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Canonical duality for unconditioned strong d-sequences

By

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§ I. Introduction.

Let A be a commutative ring with $1\neq 0$ and E an A-module. In this paper we present a theorem about an unconditioned strong d-sequence (abbrev. u. s. d-sequence) on E to which we refer as the *canonical duality theory for u. s. d-sequence.*

Our main theorem is:

(1.1) Theorem. *Let A be a commutative ring with $1\neq 0$ and E an A-module. Assume that a sequence $\boldsymbol{a}=a_1, \cdots, a_s$ of elements in A forms a u. s. d-sequence on E. Then for any injective A-module I, the sequence forms a u. s. d-sequence on*

$$\mathrm{Hom}_A(H_{\boldsymbol{a}}^s(E), I),$$

where $H_{\boldsymbol{a}}^s(E)$ stands for the limit of the direct system of Koszul (co-) homology modules

$$H^i(a_1^n, \cdots, a_s^n; E)$$

and mappings

$$\phi^{n,n+1}: H^i(\boldsymbol{a}^n; E) \longrightarrow H^i(\boldsymbol{a}^{n+1}; E),$$

where $\boldsymbol{a}^m$ denotes the system of elements $a_1^m, \cdots, a_s^m$, for an integer $m>0$.

Here we define the (u. s.) d-sequence as;

(1.2) Definition (cf. [Hu]). Let A and E be as in the theorem above. A sequence of elements $\boldsymbol{a}=a_1, \cdots, a_s$ in A is called a *d-sequence* on E if for each $i=1, \cdots, s$ and for any j with $i\leqq j\leqq s$ the following holds,

$$[(a_1, \cdots, a_{i-1})E : a_i a_j]=[(a_1, \cdots a_{i-1})E : a_j].$$

A sequence $\boldsymbol{a}$ is called a *strong* d-sequence on E, if for any integers

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$n_1, \cdots, n_s > 0$, the sequence

$$a_1^{n_1}, \cdots, a_s^{n_s}$$

forms a d-sequence on E.

If besides each of the properties is stable under any permutation of the sequence, the term *unconditioned* is attached.

It seems not off the point to say that *sequence property* of system of parameters links ring theory with homology theory. Indeed Koszul complex is nothing but the representation of sequence conditions in terms of homology. Along the developement of the theory of local rings, at each stage, sequence property provided tools for the construction of mappings, the calculation of homology or for the definition of various invariants.

More explicitly; regular sequences play key roles in the research of Cohen-Macaulay rings and rings related to it. To Buchsbaum rings, the notion of weakly-regular sequences is attached and to rings with finite local cohomology (or generalized Cohen-Macaulay rings, for the definition see § 3.), the notion of a standard system of parameters, ([T]).

Note that in each cases those notions describe the properties of a system of parameters of each rings/modules. We however wanted to describe the properties only in terms of sequence conditions free from a system of parameters but still ruling all the mechanisms of the behavior of the system of parameters. Then the notion of a '*u. s. d-sequence*' is the most acceptable one. This paper is one of the results of the research based on such recognition shared with S. Goto and K. Yamagishi.

On the other hand the canonical module also played essential roles in the developement of theory of local rings as a module invariant. Above all *the canonical duality theory* provided a good target to attack for the attempt of better understanding of the rings. In our case also we want to establish the *canonical duality theory for u. s. d-sequences* in its most general form possible.

The main theorem above was up-versioned from the *canonical duality theory for Buchsbaum rings* ([S_5] and (1.3), below). By this up-versioning the reader may realize that the theory of u. s. d-sequences unifies the theory of Buchsbaum rings and the theory of rings with finite local cohomology. In fact by the observation of the u. s. d-sequence property of an s. o. p. for the ring with finite local cohomology, we obtain as a corollary to our main theorem the canonical duality theory for Buchsbaum rings:

(1.3) Theorem ([S_5]). *Let A be a Noetherian local ring with its maximal ideal $\mathfrak{m}$ and the residue field $\boldsymbol{k}$. Assume that a finitely generated A-module E possesses the canonical module K_E.*

If E is a Buchsbaum module, then K_E is also a Buchsbaum module.

Here we define the canonical module as below.

(1.4) Definition (cf. [H=K]). Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field $\boldsymbol{k}$ and E a finitely generated A-module. A finitely generated A-module K is called *the canonical module* of E denoted by K_E if the completion of K is isomorphic to

$$\operatorname{Hom}_A(H_{\mathfrak{m}}^s(E),\ E_A(\boldsymbol{k})),$$

where $s=\dim A$ and $E_A(\boldsymbol{k})$ denotes the injective envelope of $\boldsymbol{k}$ over A.

The succeeding section, §2. is devoted to some preliminaries for the basic properties of (u. s.) d-sequences and also to the proof of the main theorem. In §3., we give some basic characterizations of rings with finite local cohomology and, following to it, the proof of the canonical duality theorem of Buchsbaum rings will be given. Beside that some example and remarks will be given in the last part of the section.

Acknowledgement
The key idea of this paper is obtained from joint work with S. Goto. The author wishes to express his thanks to S. Goto for his advice.

§2. The u. s. d-sequences and the proof of the main theorem.

In this section we at first present some basic properties of u. s. d-sequences which are neccessary for the proof of the main theorem (1.1), given in the latter half of this section. More general argument will be developed in [G=Y].

Throughout this section, let A denote a commutative ring with $1\neq 0$ and E an A-module, unless specified otherwise. For a system of elements $\boldsymbol{a}=a_1, \cdots, a_s$ of A, let $K_*(\boldsymbol{a}; E)$, $Z_*(\boldsymbol{a}; E)$, $B_*(\boldsymbol{a}; E)$ and $H_*(\boldsymbol{a}; E)$ denote the Koszul complex generated by $\boldsymbol{a}$ over E, the cycle, the boundary and the homology module, respectively.

We begin with:

(2.1) Lemma. *Assume that $\boldsymbol{a}=a_1, \cdots, a_s$ is a d-sequence on E, then:*

(1) *The sequence $a_i, \cdots, a_s$ is a d-sequence on $E/(a_1, \cdots, a_{i-1})E$ for any $i(1<i\leqq s)$.*

(2) *For any integer $n>0$,*

$$[0 \underset{E}{:} a_1^n]=[0 \underset{E}{:} a_1],$$

and consequently

$$[0 \underset{E}{:} a_1]=H_{\boldsymbol{a}}^0(E).$$

(3) *Let $q=(\boldsymbol{a})A$. Then*

$$H_q^0(E)\cap qE=0.$$

(4) *If $\boldsymbol{a}$ is a u.s. d-sequence on E, then*

$$[0 \underset{E}{:} (\boldsymbol{a})A]=[0 \underset{E}{:} a_i],$$

for any $i(1\leqq i\leqq s)$.

(5) *If $\boldsymbol{a}$ is a u.s. d-sequence on E, then*

$$[0 \underset{E}{:} (a_1^{n_1}, \cdots, a_s^{n_s})A] = [0 \underset{E}{:} a_i]$$

for any $i (1 \leqq i \leqq s)$ and integers $n_1, \cdots, n_s > 0$.

Proof. (1) follows directly from the definition. (2): Note at first that for any i and j with $1 \leqq j \leqq s$, we have

$$[0 \underset{E}{:} a_1] \subset [0 \underset{E}{:} a_j].$$

In fact we have in general the following inclusion

$$[0 \underset{E}{:} a_1] \subset [0 \underset{E}{:} a_1 a_j].$$

Recall also that by the definition right-hand-side of the above equals to $[0 \underset{E}{:} a_j]$. Consequently,

$$[0 \underset{E}{:} a_1] \subset [0 \underset{E}{:} \mathfrak{q}] \subset H_{\boldsymbol{a}}^0(E),$$

where $\mathfrak{q} = (\boldsymbol{a})A$. Hence the second assertion follows from the first, since each element of $H_{\boldsymbol{a}}^0(E)$ is killed by some power of a_1.

