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A note on the function $\sum_{n=1}^{\infty} [nx+y]/n!$

By

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Let $f(x, y)$ be the function of real variables x and y defined by

$$f(x, y)=\sum_{n=1}^{\infty} \frac{[nx+y]}{n!},$$

where $[t]$ denotes the greatest integer not exceeding the real number t. In this paper we prove in §3 the linear independency over the filed Q of all rationals of the values of $f(x, y)$ for different irrationals x and in §2 their transcendency for rationals x. Also some properties of the function $f(x, y)$ are studied in §1.

1. Some properties of the function $f(x, y)$.

From the definition it follows that

$$f(x, y)=e[x]+(e-1)[y]+f(\{x\}, \{y\}), \tag{1}$$

where $\{t\}=t-[t]$. It is easily seen that

$$f(x, y)\neq f(x', y') \quad \text{if} \quad (x, y)\neq(x', y'),$$

except when $x=x'$ is a rational number, say $x=p/q$ with coprime integers $q>0$ and p, and $r/q\leqq y, y'<(r+1)/q$ for some integer r; in this special case we have

$$f(p/q, y)=f(p/q, r/q) \quad \text{if} \quad r/q\leqq y<(r+1)/q. \tag{2}$$

The quantity in the right-hand side of (2) will be expressed in Theorem 1 as a linear form of the values of the exponential function. If x is an irrational number, $f(x, y)$ is strictly increasing as a function of y. $f(x, y)$ is also strictly increasing as a function of x for any fixed y, not necessarily irrational.

The function $[x]$ satisfies the equality

$$[nx]=\sum_{r=0}^{q-1}\left[\frac{nx}{q}+\frac{r}{q}\right]$$

for any positive integer q, so that we find

$$f(x, 0)=\sum_{r=0}^{q-1} f\left(\frac{x}{q}, \frac{r}{q}\right),$$

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which may be considered as an expression of a kind of self-similarity for the function $f(x, y)$, whence we have

$$\frac{1}{q} f(qx, 0) = \frac{1}{q} \sum_{r=0}^{q-1} f(x, r/q).$$

The right-hand side above converges to the integral $\int_0^1 f(x, y)dy$, since $f(x, y)$ is Riemann integrable, for it is a nondecreasing and bounded function of y in the unit interval. But, since by (1) the left-hand side converges to ex, we have for all real number x

$$\int_0^1 f(x, y)dy = ex.$$

We discuss now the discontinuity of $f(x, y)$ which is inherited from that of the function $[x]$. We denote by $N(x, y)$ the set of all positive integers n for which $nx+y$ is an integer. Then if $N(x_0, y_0)=\emptyset$, $f(x, y)$ is continuous at (x_0, y_0). If $N(x_0, y_0)$ is nonempty and finite, x_0 must be irrational and $N(x_0, y_0)$ consists of only one point $N(x_0, y_0)=\{n_0\}$, say. Putting $m_0=n_0x_0+y_0$, we have

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ n_0x+y\geqq m_0}} f(x, y)=f(x_0, y_0)$$

and

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ n_0x+y< m_0}} f(x, y)=f(x_0, y_0)-\frac{1}{n_0!}.$$

Finally we assume that $N(x_0, y_0)$ is infinite. Then x_0 and y_0 are rational numbers and if $x_0=p/q$ with coprime integers $q>0$ and p, then $y_0=r/q$ for some integer r. Denoting by $n_0=n_0(x_0, y_0)$ the smallest integer in $N(x_0, y_0)$, we have

$$N(x_0, y_0)=\{n_0+qk \mid k=0, 1, 2,\dots\}, \quad 1\leqq n_0\leqq q$$

We put $n_k=n_0+qk$ and $m_k=n_kx_0+y_0$, so that $m_k=m_0+pk$, and define

$$D_0=\{(x, y) \mid n_0x+y\geqq m_0 \text{ and } x\geqq x_0\},$$

$$D_k=\{(x, y) \mid n_{k-1}x+y<m_{k-1} \text{ and } n_kx+y\geqq m_k\} \quad (k\geqq 1),$$

$$E_0=\{(x, y) \mid n_0x+y<m_0 \text{ and } x\leqq x_0\},$$

$$E_k=\{(x, y) \mid n_{k-1}x+y\geqq m_{k-1} \text{ and } n_kx+y<m_k\} \quad (k\geqq 1).$$

