in $\boldsymbol{R}^n$ (cf. Cartan [3] and Brelot [1]).

Let V ($\subset \boldsymbol{R}^n$, $n \geqq 2$) be a compact set (in the sense of the Euclidean topology), and for $x \in V$, ε_x be the Dirac measure. We denote by ε_x^{CV} the swept-out of ε_x on the complement CV of V, and call ε_x^{CV} the *harmonic measure* relative to the fine interior V' of V and x.

Definition 1 (Fuglede [7]). Let U be a finely open set in $\boldsymbol{R}^n$. A finely continuous function $f: U \to \boldsymbol{R}$ is called to be *finely harmonic* in U if the fine topology on U has a base consisting of finely open sets V with $\tilde{V}$ (the fine closure of $V) \subset U$ such that f is ε_x^{CV}-integrable for every $x \in V$ and

$$f(x)=\int f \, d\varepsilon_x^{CV} \qquad \text{for every} \quad x \in V.$$

Combining Fuglede's Theorem ([8], Theorem 4.1) which Debiard and Gaveau's Theorem ([5], Theorem 2), we have the following proposition.

Proposition 1. *Let U be a finely open set in $\boldsymbol{R}^n$, and f be a finely harmonic function on U. Then there exists an $\boldsymbol{R}^n$-valued measurable function $h=(h_i)_{1 \leq i \leq n}$ on U satisfying the following condition:*

For every $x \in U$, there exist a compact (in the sense of the Euclidean topology) fine neighbourhood $V(x)$ of x with $V(x) \subset U$ and a sequence of harmonic functions f_ν (each defined in some open set V_ν with $V_\nu \supset V(x)$) such that $f_\nu \to f$ uniformly on $V(x)$ and for each i, $\partial f_\nu / \partial x_i \to h_i$ in $L^2(dv, V(x))$, where dv is the n-dimensional Lebesgue measure.

This proposition means the generalized differentiability of finely harmonic functions. We denote such a generalized gradient h by $\nabla f=(\partial f / \partial x_i)_{1 \leq i \leq n}$. We remark that ∇f is independent of any selection of a compact fine neighbourhood $V(x)$ and a sequence $\{f_\nu\}_{\nu=1}^{\infty}$.

Next we state the definition of finely harmonic morphisms.

Definition 2 (Fuglede [9]). A finely continuous mapping φ from a finely open set U $(\subset \boldsymbol{R}^n)$ into $\boldsymbol{R}^m$ $(m \geqq 2)$ is called a *finely harmonic morphism* if for any finely harmonic function h defined on a finely open set W $(\subset \boldsymbol{R}^m)$, $h \circ \varphi$ is finely harmonic in $\varphi^{-1}(W)$.

We remark that Fuglede [9] called finely harmonic morphisms by finely harmonic mappings.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right-continuous and increasing family $\{\mathcal{F}_t\}_{t \geq 0}$ of sub σ-fields of $\mathcal{F}$. Let $B(t, \omega)$ $(=B(t))$ be an n-dimensional $\mathcal{F}_t$-Brownian motion (cf. Ikeda and Watanabe [12]). If, for a point $x \in \boldsymbol{R}^n$, $P(\omega \in \Omega,\ B(0, \omega)=x)=1$, $B(t, \omega)$ is called an n-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, issued from x and we denote P by P_x. The harmonic measure is characterized by a Brownian motion.

Proposition 2 (Debiard and Gaveau [4]). *Let U be a compact (in the sense of the Euclidean topology) set in $\boldsymbol{R}^n$, ε_x^{CU} the harmonic measure relative to the fine interior U' of U and a point $x (\in U')$, and τ the first exit time of $B(t)$ from U, namely $\tau=\inf\{t>0 ; B(t, \omega) \in CU\}$. Then, $d\varepsilon_x^{CU}(\zeta)=P_x(B(\tau) \in d\zeta)$.*

Remark. For $n=2$, Debiard and Gaveau proved the above proposition.

Their proof is valid for all n.

Next we state two probabilistic notions by which the finely harmonic morphism in $\boldsymbol{R}^n$ is characterized.