As to the first one, use induction argument on the integer n. The case where $n=2$ is the direct consequence of the definition. Let now $n \geqq 3$, $a = a_1$ and e be any element of $[0 \underset{E}{:} a^n]$. Then

$$a^2 a^{n-2} e = a^n e = 0.$$

Hence

$$a^{n-2} e \in [0 \underset{E}{:} a^2] = [0 \underset{E}{:} a]$$

so we have $a^{n-1} e = 0$. By the induction assumption we may conclude that $e \in [0 \underset{E}{:} a]$ as required.

(3): Induction on s, the length of the sequence. If $s=1$ then by the last assertion of (2) above we have $H_{\boldsymbol{a}}^0(E) = [0 \underset{E}{:} a_1]$. Let $a_1 e \in H_{\boldsymbol{a}}^0(E) \cap a_1 E$. Then $a_1^2 e = 0$, hence $e \in [0 \underset{E}{:} a_1]$ by (2) above and we have $a_1 e = 0$.

Let now $s \geqq 2$ and $z \in (a_1, \cdots a_s) E \cap H_{\mathfrak{q}}^0(E)$. Also let $E' = E / a_1 E$, z' denote the image of z in E' and $q' = (a_2, \cdots, a_s)A$. Then

$$z' \in H_{\mathfrak{q}'}^0(E') \cap \mathfrak{q}' E'.$$

By the induction assumption, we may conclude that $z' = 0$, i.e., $z = a_1 e$ for some $e \in E$. Since $\mathfrak{q}^n z = 0$ for some integer $n > 0$,

$$a_1^n z = a_1^{n+1} e = 0.$$

Thus we have

$$e \in [0 \underset{E}{:} a_1^{n+1}] = [0 \underset{E}{:} a_1],$$

hence $z = a_1 e = 0$.

(4) and (5) follow easily from the preceding assertions (1)-(3). (Q.E.D)

The next lemma plays an essential role in the foundation of the theory of u. s. d-sequences and its validity had been predicted by S. Goto.

(2.2) **Lemma** (Goto's lemma [S_5]). *Assume that $a_1, \cdots, a_s$ is a u. s. d-sequence on E/bE for some $b\in A$. Then for any integers $n_1, \cdots, n_s>0$, we have*

$$[(a_1^{n_1}, \cdots, a_s^{n_s})E:b]=\sum_{J\subseteq\{1,\cdots,s\}}(\prod_{j\in J}a_j^{n_j-1})[(\sum_{j\in J}a_jE)\underset{E}{:}b].$$

Proof. We go by induction on s, the length of the sequence. Let us give you the starting push of the induction argument. Let $s=1$ and $a=a_1$. Then the Lemma asserts that for any $n>0$, the following holds

$$[a^nE\underset{E}{:}b]=a^{n-1}[aE\underset{E}{:}b]+[0\underset{E}{:}b].$$

Let $x\in[a^nE:b]$. Then for some $c\in E$,

$$bx=a^nc,$$

and then $c\in[bE:a^n]$. By our assumption on a, we have

$$[bE:a^n]=[bE:a]$$

and hence for some $y\in E$,

$$by=ac,$$

nemely $y\in[aE:b]$. Thus we get

$$bx=a^nc=a^{n-1}by$$

hence

$$x-a^{n-1}y\in[0:b]$$

and the inclusion $\subset$ holds. Converse inclusion is trivially true and the remaining induction argument is an easy exercise of the basic property of u. s. d-sequences and left to the readers.

The next theorem is one of the fundamental facts for the local cohomology with respect to a u. s. d-sequence. Essentially the proof had already been given in [S_3], Prop. 4. We however give one here briefly for the convenience of readers.

(2.3) **Theorem.** *If $\boldsymbol{a}=a_1, \cdots, a_s$ is a u. s. d-sequence on E, and $\mathfrak{q}=(\boldsymbol{a})A$. Then:*

$$\mathfrak{q}H_p(a_1^n, \cdots, a_s^n; E)=0,$$

for any integers n and $p>0$.

Consequently, if $p<s$, then

$$\mathfrak{q}H_{\mathfrak{q}}^p(E)=0.$$

Proof. Since by (2.1),

$$H_s(a_1^n, \cdots, a_s^n; E)=[0\underset{E}{:}a_s],$$

the cases where $p=s$ or $s=1$ are already over.

Now let $s\geqq 2$ and $0<p<s$. We denote by d_* and e_* the differential maps of $K_*(a_1^n, \cdots, a_s^n; E)$ and $K_*(a_2^n, \cdots, a_s^n; E)$, respectively. Let (u, v) be a cycle in

$$K_p(a_1^n, \cdots, a_s^n; E)=K_{p-1}(a_2^n, \cdots, a_s^n; E)\oplus K_p(a_2^n, \cdots, a_s^n; E).$$

Then the cycle condition for (u, v) is described as:

$$-e_{p-1}(u)=0 \tag{1}$$

and

$$a_1^n u+e_p(v)=0 \tag{2}.$$

By (2), we see that v determines a cycle v' in $K_p(a_2^n, \cdots, a_s^n; E/a_1^nE)$. By the induction assumption on the length s of the sequence, we may conclude that

$$a_s v'\in B_p(a_2^n, \cdots, a_s^n; E/a_1^nE),$$

hence there exist $x\in K_p(a_2^n, \cdots, a_s^n; E)$ and $y\in K_{p-1}(a_2^n, \cdots, a_s^n; E)$ such that

(#) $$a_s v=a_1^n x+e_{p+1}(y).$$

Operating e_p, we have

$$a_s e_p(v)=a_1^n e_p(x).$$

Together with (2), we have

$$a_1^n(a_s u+e_p(x))=0,$$

hence

$$a_s u+e_p(x)\in [0: a_1^n]_{K_{p-1}(a_2^n,\cdots,a_s^n; E)}.$$

Since each component of $a_s u+e_p(x)$ belongs to $qE\cap(0 \underset{E}{:} a_1^n)$, by (2.1), we have $a_s u+e_p(x)=0$. Consequently together with (#) above, we have

$$\begin{aligned} a_s(u, v)&=(-e_p(x), a_1^n x+e_{p+1}(y)) \\ &=d_{p+1}((x, y))\in B_p(a_1^n, \cdots, a_s^n; E) \end{aligned}$$

as required. (Q. E. D.)

Note that our proof works completely for a rather general assertion: for any integers $n_i>0$ with $i(1\leqq i\leqq s)$

$$qH_j((a_i^{n_i}; i=1, \cdots, s); E)=0$$

for any $j>0$ with $q=(a_1, \cdots, a_s)A$. Since we do not need this form, so we avoid the formal generalization.

Our next job is to prove the existence of a long exact sequence of local cohomology modules with respect to the ideal $\mathfrak{q}=(\boldsymbol{a})$ generated by a d-sequence $\boldsymbol{a}$ on E.

(2.4) Proposition. *Let $s\geqq 2$ be an integer, $\boldsymbol{a}=a_1, \cdots, a_s$ be a d-sequence on E, $a=a_1$ and $\mathfrak{q}=(\boldsymbol{a})A$. Then there exists a long exact sequence*

$$0\longrightarrow H_{\mathfrak{q}}^0(E)\longrightarrow H_{\mathfrak{q}}^0(E/aE)\longrightarrow H_{\mathfrak{q}}^1(E)\xrightarrow{.a} H_{\mathfrak{q}}^1(E)\longrightarrow H_{\mathfrak{q}}^1(E/aE)\longrightarrow\cdots$$

Consequently, if $\mathfrak{a}$ *is a u.s. d-sequence on* E, *then for each* $i=0, \cdots, s-2$, *there exists a short exact sequence*

$$0 \longrightarrow H_{\mathfrak{q}}^{i}(E) \longrightarrow H_{\mathfrak{q}}^{i}(E/aE) \longrightarrow H_{\mathfrak{q}}^{i+1}(E) \longrightarrow 0$$

and

$$0 \longrightarrow H_{\mathfrak{q}}^{s-1}(E) \longrightarrow H_{\mathfrak{q}}^{s-1}(E/aE) \longrightarrow H_{\mathfrak{q}}^{s}(E) \xrightarrow{.a} H_{\mathfrak{q}}^{s}(E) \longrightarrow 0.$$

Proof. The latter half follows from the former and (2.3).