Then we have

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (x,y)\in D_0}} f(x, y)=f(x_0, y_0),$$

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ (x,y)\in D_k}} f(x, y)=f(x_0, y_0)-\sum_{m=0}^{k-1}\frac{1}{(mq+n_0)!} \quad (k\geqq 1),$$

and

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in E_k}} f(x, y)=f(x_0, y_0)-\sum_{m=k}^{\infty}\frac{1}{(mq+n_0)!} \quad (k\geqq 0),$$

Especially, we find

$$\sum_{q=1}^{\infty}\varphi(q)\sum_{m=1}^{\infty}\frac{1}{(mq)!}=e,$$

where $\varphi(q)$ denotes the Euler function, since $f(1, 0)-f(0, 0)=e$ and $f(x, 0)$ is an increasing function increasing only by jumps occuring at rational points.

The function $f(x, y)$ satisfies some functional equations. It follows from the relation

$$[t]+[-t]+1=\begin{cases} 1 & \text{if} \quad t\equiv 0 \pmod 1, \\ 0 & \text{otherwise,}\end{cases}$$

that

(3)

$$f(x, y)+f(-x, -y)+e-1=\sum_{n\in N(x,y)}\frac{1}{n!}=\begin{cases} 0 & \text{if} \quad N(x, y)=\phi, \\ \dfrac{1}{n_0!} & \text{if} \quad N(x, y)=\{n_0\}, \\ \displaystyle\sum_{m=0}^{\infty}\frac{1}{(mq+n_0)!} & \text{if} \quad N(x, y) \text{ is infinite,}\end{cases}$$

where n_0 and q are as above. Here, for any pair of integers $q\geqq 1$ and r with $0\leqq r\leqq q-1$, we have

(4) $$\sum_{m=0}^{\infty}\frac{1}{(mq+r)!}=\frac{1}{q}\sum_{k=1}^{q}\zeta^{-kr}e^{\zeta^k}, \qquad \zeta=e^{2\pi i/q},$$

in view of the formula

$$\sum_{k=1}^{q}\zeta^{k(n-r)}=\begin{cases} q & \text{if} \quad n\equiv r \pmod q, \\ 0 & \text{otherwise.}\end{cases}$$

Especially

$$f\left(x, \frac{1}{2}\right)+f\left(-x, \frac{1}{2}\right)=\begin{cases} \displaystyle\sum_{m=0}^{\infty}\frac{1}{(2^{b-1}(2m+1))!} & \text{if} \quad x=a/2^b,\ a,\ b(\geqq 1)\in \boldsymbol{Z},\ (a, 2)=1, \\ 0 & \text{otherwise,}\end{cases}$$

and

$$f(x, 0)+f(-x, 0)+e-1=\begin{cases} \displaystyle\frac{1}{q}\sum_{k=1}^{q}e^{\zeta^k} & \text{if} \quad x \text{ is rational,} \\ 0 & \text{otherwise,}\end{cases}$$

where $\zeta=\exp(2\pi i/q)$ and q is the positive denominator of x in its lowest term.

The function $f(x, y)$ also has an interesting expression. Namely, we have

$$(5)\qquad \sum_{n=1}^{\infty}\frac{[nx+y]}{n!}=ex+ey-[y]-\frac{e}{2}+\frac{1}{2}\sum_{n\in N(x,y)}\frac{1}{n!}+\lambda(y)$$

$$+\frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{1}{m}(e^{2\pi imy+e^{2\pi imx}}-e^{-2\pi imy+e^{-2\pi imx}}),$$

provided $nx+y\neq 0$ for all positive integer n, where

$$\lambda(t)=\begin{cases}\frac{1}{2} & \text{if } t\equiv 0 \pmod 1),\\ & \text{otherwise.}\end{cases}$$

Indeed, it follows from the Fourier expansion

$$\{\theta\}=\frac{1}{2}-\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin 2m\pi\theta-\lambda(\theta)$$

that

$$(6)\qquad f(x, y)=\sum_{n=1}^{\infty}\frac{nx+y}{n!}-\sum_{n=1}^{\infty}\frac{\{nx+y\}}{n!}$$

$$=ex+(e-1)\left(y-\frac{1}{2}\right)+\frac{1}{2}\sum_{n\in N(x,y)}\frac{1}{n!}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{\infty}\frac{\sin 2m\pi(nx+y)}{m}$$