Definition 3 (Bernard, Campbell and Davie [2], and Øksendal [15]). Let U be a finely open set in $\boldsymbol{R}^n$. Let $\varphi: U \to \boldsymbol{R}^m$ ($m \geqq 2$) be a finely continuous mapping. Then we say that φ *preserves the paths of Brownian motion* if, for each $x_0 \in U$ and for each Brownian motion $B(t, \omega)$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P_{x_0})$, issued from x_0, the following conditions are fulfiled:

(i) There is a mapping $\sigma(t, \omega)$ $(=\sigma(t)): [0, \infty) \times \Omega \to [0, \infty]$ such that for each $\omega \in \Omega$, the function $\sigma(*, \omega): [0, \tau(\omega)) \to [0, \infty)$ is continuous and strictly increasing, and such that for each $t \geqq 0$, the mapping $\sigma(t, *): \Omega \to [0, \infty]$ is measurable with respect to $\mathcal{F}_{t\wedge\tau}$, where τ is the first exit time of $B(t)$ from U, $t\wedge\tau = \min\{t, \tau\}$ and $\mathcal{F}_{t\wedge\tau} = \{A \in \mathcal{F}; A \cap \{t\wedge\tau \leqq s\} \in \mathcal{F}_s$ for all $s \in [0, \infty)\}$.

(ii) $\varphi^*(\omega) = \lim_{t\to\tau-0} \varphi(B(t, \omega))$ exists for almost every $\omega \in \{\omega; \sigma(\tau(\omega), \omega)$ $(=\lim_{t\to\tau-0} \sigma(t, \omega)) < +\infty\}$.

(iii) There exists an m-dimensional Brownian motion $\hat{B}(t, \hat{\omega})$ defined on a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}_0)$, issued from 0 such that the stochastic process $\{A(t, \omega, \hat{\omega})\}_{t\geqq0}$, on the product probability space $(\Omega \times \hat{\Omega}, \mathcal{F} \times \hat{\mathcal{F}}, \mathcal{F}_{\sigma^{-1}(t)\wedge\tau} \times \hat{\mathcal{F}}_t, P_{x_0} \times \hat{P}_0)$, defined for $t \geqq 0$ and $(\omega, \hat{\omega}) \in \Omega \times \hat{\Omega}$ by

$$A(t, \omega, \hat{\omega}) = \begin{cases} \varphi(B(\sigma^{-1}(t, \omega), \omega)), & \text{if } t < \sigma(\tau(\omega), \omega) \\ \varphi^*(\omega) + \hat{B}(t) - \hat{B}(\sigma(\tau(\omega), \omega), \hat{\omega}), & \text{if } t \geqq \sigma(\tau(\omega), \omega) \end{cases}$$

is an m-dimensional Brownian motion issued from $\varphi(x_0)$.

Before we introduce another probabilistic notion, we state the following proposition.

Proposition 3 (Meyer [14]). *Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right-continuous and increasing family $\{\mathcal{F}_t\}_{t\geqq0}$ of sub σ-fields of $\mathcal{F}$. Let $\{X(t, \omega)\}_{t\geqq0}$ and $\{Y(t, \omega)\}_{t\geqq0}$ be real continuous martingales with respect to $\{\mathcal{F}_t\}_{t\geqq0}$. Then there exists the unique process $\{A(t, \omega)\}_{t\geqq0}$ satisfying conditions*: (i) *For almost every $\omega \in \Omega$, $A(0, \omega) = 0$ and $A(t, \omega)$ is of bounded variation.* (ii) *$\{X(t, \omega)Y(t, \omega) - A(t, \omega)\}_{t\geqq0}$ is a real continuous local martingale with respect to $\{\mathcal{F}_t\}_{t\geqq0}$.*

We denote $A(t, \omega)$ in Proposition 3 by $\langle X, Y\rangle(t, \omega)$. When $\{X(t, \omega)\}_{t\geqq0} = \{Y(t, \omega)\}_{t\geqq0}$, $\langle X, Y\rangle(t, \omega)$ is denoted by $\langle X\rangle(t, \omega)$.