Consider the following commutative diagram with exact sequences

$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \uparrow & & & & & & \\ 0 & \longrightarrow & aE & \longrightarrow & E & \longrightarrow & E/aE & \longrightarrow & 0 \\ & & \uparrow & \nearrow {\scriptstyle .a} & & & & & \\ & & E & & & & & & \\ & & \uparrow & & & & & & \\ & & [0:a]E & & & & & & \\ & & \uparrow & & & & & & \\ & & 0. & & & & & & \end{array}$$

Applying the functor $H_{\mathfrak{q}}^{0}(\#)$ to the vertical exact sequence, by (2.1) for each $i \geqq 1$, there exists an isomorphism

$$H_{\mathfrak{q}}^{i}(E) \cong H_{\mathfrak{q}}^{i}(aE).$$

Together with these isomorphism, applying the functor $H_{\mathfrak{q}}^{0}(\#)$ to the horizontal exact sequence, there induced a commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & H_{\mathfrak{q}}^{i}(aE) & \longrightarrow & H_{\mathfrak{q}}^{i}(E) & \longrightarrow & H_{\mathfrak{q}}^{i}(E/aE) & \longrightarrow \cdots \\ & \uparrow \wr\wr & \nearrow {\scriptstyle .a} & & & & \\ & H_{\mathfrak{q}}^{i}(E) & & & & & \end{array}$$

hence we have the required sequence. (Q. E. D.)

We are now over the general discussion on u.s. d-sequences. Next we prepare the following lemma, which is a key to the proof of our main theorem.

(2.5) Lemma, ([S_5]). *Let* A, E *and* $\mathfrak{a}$ *be as the statement of the theorem* (1.1) *and* $s \geqq 3$. *Let* P *denote the* A*-linear mapping*

$$H_{\mathfrak{q}}^{s-1}(E) \longrightarrow H_{\mathfrak{q}}^{s-1}(E/a_1E)$$

induced from the natural mapping

$$E \longrightarrow E/a_1E$$

and I be any A-mdule.

Suppose that $g_2, \cdots, g_s$ are A-linear mappings of $H_q^{s-1}(E/a_1E)$ into I satisfying the following equation:

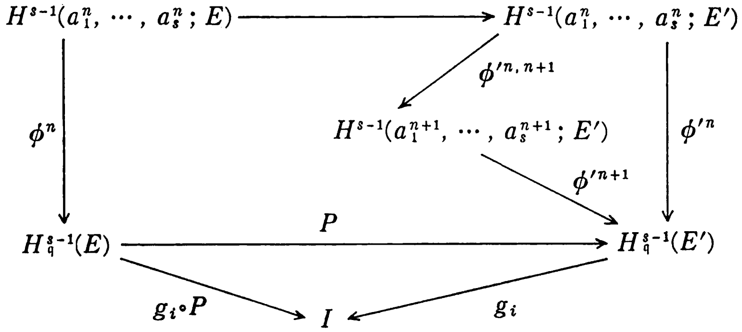
$$a_2g_2 + \cdots + a_sg_s = 0.$$

Then for each $i=2, \cdots, s$ the composition mapping is a zero mapping: $g_i \circ P = 0$.

Proof. Fix i arbitrarily with $2 \leqq i \leqq s$. We have to show that $g_i \circ P = 0$. Let z be any element of $H_q^{s-1}(E)$. For sufficiently large $n>0$, we have the following commutative diagram and can find a cycle

$$c = c_1X_1 + \cdots + c_sX_s \in Z^{s-1}(a_1^n, \cdots, a_s^n; E)$$

which determines a homology class $[c]$ in $H^{s-1}(a_1^n, \cdots, a_s^n; E)$ such that $\phi^n([c]) = z$.



where $E' = E/a_1E$.

Let $c' = c_1'X_1 + \cdots + c_s'X_s$ denote the image of c in $Z^{s-1}(a_1^n, \cdots a_s^n; E')$. It suffices to show that we have

$$g_i \circ \phi'^n([c']) = 0.$$

For that note at first that

$$\begin{aligned} \phi'^{n,n+1}([c']) &= [\textstyle\sum_{l=1}^s (\prod_{m \neq l} a_m) c_l' X_l] \\ &= [(a_2 \cdots a_s) c_1' X_1] \\ &= (a_2 \cdots a_s)[c_1' X_1], \end{aligned}$$

since a_1 kills c_l' for any $l=1, \cdots, s$. Thus we have

$$\begin{aligned} g_i \circ \phi'^n([c']) &= g_i \circ \phi'^{n+1}((a_2 \cdots a_s)[c_1'X_1]) \\ &= a_i(g_i \circ \phi'^{n+1}((a_2 \cdots a_{i-1}a_{i+1} \cdots a_s)[c_1'X_1]) \\ &= (-\textstyle\sum_{\substack{l=2 \\ l \neq i}}^s a_l \circ g_l) \circ \phi'^{n+1}((a_2 \cdots a_{i-1}a_{i+1} \cdots a_s)[c_1'X_1]) \\ &= -\textstyle\sum_{\substack{l=2 \\ l \neq i}}^s g_l \circ \phi'^{n+1}((a_2 \cdots a_{i-1}a_{i+1} \cdots a_l^2 \cdots a_s)[c_1'X_1]). \end{aligned}$$

If we can choose $n=1$ from the start, each of the last term vanishes, because a_1^2 kills the homology $H^{s-1}(a_1^2, \cdots, a_s^2; E')$. We can in fact do so by applying a result stated in [G=S]. But here we apply (2.2) to the final touch of the proof.

Recall that $c=c_1X_1+\cdots+c_sX_s$ is a cycle, hence

$$c_1a_1^n+\cdots+c_sa_s^n=0$$

and

$$c_1\in[(a_2^n, \cdots, a_s^n)E \underset{E}{:} a_1^n]=[(a_2^n, \cdots, a_s^n)E \underset{E}{:} a_1].$$

By (2.2) we have an expression of c_1 as below;

$$c_1=\sum_{J\subseteqq\{2,\cdots,s\}}(\prod_{j\in J}a_j^{n-1})z_J$$

with $z_J\in[(\sum_{j\in J}a_jE):a_1]$ for each $J\subseteqq\{2, \cdots, s\}$. Let $J\subsetneqq\{2, \cdots, s\}$ and $p\in\{2, \cdots, s\}-J$. By (2.1), we have

$$[(a_j; j\in J)E \underset{E}{:} a_1]=[(a_j; j\in J)E \underset{E}{:} a_p],$$

hence

$$\begin{aligned}&(a_2\cdots a_s)(\prod_{j\in J}a_j^{n-1})z_J\in\\ &(a_2\cdots \check{a}_p\cdots a_s)(\prod_{j\in J}a_j^{n-1})a_p[(a_j; j\in J)E \underset{E}{:} a_p]\subset\\ &(a_2\cdots \check{a}_p\cdots a_s)(\prod_{j\in J}a_j^{n-1})(a_j; j\in J)E\subset(a_j^{n+1}; j\in J)E,\end{aligned}$$

where the character with $\check{}$ means that it is deleted. So in $H^{s-1}(a_1^{n+1}, \cdots, a_s^{n+1}; E')$ the following holds

$$(a_2\cdots a_s)[(\prod_{j\in J}a_j^{n-1})z'_JX_1]=0,$$

where z'_J denotes the image of z_J in E'. Consequently we have

$$c'_1=(a_2\cdots a_s)^{n-1}z'_J$$

with $J=\{2, \cdots, s\}$ and for some $z_J\in[(a_j; j\in J)E \underset{E}{:} a_1]$. Thus we have for each $l=2, \cdots, s$ with $l\neq i$,

$$\begin{aligned}&g_l\circ\phi'^{n+1}(a_2\cdots \check{a}_i\cdots a_l^2\cdots a_s)[c'_1X_1]\\ &=g_l\circ\phi'^{n+1}(a_2\cdots \check{a}_i\cdots a_s)[(a_2\cdots a_s)^{n-1}z'_JX_1]\\ &=g_l\circ\phi'^{n+1}(a_2\cdots \check{a}_i\cdots \check{a}_l\cdots a_s)(a_2\cdots \check{a}_l\cdots a_s)^{n-1}(a_l^{n+1})[z'_JX_1]\\ &=0. \qquad\qquad \text{(Q.E.D.)}\end{aligned}$$