Applying now the Euler-Maclaurin formula, we have

$$\sum_{m=M+1}^{N}\frac{\sin\omega m}{m}=\int_{M}^{N}\frac{\sin\omega t}{t}dt+\int_{M}^{N}\left(\{t\}-\frac{1}{2}\right)\frac{\omega\cos\omega t-\sin\omega t}{t^2}dt$$

$$+\frac{1}{2}\left(\frac{\sin\omega N}{N}-\frac{\sin\omega M}{M}\right),\quad \omega=2\pi(nx+y),$$

so that

$$\sum_{m=M+1}^{\infty}\frac{\sin 2m\pi(nx+y)}{m}$$

$$=\int_{M}^{\infty}\frac{\sin 2\pi(nx+y)t}{t}dt+0\left(\frac{n}{M}\right)=0\left(\frac{1}{(nx+y)M}\right)+0\left(\frac{n}{M}\right),$$

where the constants implied in 0-symbols are independent of n and M. Hence, noticing that $nx+y\neq 0$ for all positive integer n, we see

$$\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=M+1}^{\infty}\frac{\sin 2m\pi(nx+y)}{m}\longrightarrow 0\quad as\quad M\longrightarrow\infty$$

Therefore we can change the order of summations in the double sum in (6) and obtain

$$\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{m=1}^{\infty}\frac{\sin 2m\pi(nx+y)}{m}$$

$$=\frac{1}{2i}\sum_{m=1}^{\infty}\frac{1}{m}\sum_{n=1}^{\infty}\frac{1}{n!}(e^{2\pi im(nx+y)}-e^{-2\pi im(nx+y)})$$

$$=\frac{1}{2i}\sum_{m=1}^{\infty}\frac{1}{m}(e^{2\pi imy+e^{2\pi imx}}-e^{-2\pi imy+e^{-2\pi imx}})-\sum_{m=1}^{\infty}\frac{\sin 2m\pi y}{m},$$

which together with (6) yields (5).

2. The values of the function $f(x, y)$ for rational x.

Theorem 1. *For any rational number* α *and any real number* β, *we have*

$$\sum_{n=1}^{\infty}\frac{[n\alpha+\beta]}{n!}=a_0+\sum_{k=1}^{q}a_k e^{\zeta^k},\quad \zeta=e^{2\pi i/q}, \tag{7}$$

where $q>0$ *is the denominator of* α *in its lowest term,* a_0 *is a rational number, and* a_k $(1\leqq k\leqq q)$ *are algebraic numbers given in* (9) *below.*

By the theorem of Lindemann-Weierstrass [1; Theorem 1.4], we have the following:

Corollary. *The number* $\sum_{n=1}^{\infty}[n\alpha+\beta]/n!$ *is transcendental for any rational* α *and any real* β, *except when* $\alpha=0$ *and* $0\leqq\beta<1$.

Proof of Theorem 1. We put $p/q=\{\alpha\}$ and $r=[q\{\beta\}]$, so that $0\leqq p<q$ and $0\leqq r<q$. n_0 denotes as in the preceding section the smallest positive integer n for which $np/q+r/q$ is an integer, and $m_0=n_0p/q+r/q$. Then

$$[np/q+r/q]=m_0+[(n-n_0)p/q]$$

for any positive integer n, so that we have, using (1) and (2),

$$\sum_{n=1}^{\infty}\frac{[n\alpha+\beta]}{n!}=([\alpha]+[\beta])e-[\beta]+\sum_{n=1}^{\infty}\frac{[np/q+r/q]}{n!} \tag{8}$$

$$=([\alpha]+[\beta]+m_0)-[\beta]-m_0+\sum_{n=1}^{n_0-1}\frac{[(n-n_0)p/q]}{n!}+\sum_{n=0}^{\infty}\frac{[np/q]}{(n+n_0)!}.$$

