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Precise informations on the poles of the scattering matrix for two strictly convex obstacles

By

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1. Introduction.

In the previous papers [3, 4] we considered the scattering matrix for two strictly convex obstacles. To say more precisely, let $\mathcal{O}_j$, $j=1, 2$, be bounded and strictly convex open sets in $\boldsymbol{R}^3$ with smooth boundary Γ_j. Suppose that

$$\bar{\mathcal{O}}_1 \cap \bar{\mathcal{O}}_2 = \emptyset.$$

Set $\mathcal{O}=\mathcal{O}_1 \cup \mathcal{O}_2$, $\Omega=\boldsymbol{R}^3-\bar{\mathcal{O}}$, $\Gamma=\Gamma_1 \cup \Gamma_2$. Consider an acoustic problem

$$(1.1)\qquad \begin{cases} \Box u=\dfrac{\partial^2 u}{\partial t^2}-\Delta u=0 & \text{in} \quad \Omega\times(-\infty, \infty) \\ u=0 & \text{on} \quad \Gamma\times(-\infty, \infty), \end{cases}$$

where $\Delta=\sum_{j=1}^{3} \dfrac{\partial^2}{\partial x_j^2}$. Denote by $\mathscr{S}(z)$ the scattering matrix for this problem. About the definition of the scattering matrix see for example Lax and Phillips [7, page 9].

We showed in [3, 4] the following facts:

(i) There exist positive constants c_0 and c_1 such that for any $\varepsilon>0$

$$\{z;\ \mathrm{Im}\, z \leqslant c_0+c_1-\varepsilon\} - \bigcup_{j=-\infty}^{\infty} B_j$$

contains only a finite number of poles of $\mathscr{S}(z)$, where

$$B_j=\{z;\ |z-z_j| \leqslant C(1+|j|)^{-1/2}\},$$

$$z_j=ic_0+\frac{\pi}{d}j, \quad d=\mathrm{dis}(\mathcal{O}_1, \mathcal{O}_2).$$

(ii) For large $|j|$, B_j contains at least one pole.

The purpose of this paper is to give very precise informations on the poles in B_j. Namely, we shall show the following

Theorem 1. *For large* $|j|$

(a) *every* B_j *contains exactly one pole of* $\mathscr{S}(z)$,

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(b) *denoting by* p_j *the pole in* B_j *we have an asymototic expansion*

$$p_j \sim z_j + \beta_1 j^{-1} + \beta_2 j^{-2} + \cdots \qquad \text{for} \quad |j| \longrightarrow \infty, \tag{1.2}$$

where $\beta_1, \beta_2, \ldots,$ *are complex constants determined by* $\mathcal{O}$,
(c) *in* B_j $\mathcal{S}(z)$ *is represented as*

$$\mathcal{S}(z)f = \frac{n_j}{z - p_j}(f, \psi_j) + \mathcal{H}_j(z)f \qquad \text{for all } f \in L^2(S^2)$$

where n_j *and* $\psi_j \in L^2(S^2)$ *such that* $n_j \neq 0$, $\psi_j \neq 0$, $(\cdot, \cdot)$ *stands for the scalar product of* $L^2(S^2)$ *and* $\mathcal{H}_j(z)$ *is an* $\mathcal{L}(L^2(S^2), L^2(S^2))$*-valued holomorphic function in* B_j.

In order to prove Theorem 1 we adopt a means to consider a boundary value problem with a complex parameter μ

$$\begin{cases} (\mu^2 - \Delta)u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \Gamma \end{cases} \tag{1.3}$$

for $g \in C^\infty(\Gamma)$. For $\mathrm{Re}\,\mu > 0$ (1.3) has a solution u uniquely in $\bigcap_{m>0} H^m(\Omega)$. Denote the solution by $U(\mu)g$. Then $U(\mu)$ is holomorphic in $\mathrm{Re}\,\mu > 0$ as $\mathcal{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$-valued function. We shall prove the following theorem on $U(\mu)$.