Proof of Theorem (1.1).

Let $\mathfrak{q}=(\boldsymbol{a})A$ and $L=\mathrm{Hom}_A(H_{\mathfrak{q}}^s(E), I)$. We must show that the powered sequence $a_1^{n_1}, \cdots, a_s^{n_s}$ is d-sequence on L in any order for any integers $n_1, \cdots, n_s>0$. But this sequence is still a u.s. d-sequence on E, by our starting assumption on the sequence. So considering the sequence $\boldsymbol{a}$ as the powered and permuted sequence of itself, it suffices to prove that it is a d-sequence on L. Hence we prove the following equality for each i and j with $1\leqq i\leqq j\leqq s$,

$$[(a_1, \cdots, a_{i-1})L \underset{L}{:} a_i a_j] = [(a_1, \cdots, a_{i-1})L \underset{L}{:} a_j].$$

There exists an exact sequence (c. f. (2.4))

$$(\sharp) \qquad 0 \longrightarrow H_{\mathfrak{q}}^{s-1}(E) \xrightarrow{P} H_{\mathfrak{q}}^{s-1}(E/a_1E) \xrightarrow{T} H_{\mathfrak{q}}^{s}(E) \xrightarrow{.a_1} H_{\mathfrak{q}}^{s}(E) \longrightarrow 0$$

and the I-dual sequence

$$(\sharp 2) \qquad 0 \longrightarrow L \xrightarrow{.a_1} L \xrightarrow{T^*} L' \xrightarrow{P^*} \mathrm{Hom}_A(H_{\mathfrak{q}}^{s-1}(E), I) \longrightarrow 0$$

where $L' = \mathrm{Hom}_A(H_{\mathfrak{q}}^{s-1}(E/a_1E), I)$, which is also exact.

We at first treat the leading two elements in the sequence. From (♯2) it follows that a_1 is a non-zero-divisor on L. Also a_2 must be a non-zero-divisor on L' by the same reason, because $a_2, \cdots, a_s$ form a u. s. d-sequence on E/a_1E, hence a_2 acts as a non-zero-divisor on the submodule L/a_1L of L'.

Note that we have already finished the cases where $i=1$ and 2, in general.

Let us continue the proof of (1.1). The remaining cases are $s \geqq 3$ and $i \geqq 3$. Let

$$f \in [(a_1, \cdots, a_{i-1})L \underset{L}{:} a_i a_j].$$

Then we have

$$a_i a_j f \in (a_1, \cdots, a_{i-1})L,$$

and by operating T^*

$$a_i a_j T^*(f) \in (a_1, \cdots, a_{i-1})L' = (a_2, \cdots, a_{i-1})L'.$$

By the induction hypothesis on the length s of the sequence, we may conclude that

$$(\sharp 3) \qquad a_j T^*(f) = \textstyle\sum_{l=2}^{i-1} a_l g_l$$

for some g_l's $\in L'$. By (2.5) above, for each $l=2, \cdots, i-1$, we have

$$P^*(g_l) = g_l \circ P = 0.$$

By the exactness of (♯2), for each $l=2, \cdots, i-1$, there exists $f_l \in L$ such that $T^*(f_l) = g_l$. Substituting them to (♯3), we have

$$T^*(a_j f) = \textstyle\sum_{l=2}^{i-1} a_l T^*(f_l)$$

and hence

$$T^*(a_j f - \textstyle\sum_{l=2}^{i-1} a_l f_l) = 0.$$

Again by exactness of (♯2), there must exist $f_1 \in L$ such that

$$a_1 f_1 = a_j f - \textstyle\sum_{l=2}^{i-1} a_l f_l,$$

namely as required we have

$$a_j f \in (a_1, \cdots, a_{i-1})L. \qquad \text{(Q. E. D.)}$$

§ 3. Modules with finite local cohomology.

In this section we study the u. s. d-sequence property of a system of parameters for finitely generated modules over a noetherian local ring. As a consequence, we give the proof of theorem (1.3).

Throughout the remaining part of this paper, let A denote a noetherian local ring, $\mathfrak{m}$ its maximal ideal and $\boldsymbol{k}=A/\mathfrak{m}$, if it is not specified otherwise.

(3.0) **Definition.** Let M be a finitely generated A-module and $\boldsymbol{a}=a_1, \cdots, a_s$ be a system of parameters for M. We define a numerical function $I(*, M)$ by;

$$I(\boldsymbol{a}\,; M):=l_A(M/qM)-e_0(q\,; M)$$

with $\mathfrak{q}=(\boldsymbol{a})A$, where $e_0(\sharp)$ denotes the multiplicity symbol.

Note that, since $e_0(\mathfrak{q}\,; M)$ coincides with the Euler characteristic of the Koszul Complex $K_*(\mathfrak{q}\,; M)$, the function $I(\boldsymbol{a}\,; M)$ has the following expression;

$$I(\boldsymbol{a}\,; M)=\sum_{i=1,\cdots,s}(-1)^{i+1}h_i(\boldsymbol{a}\,; M),$$

where $h_i(\sharp)$ denotes the length of the i-th Koszul homology.

If we have, for all $i\neq s=\dim M$,

$$l_A(H^i_{\mathfrak{m}}(M))<\infty,$$

then we refer to such a module M as a *module with finite local cohomology* (abbrev. F. L. C.) and define a numerical invariant of M by

$$I(M)=\sum_{i=0}^{s-1}\binom{s-1}{i}h^i(M),$$

where $h^i(M)$ denotes the length of $H^i_{\mathfrak{m}}(M)$.

We begin with

(3.1) **Lemma** (cf. [S_3], [S_6] and [A=B]). *Let M be an A-module with F. L. C. of dimension s. Then $M_{\mathfrak{p}}$ is Cohen-Macauley $A_{\mathfrak{p}}$ module of dimension $s-\dim(A/\mathfrak{p})$ for all prime $\mathfrak{p}\neq\mathfrak{m}$ and for any parameter element a for M,*

$$[0 \underset{M}{:} a]\subset H^0_{\mathfrak{m}}(M).$$

Consequently, for any s. o. p. $\boldsymbol{a}=a_1, \cdots, a_s$ for M, we have the following;

(1) *For any r with $0<r<s$ and for each $p<s-r$,*

$$h^p(M/a_1, \cdots, a_r)M)\leqq\sum_{i=0}^{r}\binom{r}{i}h^{i+p}(M).$$

(2) *For r as above and for any $p>0$,*

$$h_p(a_1, \cdots, a_r\,; M)\leqq\sum_{i=0}^{r-p}\binom{r}{p+i}h^i(M).$$

(3) $I(\boldsymbol{a}\,; M)=l_A([0: a_s]_{M/(a_1,\cdots,a_{s-1})M})$.