We assume from now on $p\neq 0$ and define for any positive integer h

$$\nu(h)=\begin{cases} hq/p & \text{if } hq/p \text{ is an integer,}\\ [hq/p]+1 & \text{otherwise,}\end{cases}$$

so that $\nu(1)<\nu(2)<\cdots$ and

$$[np/q]=h \quad \text{if} \quad \nu(h)\leqq n<\nu(h+1),$$

since $hq/p\leqq\nu(h)\leqq n\leqq\nu(h+1)-1<(h+1)q/p$. Then

$$\sum_{n=0}^{\infty}\frac{[np/q]}{(n+n_0)!}=\sum_{h=0}^{\infty}\sum_{n=\nu(h)}^{\nu(h+1)-1}\frac{[np/q]}{(n+n_0)!}=\sum_{h=0}^{\infty}\sum_{l=0}^{\nu(h+1)-\nu(h)-1}\frac{h}{(\nu(h)+l+n_0)!}.$$

Writing $h=mp+j$ with $0\leqq j<p$, we find $\nu(mp+j)=mq+\nu(j)$, and so,

$$\sum_{n=0}^{\infty}\frac{[np/q]}{(n+n_0)!}=\frac{p}{q}\sum_{j=0}^{p-1}\sum_{l=0}^{\nu(j+1)-\nu(j)-1}\sum_{m=0}^{\infty}\frac{mq+\nu(j)+l+n_0+jq/p-\nu(j)-l-n_0}{(mq+\nu(j)+l+n_0)!}$$

$$=\frac{p}{q}\sum_{j=0}^{p-1}\sum_{l=0}^{\nu(j+1)-\nu(j)-1}\Big(\sum_{m=0}^{\infty}\frac{mq+r(j,l)}{(mq+r(j,l))!}$$

$$+\Big(\frac{jq}{p}-\nu(j)-l-n_0\Big)\sum_{m=0}^{\infty}\frac{1}{(mq+r(j,l))!}\Big)-B$$

with

$$B=\frac{p}{q}\sum_{j=0}^{p-1}\sum_{l=0}^{\nu(j+1)-\nu(j)-1}\sum_{m=0}^{m(j,l)-1}\frac{mq+r(j,l)+jq/p-\nu(j)-l-n_0}{(mq+r(j,l))!},$$

where $m(j,l)$ and $r(j,l)$ are nonnegative integers such that

$$m(j,l)q+r(j,l)=\nu(j)+l+n_0,\quad 0\leqq r(j,l)<q.$$

Therefore, using (4), we obtain

$$\sum_{n=0}^{\infty}\frac{[np/q]}{(n+n_0)!}=\frac{p}{q^2}\sum_{k=1}^{q}e^{\zeta^k}\sum_{j=0}^{p-1}\sum_{l=0}^{\nu(j+1)-\nu(j)-1}\Big(\zeta^k+\frac{jq}{p}-\nu(j)-l-n_0\Big)\zeta^{-kr(j,l)}-B.$$

This together with (8) yields (7) with

$$(9)\quad\begin{cases} a_0=-[\beta]-m_0-B+\displaystyle\sum_{n=1}^{n_0}\frac{[(n-n_0)p/q]}{n!}, \\ a_k=\displaystyle\frac{p}{q^2}\sum_{j=0}^{p-1}\sum_{l=0}^{\nu(j+1)-\nu(j)-1}\Big(\zeta^k+\frac{jq}{p}-\nu(j)-l-n_0\Big)\zeta^{-kr(j,l)}\quad(1\leqq k<q), \\ a_q=[\alpha]+[\beta]+m_0+\displaystyle\frac{p}{q^2}\sum_{j=0}^{p-1}\sum_{l=0}^{\nu(j+1)-\nu(j)-1}\Big(1+\frac{jq}{p}-\nu(j)-l-n_0\Big), \end{cases}$$

Where, by means of the simple continued fractin expansion

$$\frac{p}{q}=\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_s},\quad s\ \text{odd},$$

the integer n_0 can be expressed as

$$n_0=q\Big\{\frac{-r}{b_s}+\frac{1}{b_{s-1}}+\cdots+\frac{1}{b_1}\Big\}.$$

Example 1. As the expression (7) in terms of (9) looks rather complicated, we give here some simple examples.