Definition 4. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a right-continuous and increasing family $\{\mathcal{F}_t\}_{t\geqq0}$ of sub σ-fields of $\mathcal{F}$. Then an $\boldsymbol{R}^m$-valued stochastic process $\{Z(t, \omega) = Z(0, \omega) + (X_i(t, \omega))_{1\leqq i\leqq m}\}_{t\geqq0}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ is called a *diagonal martingale* with respect to $\{\mathcal{F}_t\}_{t\geqq0}$ if $\{X_i(t, \omega)\}_{t\geqq0}$ $(1 \leqq i \leqq m)$ are real continuous local martingales with $\{\mathcal{F}_t\}_{t\geqq0}$ and

$$\langle X_i, X_j\rangle(t, \omega)=\delta_{ij}\langle X_1\rangle(t, \omega) \qquad \text{a.e. on} \quad [0, \infty)\times\Omega,$$

where δ_{ij} is Kronecker's delta.

Remark. This notion can be extended to the case of processes on $[0, \tau)$, where τ is a previsible stopping time (cf. Getoor and Sharpe [11], p. 279 and p. 297). In the next section, such a extended process is considered.

§ 2. Main Theorem and the proof.

We summarize Theorems 1 and 2 as follows.

Main Theorem. *Let U be a fine domain (finely open and finely connected set) in $\boldsymbol{R}^n$. Then, for a non-constant finely continuous mapping $\varphi: U\to\boldsymbol{R}^m$ $(m\geqq 2)$, the followings are equivalent:*

(i) *For each $x\in U$ and for each Brownian motion $B(t, \omega)$ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$, issued from x, $\{\varphi(B(t))\}_{0\leq t<\tau}$ is a diagonal martingale with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$, where τ is the first exit time of $B(t)$ from U, $t\wedge\tau=\min\{t, \tau\}$ and $\mathcal{F}_{t\wedge\tau}=\{A\in\mathcal{F}, A\cap\{t\wedge\tau\leqq s\}\in\mathcal{F}_s$, for all $s\in[0, \infty)\}$.*

(ii) *φ preserves the paths of Brownian motion.*

(iii) *φ is a finely harmonic morphism on U.*

(iv) *The components φ_i $(1\leqq i\leqq m)$ of φ, $\varphi_i\varphi_j$ $(i\neq j)$ and $\varphi_i^2-\varphi_j^2$ are finely harmonic in U.*

(v) *The components φ_i $(1\leqq i\leqq m)$ of φ are finely harmonic in U and, for almost every $x\in U$ with respect to the n-dimensional Lebesgue measure dv,*

$$\nabla\varphi_i\nabla\varphi_j=\delta_{ij}|\nabla\varphi_1|^2,$$

where δ_{ij} is Kronecker's delta.

Proof. (i)$\Rightarrow$(iv): For each $z\in U$, let $B(t, \omega)$ be a Brownian motion issued from z and τ be the first exit time of $B(t)$ from U. Suppose that the statement of (i) holds. From Proposition 3, we see that $\{\varphi_i(B(t))\}_{0\leq t<\tau}$ $(1\leqq i\leqq m)$, $\{\varphi_i(B(t))\varphi_j(B(t))\}_{0\leq t<\tau}$ $(i\neq j)$ and $\{\varphi_i^2(B(t))-\varphi_j^2(B(t))\}_{0\leq t<\tau}$ are continuous local martingales with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$. Consider the stochastic process $\{\varphi_i(B(t))\}_{0\leq t<\tau}$. Since it is a continuous local martingale with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$, there exists a sequence of stopping times τ_ν $(<\tau)$ such that for almost every $\omega\in\Omega$, $\tau_\nu(\omega)\to\tau(\omega)$ increasingly and $\{\varphi_i(B(t\wedge\tau_\nu))\}_{t\geq 0}$ is a continuous martingale with respect to $\{\mathcal{F}_{t\wedge\tau_\nu}\}_{t\geq 0}$. Since φ_i is finite and finely continuous, for each $x\in U$, there exists a compact fine neighbourhood $U(x)$ of x such that φ_i is bounded on $U(x)$. Now we restrict z to a point containing in the fine interior $U(x)'$ of $U(x)$. Let τ' be the first exit time of $B(t)$ from $U(x)$. Then, for each ν, $\{\varphi_i(B(t\wedge\tau_\nu\wedge\tau'))\}_{t\geq 0}$ is a continuous martingale with respect to $\{\mathcal{F}_{t\wedge\tau_\nu\wedge\tau'}\}_{t\geq 0}$. By Lebesgue's bounded convergence theorem and Proposition 2, we have