Proof. As to the first part, by the conventional argument we may reduce to the case where A is a complete local Gorenstein ring of dimension s. Then for any $\mathfrak{p}\neq\mathfrak{m}$ and for any $i\neq s$ we have

$$\begin{aligned}0&=(\mathrm{Hom}_A(H_{\mathfrak{m}}^i(M),\ E_A(A/\mathfrak{m})))_{\mathfrak{p}}\\ &\cong(\mathrm{Ext}_A^{s-i}(M,\ A))_{\mathfrak{p}}\\ &\cong\mathrm{Ext}_A^{s-i}(M_{\mathfrak{p}},\ A_{\mathfrak{p}}).\end{aligned}$$

Since $A_{\mathfrak{p}}$ is a Gorenstein local ring, by the local duality theorem, this leads that $M_{\mathfrak{p}}$ is a Cohen-Macaulay module of maximal possible dimension over $A_{\mathfrak{p}}$, which is what we must show at first. Now let a be any element of $\mathfrak{m}$ such that $\dim(M/aM)<\dim(M)$ and $\mathfrak{p}$ be any prime ideal in the support of M different from $\mathfrak{m}$. Suppose that

$$[0\underset{M}{:}a]_{\mathfrak{p}}\neq(0) \qquad (\sharp),$$

contrary to our assertion. By the preceding result, $M_{\mathfrak{p}}$ is a Cohen-Macaulay module, hence if a is supposed to be a parameter for $M_{\mathfrak{p}}$, a is a non-zero-devisor for $M_{\mathfrak{p}}$ and this contradicts to ($\sharp$) above. Thus there exists a prime ideal $\mathfrak{q}$ of A such that $a\in\mathfrak{q}$, $\mathfrak{q}\subset\mathfrak{p}$ and

$$\dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}=\dim M_{\mathfrak{p}}=\dim A_{\mathfrak{p}}.$$

Then

$$\dim A/\mathfrak{q}=\dim A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}+\dim A/\mathfrak{p}=\dim A_{\mathfrak{p}}+\dim A/\mathfrak{p}=\dim A=\dim M,$$

(recall that A is a Gorenstein local ring). But this contradicts the fact that a is a parameter for M.

(1): Since $\mathfrak{q}=(\boldsymbol{a})A$ is a parameter ideal for M, $H_{\mathfrak{q}}^*(M)\cong H_{\mathfrak{m}}^*(M)$. Hence by the preceding results on the parameter ideal for M, we may apply the same argument as in the proof of (2.4) to obtain the similar long exact sequence of local cohomology modules as in the statement of (2.4), although $\boldsymbol{a}$ is not necessarily a d-sequence on M. Then the assertion follows by standard induction argument on r together with calculation of binomial coefficients.

(2): We generally have the following exact sequence of Koszul homology modules

$$0\longrightarrow H_p(\boldsymbol{a}';M)/a_rH_p(\boldsymbol{a}';M)\longrightarrow H_p(a;M)\longrightarrow[0:a_r]_{H_{p-1}(\boldsymbol{a}';M)}\longrightarrow 0,$$

where $\boldsymbol{a}'=a_1,\cdots,a_{r-1}$ and $\boldsymbol{a}=a_1,\cdots,a_r$. From this and (1) above, the assertion follows by induction on r.

(3): Note at first that for any $i<s$ with $q_i=(a_1,\cdots,a_i)A$

$$l_A([0:a_{i+1}]_{M/q_iM})<\infty.$$

Consequently we have

$$e_0(a_{i+1},\cdots,a_s;M/q_iM)=e_0(a_i,\cdots,a_s;M/q_{i-1}M).$$

Hence we have

$$I(\boldsymbol{a};M)=I(a_s;M/q_{s-1}M)$$

and the right hand side of the above equality is nothing but the one of (3).
(Q. E. D.)

Next we want to characterize an s. o. p. for a module with F. L. C. which is a u. s. d-sequence on the module.

(3.2) Proposition. *Let M be a finitely generated A-module with F. L. C.. Then an s. o. p. $\boldsymbol{a}=a_1, \cdots, a_s$ for M is a u. s. d-sequence on M if and only if the following holds:*

$$l_A(M/\mathfrak{q}M)-e_0(\mathfrak{q}\,;M)=I(M),$$

with $\mathfrak{q}=(\boldsymbol{a})A$ the parameter ideal.

Moreover there indeed exists an s. o. p. which is a u. s. d-sequence on M.

Proof. By (1) and (3) of (3.1), we generally have

$$l_A([0:a_s]_{M/(a_1,\cdots,a_{s-1})M})\leqq h^0(M/(a_1, \cdots, a_{s-1})M).$$

We go by induction on s. Let $s=1$. Assume that the equality holds for some parameter a for M. The we have

$$[0\underset{M}{:}a]=H^0_{\mathfrak{m}}(M).$$

Since for any $n>0$,

$$[0\underset{M}{:}a]\subset[0\underset{M}{:}a^n]\subset H^0_{\mathfrak{m}}(M),$$

we have

$$[0\underset{M}{:}a^n]=[0\underset{M}{:}a],$$

for all $n>0$. This means that a is a (u.)s. d-sequence on M.

Conversely, if a is a (strong) d-sequence on M, then by (2.1) (2),

$$H^0_{\mathfrak{m}}(M)=[0\underset{M}{:}a]$$

hence

$$I(a\,;M)=l_A([0\underset{M}{:}a])=h^0(M)=I(M).$$

Now let $s>1$. Assume that an s. o. p. $\boldsymbol{a}$ for M forms a u. s. d-sequence on M. Then by the induction on the dimension of the module, we have the equality

$$I(\boldsymbol{a}'\,;M')=I(M')$$

for $\boldsymbol{a}'=a_2, \cdots, a_s$ and $M'=M/aM$ with $a=a_1$. Together with the fact that the length of $[0:a]$ is finite, we have

$$e_0(\boldsymbol{a}'\,;M')=e_0(\boldsymbol{a}\,;M),$$

(cf. say Prop. 3.2 [A=B]). On the other hand, by (2.4),

$$h^i(M')=h^i(M)+h^{i+1}(M)$$

for $i=0, \cdots, s-2$. By usual calculation of binomial coefficients, we have

$$I(M')=I(M),$$

and the only if part is over.

Let us proceed to the if part for the case where $s>1$. Let $\boldsymbol{a}=a_1, \cdots, a_s$ be an s. o. p. for M such that

$$I(\boldsymbol{a}\,; M)=I(M).$$

As remarked above, in this case also for each $i=1, \cdots, s$, we have

$$I(a_i, \cdots, a_s\,; M/(a_1, \cdots, a_{i-1})M)=I(\boldsymbol{a}\,; M).$$

Note at first that for any integers $n_1, \cdots, n_s>0$ we have

$$I(a_1^{n_1}, \cdots a_s^{n_s}\,; M)=I(M) \qquad (\sharp)$$

In fact, once we have proved for all $n>0$,

$$I(a_1, \cdots, a_{s-1}, a_s^n\,; M)=I(M) \qquad (\sharp\sharp),$$

then starting from the s. o. p. $a_1, \cdots, a_{s-1}, a_s^n$ for M, we can repeat the same argument until we have ($\sharp$). So it suffices to prove ($\sharp\sharp$) to obtain ($\sharp$).