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty}\Big[\frac{1}{2}n\Big]/n!=\frac{1}{2}\cosh 1, \\ \displaystyle\sum_{n=1}^{\infty}\Big[\frac{1}{2}n+\frac{1}{2}\Big]/n!=\frac{1}{2}\cosh 1+\sinh 1, \end{cases}$$

$$\begin{cases} \sum_{n=1}^{\infty}\left[\frac{1}{3}n\right]/n!=\frac{1}{3}\left(\frac{1}{\sqrt{e}}\cos\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3e}}\sin\frac{\sqrt{3}}{2}\right) \\ \sum_{n=1}^{\infty}\left[\frac{1}{3}n+\frac{1}{3}\right]/n!=\frac{1}{3}\left(-\frac{2}{\sqrt{3e}}\sin\frac{\sqrt{3}}{2}+e\right) \\ \sum_{n=1}^{\infty}\left[\frac{1}{3}n+\frac{2}{3}\right]/n!=\frac{1}{3}\left(-\frac{1}{\sqrt{e}}\cos\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3e}}\sin\frac{\sqrt{3}}{2}+2e\right) \end{cases}$$

$$\begin{cases} \sum_{n=1}^{\infty}\left[\frac{2}{3}n\right]/n!=\frac{1}{3}\left(\frac{1}{\sqrt{e}}\cos\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3e}}\sin\frac{\sqrt{3}}{2}+e\right) \\ \sum_{n=1}^{\infty}\left[\frac{2}{3}n+\frac{1}{3}\right]/n!=\frac{1}{3}\left(\frac{2}{\sqrt{3e}}\sin\frac{\sqrt{3}}{2}+2e\right) \\ \sum_{n=1}^{\infty}\left[\frac{2}{3}n+\frac{2}{3}\right]/n!=\frac{1}{3}\left(-\frac{1}{\sqrt{e}}\cos\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3e}}\sin\frac{\sqrt{3}}{2}+3e\right) \end{cases}$$

$$\begin{cases} \sum_{n=1}^{\infty}\left[\frac{1}{4}n\right]/n!=\frac{1}{4}(\cos 1+\sin 1-\sinh 1) \\ \sum_{n=1}^{\infty}\left[\frac{1}{4}n+\frac{1}{4}\right]/n!=\frac{1}{4}(\cos 1-\sin 1+\sinh 1) \\ \sum_{n=1}^{\infty}\left[\frac{1}{4}n+\frac{1}{2}\right]/n!=\frac{1}{4}(-\cos 1-\sin 1+\cosh 1+e) \\ \sum_{n=1}^{\infty}\left[\frac{1}{4}n+\frac{3}{4}\right]/n!=\frac{1}{4}(-\cos 1+\sin 1+\sinh 1+2e) \end{cases}$$

$$\begin{cases} \sum_{n=1}^{\infty}\left[\frac{3}{4}n\right]/n!=\frac{1}{4}(\cos 1-\sin 1+\cosh 1+e) \\ \sum_{n=1}^{\infty}\left[\frac{3}{4}n+\frac{1}{4}\right]/n!=\frac{1}{4}(\cos 1+\sin 1+\sinh 1+2e) \\ \sum_{n=1}^{\infty}\left[\frac{3}{4}n+\frac{1}{2}\right]/n!=\frac{1}{4}(-\cos 1+\sin 1+\cosh 1+3e) \\ \sum_{n=1}^{\infty}\left[\frac{3}{4}n+\frac{3}{4}\right]/n!=\frac{1}{4}(-\cos 1-\sin 1+\sinh 1+4e) \end{cases}$$

Example 2.

$$\sum_{n=1}^{\infty}\left[\frac{1}{q}n\right]/n!=-\frac{1}{p^2}\sum_{h=1}^{q-1}\sum_{l=1}^{q-1}l\zeta^{-hl}e^{\zeta^h}+\frac{1}{2q}(3-q)e, \qquad \zeta=e^{2\pi i/q}.$$