$$\begin{aligned}\varphi_i(z) &= \lim_{\nu\to\infty}\lim_{t\to\infty} E_z(\varphi_i(B(t\wedge\tau_\nu\wedge\tau'))) \\ &= E_z(\varphi_i(B(\tau'))) \\ &= \int\varphi_i(\zeta)d\varepsilon_z^{CU(x)}(\zeta), \qquad \text{for all} \quad z\in U(x)'.\end{aligned}$$

This means that φ_i $(1\leqq i\leqq m)$ are finely harmonic in U. Using the same argument as above, we see that $\varphi_i\varphi_j$ $(i\neq j)$ and $\varphi_i^2-\varphi_j^2$ are finely harmonic in U, since $\{\varphi_i(B(t))\varphi_j(B(t))\}_{0\leq t<\tau}$ $(i\neq j)$ and $\{\varphi_i^2(B(t))-\varphi_j^2(B(t))\}_{0\leq t<\tau}$ are continuous local martingales with respect to $\{\mathcal{F}_{t\wedge t}\}_{t\geq 0}$. Thus (iv) is established.

(iv)$\Rightarrow$(i): Assume the statement of (iv). Since each component φ_i of φ is a finely harmonic in U, by Proposition 1, for any $x\in U$, there exists a compact fine neighbourhood $V(x)$ of x such that, for each $z\in V(x)'$ and for each Brownian motion $B(t, \omega)$ issued from z, $\{\varphi_i(B(t\wedge\tau'))\}_{t\geq 0}$ is a continuous martingale with respect to $\{\mathcal{F}_{t\wedge\tau'}\}_{t\geq 0}$, and moreover Debiard and Gaveau's Theorem ([5], Theorem 2) states that

$$\varphi_i(B(t\wedge\tau'))=\varphi_i(z)+\int_0^{t\wedge\tau'}\nabla\varphi_i(B(s))dB(s) \ (1\leqq i\leqq m),$$

where τ' is the first exit time of $B(t)$ from $V(x)$ and the above integral is a stochastic integral (cf. Ikeda and Watanabe [12], and Mckean [13]). From the representation of $\varphi_i(B(t\wedge\tau'))$ and Doob's quasi-Lindelöf principle [6], we see that there exists a sequence of stopping times τ_ν $(<\tau)$ with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that for almost every $\omega\in\Omega$, $\tau_\nu(\omega)\to\tau(\omega)$ increasingly and $\{\varphi_i(B(t\wedge\tau_\nu))\}_{t\geq 0}$ is a continuous martingale with respect to $\{\mathcal{F}_{t\wedge\tau_\nu}\}_{t\geq 0}$. This means that $\{\varphi_i(B(t))\}_{0\leq t<\tau}$ $(1\leqq i\leqq m)$ are continuous local martingales with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$. Using the same argument as above, we see that $\{\varphi_i(B(t))\varphi_j(B(t))\}_{0\leq t<\tau}$ $(i\neq j)$ and $\{\varphi_i^2(B(t))-\varphi_j^2(B(t))\}_{0\leq t\leq\tau}$ are continuous local martingales with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$ since $\varphi_i\varphi_j$ $(i\neq j)$ and $\varphi_i^2-\varphi_j^2$ are finely harmonic in U. Hence by Proposition 3, we have

$$\langle\varphi_i(B(*)),\ \varphi_j(B(*))\rangle(t)=\delta_{ij}\langle\varphi_1(B(*))\rangle(t) \qquad \text{a.e. on} \quad [0,\tau)\times\Omega. \qquad (1)$$

Thus (i) is established.