By (3) of (3.1),

$$I(a_1, \cdots, a_{s-1}, a_s^n\,; M)=l_A([0: a_s^n]_{M/a_1,\cdots,a_{s-1})M})$$
$$\geqq l_A([0: a_s]_{M/(a_1,\cdots,a_{s-1})M}).$$

Thus we have

$$I(a_1, \cdots, a_{s-1}, a_s^n\,; M)\geqq I(M).$$

Since $\leqq$ holds in general by (3.1), we have the equality ($\sharp\sharp$) and hence ($\sharp$). Note also that ($\sharp$) is independent of the order of the sequence. So the permuted and powered sequence of $\boldsymbol{a}$ still preserves the same property. Considering $\boldsymbol{a}$ as the permuted and powered sequence of itself, we need only to show that $a_1, \cdots, a_s$ is a d-sequence on M, i. e., for each i and j with $1\leqq i\leqq j\leqq s$, the following equality holds

$$[(a_1, \cdots, a_{i-1})M \underset{M}{:} a_i a_j]=[(a_1, \cdots, a_{i-1})M \underset{M}{:} a_j].$$

By (1) of (3.1),

$$I(M/(a_1, \cdots, a_{i-1})M)\leqq I(M).$$

Furthermore,

$$I(M)=I(\boldsymbol{a}\,; M)=I(a_i, \cdots, a_s\,; M/(a_1, \cdots, a_{i-1})M),$$

hence

$$I(a_i, \cdots, a_s\,; M/(a_1, \cdots, a_{i-1})M)=I(M/(a_1, \cdots, a_{i-1})M)=I(M) \qquad (\sharp\sharp\sharp)$$

Consequently for the case where $i>1$, by the induction on the dimension of the module, we can conclude that $a_i, \cdots, a_s$ is a d-sequence on $M/(a_1, \cdots, a_{i-1})M$ and the remaining is to prove the following:

$$[0 \underset{M}{:} a_1 a_j]=[0 \underset{M}{:} a_j]$$

for all $j=1, \cdots, s$. But this is true if we see that

$$l_A([0 \underset{M}{:} a_j]) = h^0(M)$$

for each $j=1, \cdots, s$, since $a_1 a_j$ is also a parameter for M and by (3.1),

$$[0 \underset{M}{:} a_1 a_j] \subset H^0_{\mathfrak{m}}(M).$$

Suppose now that for some a_j, say $a=a_1$, we have

$$l_A([0 \underset{M}{:} a]) < h^0(M).$$

Then we deduce from the exact seqence

$$0 \longrightarrow [0 \underset{M}{:} a] \longrightarrow M \longrightarrow aM \longrightarrow 0,$$

$$h^0(aM) = h^0(M) - l_A([0 \underset{M}{:} a]) > 0$$

and

$$h^1(aM) = h^1(M).$$

From the long exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(aM) \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow H^0_{\mathfrak{m}}(M/aM) \longrightarrow H^1_{\mathfrak{m}}(aM) \longrightarrow \cdots$$

we have

$$h^0(M/aM) \leqq h^0(M) - h^0(aM) + h^1(aM) < h^0(M) + h^1(M).$$

Hence with $M' = M/aM$, we have

$$I(q'; M') = \sum_{i=0}^{s-2} \binom{s-1}{i} h^i(M') < \sum_{i=0}^{s-2} \binom{s-2}{i} [h^i(M) + h^{i+1}(M)] = I(M).$$

But in our case $I(q'; M') = I(q; M) = I(M)$, a contradiction. We are now over the former half.

There still remains to show the existence of an s. o. p. for M which is a u. s. d-sequence on M. Induction on s. Let $s=1$. Then for any parameter a for M, there exists $n>0$ such that $a^n H^0_{\mathfrak{m}}(M) = 0$ and for such n we have

$$[0 \underset{M}{:} a^n] = H^0_{\mathfrak{m}}(M).$$

Clearly this a^n is a strong d-sequence on M.

Let $s>1$. Choose a parameter a for M so that

$$aH^i_{\mathfrak{m}}(M) = 0$$

for all $i=0, \cdots, s-1$. Then by the same argument as in the proof of (2.4), we have

$$h^i(M/aM) = h^i(M) + h^{i+1}(M) \tag{$\sharp$}$$

for $i=0, \cdots, s-2$. On the other hand, by the induction assumption, there exists an s. o. p. $\boldsymbol{a}' = a_2, \cdots, a_s$ for $M' = M/aM$ such that $I(\boldsymbol{a}'; M') = I(M')$. Then by ($\sharp$) above

$$I(M') = I(M).$$

Still now we have $I(\boldsymbol{a}'; M) = I(\boldsymbol{a}; M)$ with $\boldsymbol{a} = a, a_2, \cdots, a_s$ hence this $\boldsymbol{a}$ is the required s. o. p. for M. (Q. E. D.)

We have proved the existence of the upper bound for the function $I(\boldsymbol{a}; M)$ for s.o.p.'s for a module M with F.L.C.. Conversely the existence of such upper bound characterizes the modules with F.L.C.:

(3.3) **Proposition.** *Let M be a finitely generated A-module of dimension $s>0$. Assume that there exists a numerical invariant $J(M)<\infty$ such that*

$$J(M) \geqq I(\boldsymbol{a}; M)$$

for any s.o.p. $\boldsymbol{a}$ for M. Then M is with F.L.C..

Proof. Note at first that by (3.2) once M is proved to be with F.L.C., then we may choose $I(M)$ as the invariant $J(M)$.

Let us go by induction on s. If $s=1$ we have nothing to do any more. Let $s>1$. Since we are interested in the local cohomology module of positive degree, we may assume that depth $M>0$. Note that for any s.o.p. $\boldsymbol{a}=a_1, \cdots, a_s$ for M, we have the following inequality, by the general theory of multiplicity (c.f. say [A=B]),

$$l_A(M/\mathfrak{q}M)-e_0(\mathfrak{q}; M) \geqq l_A(M'/\mathfrak{q}'M')-e_0(\mathfrak{q}'; M'),$$

where $\mathfrak{q}=(\boldsymbol{a})A$, $\mathfrak{q}'=(a_2, \cdots, a_s)A$ and $M'=M/a_1M$. So $J(M)$ bounds the function $I(\boldsymbol{a}', M')$ from above, hence by the induction assumption, M' must be with F.L.C.. Moreover the invariant $J(M')=I(M')$ which is not greater than $J(M)$ bounds the length of the local cohomology modules of M' of degree $i<s-1$; i.e., for $i<s-1$, we have

$$J(M) \geqq J(M') \geqq h^i(M').$$

Hence we see that there evists an integer N independent of the parameter element a_1 such that the N-th power of $\mathfrak{m}$ kills all the local cohomology module of M' of degree $i<s-1$. Now choose an M-regular element a and let $M'=M/aM$, then from the exact sequence

$$0 \longrightarrow M \xrightarrow{\ .a\ } M \longrightarrow M/aM \longrightarrow 0,$$

we have an exact sequence for each $i=1, \cdots, s-1$,

$$H_{\mathfrak{m}}^{i-1}(M') \longrightarrow H_{\mathfrak{m}}^{i}(M) \xrightarrow{\ a\ } H_{\mathfrak{m}}^{i}(M).$$

This leads

$$\mathfrak{m}^N[0: a]_{H_{\mathfrak{m}}^i(M)}=(0),$$

whence

$$\mathfrak{m}^N[0: a^r]_{H_{\mathfrak{m}}^i(M)}=(0),$$

for any integer $r>0$. Thus

$$\mathfrak{m}^N H_{\mathfrak{m}}^i(M)=(0),$$

because

$$H_{\mathfrak{m}}^i(M)=\bigcup_{r>0}[0: a^r]_{H_{\mathfrak{m}}^i(M)}. \qquad \text{(Q.E.D.)}$$

Summarizing the preceding discussion, we have the following key theorem in this section.

(3.4) Theorem, (cf. [S=C=T], [G=Y]). *Let M be a finitely generated A-module of dimension s. Then the following conditions are equivalent.*

(1) *There exists an s. o. p. for M that forms a u. s. d-sequence on M.*

(2) $\sup\{I(\mathfrak{q}; M); \mathfrak{q}$ *is a parameter ideal for* $M\}<\infty$.