Example 3. As we have seen in the proof of Theorem 1, the values $f(p/q,\ r/q)$ can be written as a linear form of the numbers

$$e_{q,r}=\sum_{m=0}^{\infty}\frac{1}{(mq+r)!} \qquad (r=0,\ 1,\ldots,\ q-1).$$

For $p=1$, we have the following simple relation; however, in general, it could be complicated. We have

$$
(10)\qquad
\begin{pmatrix}
f(1/q,\,0)\\
f(1/q,\,1/q)\\
f(1/q,\,2/q)\\
\vdots\\
\vdots\\
f(1/q,\,(q-3)/q)\\
f(1/q,\,(q-2)/q)\\
f(1/q,\,(q-1)/q)
\end{pmatrix}
=\frac{1}{q}
\begin{pmatrix}
1 & 0 & -1 & -2\cdots-q+4 & -q+3 & -q+2\\
1 & 0 & -1 & -2\cdots-q+4 & -q+3 & 2\\
1 & 0 & -1 & -2\cdots-q+4 & 3 & 2\\
\vdots & \vdots & \vdots & \vdots\qquad 4 & 3 & \vdots\\
\vdots & \vdots & \vdots & -2\qquad\ \vdots & \vdots & \vdots\\
1 & 0 & -1 & q-2\cdots\qquad 4 & 3 & 2\\
1 & 0 & q-1 & q-2\cdots\qquad 4 & 3 & 2\\
1 & q & q-1 & q-2\cdots\qquad 4 & 3 & 2
\end{pmatrix}
\begin{pmatrix}
e_{q,0}\\
e_{q,1}\\
e_{q,2}\\
\vdots\\
\vdots\\
e_{q,q-3}\\
e_{q,q-2}\\
e_{q,q-1}
\end{pmatrix}
$$

and by (4)

$$
\begin{pmatrix}
e_{q,0}\\
e_{q,1}\\
\vdots\\
e_{q,q-1}
\end{pmatrix}
=\frac{1}{q}
\begin{pmatrix}
1 & 1 & \cdots\cdots 1\\
\zeta^{-1} & \zeta^{-2} & \cdots\cdots\zeta^{-(q-1)}\\
\vdots & \vdots & \cdots\cdots\ \vdots\\
\zeta^{-(q-1)} & \zeta^{-2(q-1)} & \cdots\cdots\zeta^{-(q-1)^2}
\end{pmatrix}
\begin{pmatrix}
e\\
e^{\zeta}\\
\vdots\\
e^{\zeta^{q-1}}
\end{pmatrix}
$$

The determinant of the former matrix is $\varepsilon q^{q-2}\neq 0$, where $\varepsilon=1$ if $q\equiv 1$ or $2 \pmod 4$ and $\varepsilon=-1$ otherwise, and that of the latter is also nonzero, since it is Vandermonde's determinant. Thus q numbers $f(1/q,\,0)$, $f(1/q,\,1/q),\ldots,f(1/q,\,(q-1)/q)$ are algebraically dependent.

Proof of (10). If $1\leqq r\leqq q-1$, we have

$$
\begin{aligned}
f(1/q,\,r/q)&=\sum_{m=0}^{\infty}\sum_{s=0}^{q-1}\frac{[(mq+s)/q+r/q]}{(mq+s)!}\\
&=\sum_{m=0}^{\infty}\left(\sum_{s=0}^{q-r-1}\frac{m}{(mq+s)!}+\sum_{s=q-r}^{q-1}\frac{m+1}{(mq+s)!}\right)\\
&=\frac{1}{q}\sum_{m=1}^{\infty}\frac{1}{(mq-1)!}+\frac{1}{q}\sum_{m=0}^{\infty}\sum_{s=0}^{q-1}\left(\frac{1}{(mq+s-1)!}-\frac{s}{(mq+s)!}\right)+\sum_{s=q-r}^{q-1}\frac{1}{(mq+s)!}\\
&=\frac{1}{q}\Big(e_{q,q-1}+\sum_{s=1}^{q-1}(e_{q,s-1}-se_{q,s})+\sum_{s=q-r}^{q-1}qe_{q,s}\Big)\\
&=\frac{1}{q}\Big(e_{q,0}+\sum_{s=1}^{q-r-1}(1-s)e_{q,s}+\sum_{s=q-r}^{q-2}(q-s+1)e_{q,s}+2e_{q,q-1}\Big).
\end{aligned}
$$

Similarly we can write $f(1/q,\,0)$ as a linear form of $e_{q,0},\,e_{q,s},\ldots,\,e_{q,q-1}$.