(i)$\Leftrightarrow$(v): From the well-known results of stochastic integrals (Ikeda and Watanabe [12], Chap. II Example 2.1 and Proposition 2.3), we see that (i) is equivalent to the following:

(i)$'$ For each $x\in U$ and for each Brownian motion $B(t, \omega)$ issued from x, $\{\varphi_i(B(t))\}_{0\leq t<\tau}$ $(1\leqq i\leqq m)$ are continuous local martingale with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$ and

$$\nabla\varphi_i(B(t))\nabla\varphi_j(B(t))=\delta_{ij}|\nabla\varphi_1(B(t))|^2 \qquad \text{a.e. on} \quad [0,\tau)\times\Omega,$$

where τ is the first exit time of $B(t)$ from U, $t\wedge\tau=\min\{t, \tau\}$ and

$$\mathcal{F}_{t\wedge\tau}=\{A\in\mathcal{F};\ A\cap\{t\wedge\tau\leqq s\}\in\mathcal{F}_s \text{ for all } s\in[0,\infty)\}.$$

Hence we prove that (i)$'$ and (v) are equivalent.

(v)⇒(i)′: Using the same argument as in the proof of implication: (iv)⇒(i), we see that the former part of (i)′ holds. We need to prove the latter part of (i)′.

For any point $x \in U$, let $B(t, \omega)$ be a Brownian motion issued from x. We take a compact fine neighbourhood $V(x)$ of x as in Proposition 1. Let τ' be the first exit time of $B(t)$ from $V(x)$. Then it is easily seen that $P_x(B(t) \in dz, t<\tau')$ is absolutely continuous with respect to the n-dimensional Lebesgue measure $dv(z)$. We denote by $p(t, x, z)$ the density of $P_x(B(t) \in dz, t<\tau')$ with respect to $dv(z)$. For $i, j=1, 2, \cdots, m$, we set

$$E_1^{ij}=\{(t, \omega) \in [0, \tau'(\omega)) \times \Omega\,;\ \nabla\varphi_i(B(t)) \cdot \nabla\varphi_j(B(t)) > \delta_{ij}|\nabla\varphi_1(B(t))|^2\},$$

$$E_2^{ij}=\{(t, \omega) \in [0, \tau'(\omega)) \times \Omega\,;\ \nabla\varphi_1(B(t)) \cdot \nabla\varphi_j(B(t)) < \delta_{ij}|\nabla\varphi_1(B(t))|^2\},$$

$$\hat{E}_1^{ij}=\{z \in V(x)\,;\ \nabla\varphi_i(z) \cdot \nabla\varphi_j(z) > \delta_{ij}|\nabla\varphi_1(z)|^2\},$$

and

$$\hat{E}_2^{ij}=\{z \in V(x)\,;\ \nabla\varphi_i(z) \cdot \nabla\varphi_j(z) < \delta_{ij}|\nabla\varphi_1(z)|^2\}.$$

Then we have, for $k=1, 2$,

$$(-1)^k \int\!\!\int_0^{\tau'} (\nabla\varphi_i(B(t)) \cdot \nabla\varphi_j(B(t)) - \delta_{ij}|\nabla\varphi_1(B(t))|^2) \chi(E_k^{ij})\, dt\, dP_x \tag{2}$$

$$=(-1)^k \int_0^\infty\!\!\int (\nabla\varphi_i(B(t)) \cdot \nabla\varphi_j(B(t)) - \delta_{ij}|\nabla\varphi_1(B(t))|^2) \chi(E_k^{ij})\, dP_x\, dt$$

$$=(-1)^k \int (\nabla\varphi_i(z) \cdot \nabla\varphi_j(z) - \delta_{ij}|\nabla\varphi_1(z)|^2) \chi(\hat{E}_k^{ij}) \int_0^\infty p(t, x, z)\, dt\, dv(z), \tag{3}$$

where $\chi(E)$ generally stands for the defining function of a set E. If, for some i, j and k, $\int_0^\infty\!\int \chi(E_k^{ij})\, dP_x\, dt>0$, (2) is positive, while (3) is zero from the latter part of (v). This is a contradiction. Thus (i) is established.

(i)′⇒(v): Using the same argument as in the proof of implication: (i)⇒(iv), we see from Proposition 2 that the former part of (v) holds. Using the equation (2)=(3) in the proof of implication: (v)⇒(i)′, we can prove the latter part of (v), since $\int_0^\infty p(t, x, z)\, dt>0$, for all z which is contained in the finely connected component of $V(x)'$ containing x, and the n-dimensional Lebesgue measure of a finely open set in $\boldsymbol{R}^n$ is positive. Thus (v) is established.