(3) *The local cohomology module $H_{\mathfrak{m}}^{i}(M)$ is of finite length for any $i\neq s$* $(=\dim M)$.

If this is the case, the supremum mentioned in (2) *above coincides with the invariant $I(M)$ defined in* (3.0):

$$I(M)=\sum_{i=0}^{s-1}\binom{s-1}{i}h^{i}(M).$$

Proof. (1)⇒(3): Let $a_1, \cdots, a_s$ be a system of parameters for M which forms a u. s. d-sequence on M and $q=(a_1, \cdots, a_s)A$. Note that

$$H_{\mathfrak{m}}^{*}(M)\cong H_{q}^{*}(M).$$

Then by (2.3), we conclude that

$$\mathfrak{q}H_{\mathfrak{m}}^{i}(M)\cong\mathfrak{q}H_{\mathfrak{q}}^{i}(M)=0,$$

for any $i<s$. This means that $H_{\mathfrak{m}}^{i}(M)$ is of finite length for each $i<s$.

The implications (3)⇒(2) and (3)⇒(1) follow directly from (3.2). (2)⇒(3) has been proved in (3.3). (Q. E. D.)

By the preceding characterization of modules with F. L. C., applying theorem (1.1) we have

(3.5) Corollary. *Let M be a module with F. L. C. of dimension s. Assume that there exists the canonical module K_M of M. Then K_M is also a module with F. L. C..*

Next we give a duality theorem for the local cohomology modules of a module with F. L. C. and the ones of its canonical module, which we will make use of in the last part of this section. In [S_2] the author gave a proof of it applying the theory of generalized local cohomology studied in [S_1] and the proof was based on the spectral sequence argument. But once we view the F. L. C.-modules as derived ones from Cohen-Macaulay rings/modules, we can give quite an elementary proof by reducing to the Cohen-Macaulay case. We will give such a proof below.

(3.6) Theorem (cf. [S_2] and [S_7]). *Let $(A, \mathfrak{m}, \boldsymbol{k})$ be a local ring and M a finitely generated A-module of dimension s. Suppose that M is with F. L. C.. Let us define the functor*

$$D^{p}(\sharp):=\mathrm{Hom}_{A}(H_{\mathfrak{m}}^{p}(\sharp), E_{A}(\boldsymbol{k})).$$

Then in the case $s\geqq 2$, *we have the following exact sequence*

$$0 \longrightarrow D^0D^0(M) \longrightarrow \hat{M} \longrightarrow D^sD^s(M) \longrightarrow D^0D^1(M) \longrightarrow 0$$

where $\hat{M}$ *denotes the completion of* M, *and the isomorphisms*

$$D^1D^s(M)\cong D^0D^s(M)=0\,,$$

$$D^iD^s(M)\cong D^0D^{s-i+1}(M)$$

for $i=2, \cdots, s-1$.

If $s=1$ *then* $D^0D^1(M)=0$.

Proof. Without loss of generality, we may assume that A is a complete Gorenstein local ring of dimension $s=\dim M$. If M is a C.-M. module, the validity of our assertions is well known, (cf. say Satz 6.1 of [H=K]). Furthermore we can apply the starting argument of the proof of Theorem (1.1) to see that K_M is C.-M., if $s\leqq 2$. So we treat the case where $s\geqq 3$.

We go by descending induction on depth$M$. By replacing M by $M/H^0_{\mathfrak{m}}(M)$, we may assume that depth$M>0$. Let ( )* denote the functor $\mathrm{Hom}_A(\ ,A)$. Consider the exact sequences

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0 \qquad (\sharp 1)$$

and

$$0 \longrightarrow L \longrightarrow G \longrightarrow N \longrightarrow 0 \qquad (\sharp 2)$$

with finite free A-modules F and G. Then there induced an exact sequence

$$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow N^* \longrightarrow \mathrm{Ext}^1_A(M, A) \longrightarrow 0 \qquad (\sharp 1)^*$$

and isomorphisms, for all $i\geqq 1$,

$$\mathrm{Ext}^i_A(N, A)\cong \mathrm{Ext}^{i+1}_A(M, A) \qquad (\sharp\sharp 1).$$

Let X denote the coker$(M^*\to F^*)$, then we have exact sequences

$$0 \longrightarrow M^* \longrightarrow F^* \longrightarrow X \longrightarrow 0$$

and

$$0 \longrightarrow X \longrightarrow N^* \longrightarrow \mathrm{Ext}^1_A(M, A) \longrightarrow 0 \qquad (\sharp).$$

Applying ( )* to these, we have an exact sequence

$$0 \longrightarrow X^* \longrightarrow F^{**} \longrightarrow M^{**} \longrightarrow \mathrm{Ext}^1_A(X, A) \longrightarrow 0 \qquad (\sharp 1)^{**}$$

and isomorphisms, for all $i\geqq 1$,

$$\mathrm{Ext}^i_A(M^*, A)\cong \mathrm{Ext}^{i+1}_A(X, A) \qquad (\sharp\sharp\sharp 1),$$

moreover, isomorphisms for $i\leqq s-2$,

$$\mathrm{Ext}^i_A(N^*, A)\cong \mathrm{Ext}^i_A(X, A) \qquad (\sharp\sharp\sharp\sharp 1),$$

because $\mathrm{Ext}^1_A(M, A)\cong D^{s-1}_A(M)$ is of finite length.

Similarly from ($\sharp$2), we have an exact sequence

$$0 \longrightarrow N^* \longrightarrow G^* \longrightarrow L^* \longrightarrow \mathrm{Ext}^1_A(N, A) \longrightarrow 0 \qquad (\sharp 2)^*.$$

Let $W:=\mathrm{coker}(N^*\to G^*)$. Then from the above, there induced an exact sequence:

$$0 \longrightarrow W^* \longrightarrow G^{**} \longrightarrow N^{**} \longrightarrow \mathrm{Ext}_A^1(W, A) \longrightarrow 0 \qquad (\sharp 2)^{**}.$$

Since depth $L>$depth $M+1$ and we assumed that depth $M>0$, depth $L>2$ and we can apply our theorem to L. So we have

$$\mathrm{Ext}_A^1(W, A)\cong\mathrm{Ext}_A^1(L^*, A)\cong D^{s-1}(L^*)\cong D^2(L)=(0).$$

Moreover by ($\sharp\sharp\sharp\sharp$1) with $i=0$, we have $N^{**}=X^*$. Hence from ($\sharp$1)** and ($\sharp$2)**, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \longrightarrow & G^{**} & \longrightarrow & F^{**} & \longrightarrow & M^{**} & \longrightarrow & \mathrm{Ext}_A^1(X, A) & \longrightarrow 0 \\ & \uparrow\wr\Vert & & \uparrow\wr\Vert & & \uparrow & & & \\ \longrightarrow & G & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0. & \end{array}$$

This leads an exact sequence

$$0 \longrightarrow M \longrightarrow M^{**} \longrightarrow \mathrm{Ext}_A^1(X, A) \longrightarrow 0 \qquad (\sharp 3).$$

Now by ($\sharp$1), we see that N is also with F. L. C. and depth $N>$depth M. By the induction assumption, we obtain the isomorphisms, for $i=2, \cdots, s-1$,

$$\mathrm{Ext}_A^{s-i}(N^*, A)\cong D^iD^s(N)\cong D^0D^{s-i+1}(N)\cong\mathrm{Ext}_A^s(\mathrm{Ext}_A^{i-1}(N, A), A).$$

By ($\sharp\sharp$1) and ($\sharp\sharp\sharp\sharp$1), and above isomorphisms lead the isomorphisms, for $i=2, \cdots, s-1$,

$$\mathrm{Ext}_A^{s-i}(X, A)\cong\mathrm{Ext}_A^s(\mathrm{Ext}_A^i(M, A), A) \qquad (\sharp 4).$$

In particular for $i=s-1$, we have

$$\mathrm{Ext}_A^1(X, A)\cong\mathrm{Ext}_A^s(\mathrm{Ext}_A^{s-1}(M, A), A)\cong D^0D^1(M).$$

This leads together with ($\sharp$3), the first exact sequence in our assertion.