3. Linear independence of the values of the function $f(x,\,y)$.

We generalize a theorem of Skolem [3; Theorem 6] concerning the linear independence over Q of the numbers $1, \sum_{n=1}^{\infty}[nx_1]/n!,\dots, \sum_{n=1}^{\infty}[nx_l]/n!$ when $x_1,\dots,x_l$ are l distinct positive numbers such that 1 is not dependent on them over Q.

Theorem 2. *Let $x_1,\dots,x_l$ be as above and let $y_1,\dots,y_l$ be any choice of l real numbers. Then $1, \sum_{n=1}^{\infty}[nx_1+y_1]/n!,\dots, \sum_{n=1}^{\infty}[nx_l+y_l]/n!$ are linearly independent over Q.*

Proof. Suppose that

$$A_0+\sum_{i=1}^{l}A_i\sum_{n=1}^{\infty}\frac{[nx_i+y_i]}{n!}=0$$

for some integer A_i $(1\leqq i\leqq l)$. Denoting by H_n the integer

$$H_n=-(n-1)!(A_0+\sum_{k=1}^{n-1}\sum_{i=1}^{l}A_i[kx_i+y_i]/k!),$$

we have

$$H_n=\sum_{i=1}^{l}A_ix_i+O\left(\frac{1}{n}\right)$$

as $n\to\infty$. Hence $\sum_{i=1}^{l}A_ix_i$ is an integer, and so, by the assumption on $x_1,\dots,x_l$, we get

$$\sum_{i=1}^{l}A_ix_i=0, \tag{11}$$

so that $H_n=0$ for all $n\geqq n_0$ for some n_0. Thus

$$\sum_{i=1}^{l}A_i(\{nx_i\}-[\{nx_i\}+y])=0 \quad (n\geqq n_0). \tag{12}$$

We need now the following Lemma [3; pp. 79–80]: *If $x_1,\dots,x_l$ are l positive numbers, then there are p $(\leqq l)$ positive numbers $\xi_1,\dots,\xi_p$ linearly independent over Q such that*

$$x_i=\sum_{j=1}^{p}a_{ij}\xi_j \quad (1\leqq j\leqq l),\quad \xi_j=\sum_{i=1}^{l}b_{ji}x_i \quad (1\leqq j\leqq p), \tag{13}$$

where a's are nonnegative integers and b's are rational numbers.

We will prove $A_1=0$. For this we may assume that

$$\gamma a_{i1}/a_{11}\not\equiv 0 \pmod 1 \quad (2\leqq i\leqq l) \tag{14}$$

and

$$\gamma a_{i1}/a_{11}+y_i\not\equiv 0 \pmod 1 \quad (2\leqq i\leqq l) \tag{15}$$

where γ is the real number defined by

$$\gamma+y_1\equiv 0 \pmod 1 \quad\text{and}\quad 0<\gamma\leqq 1. \tag{16}$$

Indeed, we may assume $0<\xi_1<\xi_2<\cdots<\xi_p$. Then $\xi_j-t^{j-1}\xi_1>0$ for all $j\geqq 2$ and all t with $0<t<t_0$ for some t_0. Putting $t=r/s$ with r, s positive integers, we can write

$$x_i=\sum_{j=1}^{p} a'_{ij}\xi'_j \quad (1\leqq i\leqq l),$$

where $a'_{i1}=s^{p-1}\sum_{j=1}^{p} a_{ij}t^{j-1}$ $(1\leqq i\leqq l)$, $a'_{ij}=a_{ij}$ $(1\leqq i\leqq l,\ 2\leqq j\leqq p)$, and $\xi'_1=\xi_1/S^{p-1}$, $\xi'_j=\xi_j-t^{j-1}\xi_1$ $(2\leqq j\leqq p)$. Clearly $\xi'_1,\dots,\xi'_p$ are linearly independent over Q and can be written as linear forms of $x_1,\dots,x_l$ with rational coefficients. Since $\sum_{j=1}^{p} a_{ij}t^{j-1}$ $(1\leqq i\leqq l)$ are different as polynomials of t, we can choose $t=r/s$ so that a'_{ij}'s satisfy the required properties (14) and (15).