(ii)⇒(iii): Let u be a finely harmonic function on a finely open set W in $\boldsymbol{R}^n$. By Proposition 1, for any $x \in \varphi^{-1}(W)$, there exist a compact fine neighbourhood V of $\varphi(x)$ with $V \subset W$ and a sequence of harmonic functions u_ν (each defined in some open set V_ν with $V_\nu \supset V$) such that $u_\nu \to u$ uniformly on V. Since φ is finely continuous in U, $u \circ \varphi$ is finite finely continuous in $\varphi^{-1}(W)$. Hence there exists a compact fine neighbourhood $U(x)$ of x with $U(x) \subset \varphi^{-1}(V)$ such that $u \circ \varphi$ is bounded on $U(x)$. For any point $z \in U(x)'$, let $B(t, \omega)$ be a Brownian motion issued from z and τ' be the first exit time of $B(t)$ from $U(x)$. Then we see from (ii) that $\{(u_\nu \circ \varphi)(B(\sigma^{-1}(t) \wedge \tau'))\}_{t \geq 0}$ is a martingale with respect to $\{\mathcal{F}_{\sigma^{-1}(t)\wedge\tau'}\}_{t\geq 0}$, where $\sigma(t, \omega)$ is the same function as that of Definition 3. Hence $\{(u\circ\varphi)(B(\sigma^{-1}(t)\wedge\tau'))\}_{t\geq 0}$ is a martingale with respect to $\{\mathcal{F}_{\sigma^{-1}(t)\wedge\tau'}\}_{t\geq 0}$. By Lebesgue's bounded convergence theorem and Proposition 2, we have,

$$\begin{aligned}(u\circ\varphi)(z)&=\lim_{t\to\infty} E_z((u\circ\varphi)(B(\sigma^{-1}(t)\wedge\tau')))\\ &=E_z((u\circ\varphi)(B(\tau')))\\ &=\int(u\circ\varphi)(\zeta)d\varepsilon_z^{CU(x)}(\zeta), \qquad \text{for all} \quad z\in U(x)'.\end{aligned}$$

This means that $u\circ\varphi$ is finely harmonic in $\varphi^{-1}(W)$. Hence φ is a finely harmonic morphism on U.

(iii)$\Rightarrow$(iv): This is obvious from the definition of finely harmonic morphisms.

(iv)$\Rightarrow$(ii): Assume the statement of (iv). For each $x\in U$, let $B(t, \omega)$ be a Brownian motion issued from x and τ be the first exit time $B(t)$ from U. Since (i) and (iv) are equivalent, $\{\varphi_1(B(t))\}_{0\leq t<\tau}$ is a continuous local martingale with respect to $\{\mathcal{F}_{t\wedge\tau}\}_{t\geq 0}$. We denote $\langle\varphi_1(B(*))\rangle(t\wedge\tau, \omega)$ by $\sigma(t, \omega)$. In the proof of implication: (iv)$\Rightarrow$(i), we obtained the equation (1). Hence from the well-known results of stochastic integrals (cf. Ikeda and Watanabe [12], Chap. II Example 2.1 and Proposition 2.3), we see that $\sigma(t)=\int_0^{t\wedge\tau}|\nabla\varphi_1(B(s))|^2 ds$. On the other hand, using the same argument as in the proof of implication: (ii)$\Rightarrow$(iii), we see from Itô's formula (cf. Ikeda and Watanabe [12], Chap. II Theorem 5.1) that (iv) means (iii), since (i) and (iv) are equivalent. Therefore, from Fuglede's Theorem ([9], Theorem 6), we see that $|\nabla\varphi_1(z)|$ is positive except an finely nowhere dense set, since (iv) and (v) equivalent. This means that for almost every $\omega\in\Omega$, $\sigma(t, \omega)$ is strictly increasing. Thus, from the well-known fact (cf. Getoor and Sharpe [11], Theorem 6.7 or Ikeda and Watanabe [12], Theorem 7.3′), we see that $\sigma(t, \omega)$ satisfies the conditions of Definition 3, since (i) and (iv) are equivalent.
q. e. d.

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