On the other hand, by the isomorphisms ($\sharp\sharp\sharp\sharp$1), ($\sharp$4) leads the isomorphisms, for $i=2, \cdots, s-2$,

$$\mathrm{Ext}_A^{s-i-1}(M^*, A)\cong\mathrm{Ext}_A^s(\mathrm{Ext}_A^i(M, A), A) \qquad (\sharp\sharp 4).$$

There still remains the case where $i=1$ for the isomorphism above, i. e.,

$$\mathrm{Ext}_A^{s-2}(M^*, A)\cong\mathrm{Ext}_A^s(\mathrm{Ext}_A^1(M, A), A).$$

From ($\sharp$) there induced the following exact sequence

$$\mathrm{Ext}_A^{s-1}(N^*, A) \longrightarrow \mathrm{Ext}_A^{s-1}(X, A) \longrightarrow \mathrm{Ext}_A^s(\mathrm{Ext}_A^1(M, A), A) \longrightarrow \mathrm{Ext}_A^s(N^*, A)$$

which leads an isomorphism

$$\mathrm{Ext}_A^{s-1}(X, A)\cong\mathrm{Ext}_A^s(\mathrm{Ext}_A^1(M, A), A),$$

for, since depth $N^*\geqq 2$, both of the ends of the above exact sequence vanish. Again by ($\sharp\sharp\sharp$1), the remaied isomorphism follows. (Q. E. D.)

Remark. Only to determine the kernel and the cokernel of the mapping

$M \to D^s D^s(M)$, it suffices to see that depth $D^s D^s(M) \geqq 2$, which follows from the first of the proof of (1.1). On the other hand, the duality isomorphisms can be obtained by calculation of local cohomology of the minimal injective resolution of the canonical module $D^s(M)$. We wanted to give an elementary and unified proof of them.

Before starting the proof of the theorem (1.3), we give the definition of Buchsbaum rings.

(**3.7**) **Definition** (cf. [St=V_1], [St=V_2] and [V]). Let M be a finitely generated A-module. M is called a *Buchsbaum module* if there exists a numerical invariant I of M such that $I(\boldsymbol{a}; M)=I$ for any s. o. p. $\boldsymbol{a}$ for M.

A ring A is called a *Buchsbaum ring* if it is a Buchsbaum module over itself.

(**3.8**) **Remarks.** (1) By (3.3), Buchsbaum modules are with F. L. C.. Moreover for each $i \neq s = \dim M$,

$$\mathfrak{m} H^i_{\mathfrak{m}}(M)=0$$

(cf. [R=S=V]).

(2) Any s. o. p. for the canonical module of a module is also an s. o. p. for the module itself (cf. Section 1 of [A]).

We are now ready to give

Proof of theorem (1.3).

Let $\boldsymbol{a}$ be any s. o. p. for K_M for a Buchsbaum module M. Then by the remark above $\boldsymbol{a}$ is an s. o. p. for M. By (3.7) and (3.4), $\boldsymbol{a}$ forms a u. s. d-sequence on M. Then it follows from our main theorem (1.1) that $\boldsymbol{a}$ is also a u. s. d-sequence on K_M. Then again by (3.4) and (3.7), we see that K_M is a Buchsbaum module. (Q. E. D.)

To obtain the corollary below, we only need to remark that the completion of $H=\mathrm{End}_A(K_A)$ is isomorphic to $\mathrm{Hom}_A(H^s_{\mathfrak{m}}(K_M), E_A(\boldsymbol{k}))$, the canonical module of the canonical module K_A of A. Note that this H is essetialy the (S_2)-fication or the Cohen-Macaulayfication discussed in [A=G] and [G_5].

(**3.9**) **Corollary.** *Suppose that there exists the canonical module K of a local ring A. Let $H=\mathrm{End}_A(K)$. If $\boldsymbol{a}=a_1, \cdots, a_m$ forms a u. s. d-sequence on A, then it also forms a u. s. d-sequence on H.*

We close this note with an example which shows the best possibility of our main theorem and some rearks on d-sequences.

(**3.10**) **Erample.** The d-sequence property is not necessarily inherited by the canonical module, even if A is a ring with finite local cohomology.

Indeed let $(A, \mathfrak{m}, \boldsymbol{k})$ be a local ring of dimension $d>2$ and depth $d-1$ such that $H_{\mathfrak{m}}^{d-1}(A)\cong A/\mathfrak{m}^2$, (such ring can be constructed following the way stated in [G$_1$] or [G$_4$]). Let $\boldsymbol{a}=a_1, \cdots, a_d$ be a s.o.p. of A. Furthermore choose for some a_i with $i<d$, say $i=1$, so that a_1 is not contained in $\mathfrak{m}^2$, then, with $A'=A/(a_1, \cdots, a_{d-1})$,

$$H_{\mathfrak{m}}^0(A')\cong[0:(a_1, \cdots, a_{d-1})]_{A/\mathfrak{m}^2}=[0:a_1]_{A/\mathfrak{m}^2}=\mathfrak{m}/\mathfrak{m}^2.$$

This means that A' is a Buchsbaum ring of dimension 1, hence $\boldsymbol{a}$ forms a d-sequence on A, because $a_1, \cdots, a_{d-1}$ is a regular sequence of A. On the other hand, by the duality of local cohomology modules of canonical module (3.6),

$$H_{\mathfrak{m}}^2(K_A)\cong\mathrm{Hom}_A(H_{\mathfrak{m}}^{d-1}(A), E_A(\boldsymbol{k}))\cong\mathrm{Hom}_A(A/\mathfrak{m}^2, E_A(\boldsymbol{k}))$$

and

$$H_{\mathfrak{m}}^0(K_A/(a_1, a_2)K_A)\cong[0:(a_1, a_2)]_{H_{\mathfrak{m}}^2(K_A)}\cong\mathrm{Hom}_A(A/((a_1, a_2)+\mathfrak{m}^2), E_A(\boldsymbol{k})).$$

Let us choose a_1, a_2, a_3 so that they form a part of a minimal generating system of $\mathfrak{m}$. If they were a d-sequence on K_A, then we have

$$a_3H_{\mathfrak{m}}^0(K_A/(a_1, a_2)K_A)=0.$$

Hence a_3 belongs to the annihilator $(a_1, a_2)+\mathfrak{m}^2$. But it is impossible, by the choice of a_1, a_2, a_3.

(**3.11**) **Remark.** In the preceding example if we choose all a_i's in $\mathfrak{m}^2$, then they form a strong d-sequence: indeed for any $n_1, \cdots, n_d>0$ integers

$$H_{\mathfrak{m}}^0(A/(a_1^{n_1}, \cdots, a_{d-1}^{n_{d-1}})A)\cong A/\mathfrak{m}^2$$

which is killed by any power of a_d. Hence the powered sequence of a_i's forms a d-sequence. Since the property is stable under any permutation, the sequence is in fact a u.s. d-sequence.

(**3.12**) **Remark.** Furthermore in the same situation as above, choose a_1 in $\mathfrak{m}-\mathfrak{m}^2$ and a_i's in $\mathfrak{m}^2$ for any $i=2, \cdots, d$, then $a_2, \cdots, a_d, a_1$ is not a d-sequence. Bacause in this case,

$$H_{\mathfrak{m}}^0(A/(a_2, \cdots, a_d)A)\cong A/\mathfrak{m}^2$$

and this cannot be killed by a_1.

Namely the sequence $a_1, \cdots, a_d$ chosen in this manner is a d-sequence which is **NOT** an unconditioned d-sequence.

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