We choose ξ_j and a_{ij} as above. Then for any integer n

$$\{nx_i\}\equiv\sum_{j=1}^{p} a_{ij}\{n\xi_j\} \pmod 1 \quad (1\leqq i\leqq l) \tag{17}$$

Noticing that $1, \xi_1,\dots,\xi_p$ are linearly independent over Q, we may apply Kronecker's theorem [2; Theorem 442]: *For any real numbers* $\gamma_1,\dots,\gamma_p$ *and positive* $\varepsilon_1,\dots,\varepsilon_p$, *there are infinitely many* n *such that* $|\{n\xi_j\}-\gamma_j|<\varepsilon_j$ $(1\leqq j\leqq p)$.

We put $\gamma_1=\gamma/a_{11}-p\varepsilon$, $\gamma_j=0$ $(2\leqq j\leqq p)$, $a=\max_{1\leqq i\leqq l} a_{i1}$, and $\varepsilon_j=\varepsilon$ $(1\leqq j\leqq p)$, where ε is a fixed positive number chosen sufficiently small. Then we have

$$\gamma a_{i1}/a_{11}-2ap\varepsilon<\sum_{j=1}^{p} a_{ij}\{n\xi_j\}<\gamma a_{i1}/a_{11} \quad (1\leqq i\leqq l)$$

for infinitely many n, so that by (14) we see

$$\{nx_i\}=\sum_{j=1}^{p} a_{ij}\{n\xi_j\}-[\gamma a_{i1}/a_{11}] \quad (2\leqq i\leqq l), \tag{18}$$

$$[\{nx_i\}+y_i]=[\gamma a_{i1}/a_{11}+y_i]-[\gamma a_{i1}/a_{11}] \quad (2\leqq i\leqq l),$$

and $\{nx_1\}=\sum_{j=1}^{p} a_{1j}\{n\xi_j\}$, $[\{nx_1\}+y_1]=[\gamma+y_1]-1$, taking (14), (15), and and (16) into account. Thus we have by (12)

$$\sum_{i=1}^{l} A_i\sum_{j=1}^{p} a_{ij}\{n\xi_j\}-\sum_{i=1}^{l} A_i[\gamma a_{i1}/a_{11}]-A_1=0$$

for some n. But, since $\xi_1,\dots,\xi_p$ are linearly independent over Q, (11) and (13) imply $\sum_{j=1}^{p} A_i a_{ij}=0$ $(1\leqq j\leqq p)$, and hence $\sum_{i=1}^{p} A_i\sum_{j=1}^{p} a_{ij}\{n\xi_j\}=0$. Therefore we obtain

$$\sum_{i=1}^{l} A_i[\gamma a_{i1}/a_{11}+y_i]-A_1=0 \tag{19}$$

Next we put $\gamma_1=\gamma/a_{11}+p\varepsilon$, $\gamma_j=0$ $(2\leqq j\leqq l)$ and $\varepsilon_j=\varepsilon$ $(1\leqq j\leqq l)$. In this case we find

$$\gamma a_{i1}/a_{11}<\sum_{j=1}^{p} a_{ij}\{n\xi_j\}<\gamma a_{i1}/a_{11}+2ap\varepsilon,$$

so that we have (18) and (19) again, and for $i=1$

$$\{nx_1\}=\sum_{j=1}^{p}a_{ij}\{n\xi_j\}-[\gamma], \quad [\{nx_1\}+y_1]=[\gamma+y_1]-[\gamma]$$

for infinitely many n. Therefore in the same way as above we get

$$\sum_{i=1}^{l}A_i[\gamma a_{i1}/a_{11}+y_i]=0,$$

which together with (19) yields $A_1=0$. Repeating this argument, we obtain $A_0=A_1=\cdots=A_l=0$, and the theorem is proved.

Remark. We have established a theorem on linear independence of numbers $1, f(x_1, y_1),\dots, f(x_l, y_l)$ when $x_1,\dots, x_l$ are l distinct positive rationals such that 1 is not dependent on them over Q. However, the relation (3) shows that $1, f(1, 0)=e, f(x, y)$ and $f(-x, -y)$ are linearly dependent over Q provided that x is irrational. It may be interesting to decide whether three numbers 1, e, and $\sum_{n=1}^{\infty}[nx]/n!$ with irrational x are linearly independent over Q or not.

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