Theorem 2. *Set for* $k \in \boldsymbol{R} - \{0\}$

$$G_k = \{\mu \in \boldsymbol{C};\ |\mu + ik| \leqslant c_0 + c_1,\ \mathrm{Re}\,\mu \geqslant -c_0 - (\log|k|)^{-1}\}.$$

Then for large $|k|$, $U(\mu)$ *is represented in* $G_k \cap \{\mu \in \boldsymbol{C};\ \mathrm{Re}\,\mu > 0\}$ *as*

$$U(\mu) = \frac{\beta(x, \mu; k)}{\mathcal{P}(\mu) - \gamma(\mu, k)} F(\mu, k) + \tilde{U}(\mu, k). \tag{1.4}$$

Here
(i) $\beta(\cdot, \mu, k)$ *is a* $C^\infty(\bar{\Omega})$*-valued holomorphic function in* G_k,
(ii) $\mathcal{P}(\mu) = 1 - \lambda\tilde{\lambda}e^{-2d\mu}$,
where λ, $\tilde{\lambda}$ *are constants determined by* $\mathcal{O}$ *such that* $0 < \lambda, \tilde{\lambda} < 1$,
(iii) $\gamma(\mu, k)$ *is a complex valued holomorphic function in* G_k *such that*

$$|\gamma(\mu, k) - \sum_{l=1}^{N-1} (\sum_{h=0}^{2l} \gamma_{l,h}(\mu + ik)^h) k^{-l}| \leqslant C_N |k|^{-N} \tag{1.5}$$

holds for $\mu \in G_k$, *where* $\gamma_{l,h}$ *are complex constants,*
(iv) $F(\mu, k)$ *is a holomorphic* $\mathcal{L}(L^2(\Gamma), \boldsymbol{C})$*-valued function in* G_k,
(v) $\tilde{U}(\mu, k)$ *is a holomorphic* $\mathcal{L}(L^2(\Gamma), C^\infty(\bar{\Omega}))$*-valued function in* G_k.

It follows immediately from Theorem 2 that

Corollary. $U(\mu)$ *can be prolonged analytically as* $\mathcal{L}(C^\infty(\Gamma), C^\infty(\bar{\Omega}))$*-valued function into*

$$\bigcup_{|k|:large} (G_k - \{\mu;\ \mathcal{P}(\mu) - \gamma(\mu, k) = 0\}).$$

Another result on a boundary valued problem (1.3) is the following

Theorem 3. *Suppose that* $|k|$ *is large and that* $\mathscr{P}(\mu)-\gamma(\mu, k)=0$. *Then we have*

$$\dim\{u;\ \mu\text{-outgoing solution of } (\mu^2-\Delta)u=0 \text{ in } \boldsymbol{\Omega},\ u=0 \text{ on } \Gamma\}=1. \tag{1.6}$$

By recalling the relationships between the poles of $\mathscr{S}(z)$ and those of $U(\mu)$ shown in Lax and Phillips [7], we can derive easily Theorem 1 from Theorems 2 and 3. But we postpone the derivation of Theorem 1. Now we would like to give a remark on the method to prove Theorems 2 and 3. The procedure of the proofs is a slight modification of the one in [3, 4]. As in the previous papers, first we construct an asymptotic solution of

$$\begin{cases} \Box u=0 & \text{in} \quad \Omega\times \boldsymbol{R}, \\ u=e^{ik(\psi(x)-t)}f(x, t) & \text{on} \quad \Gamma\times \boldsymbol{R}, \\ \operatorname{supp} u\subset \bar{\Omega}\times\{t;\ t\geqslant 0\} \end{cases}$$

for $f\in C_0^\infty(\Gamma\times(0, 1))$. Here we require only a first order approximation of the boundary condition, that is,

$$|u(x, t)-e^{ik(\psi(x)-t)}f(x, t)|\leqslant Ce^{-c_0 t}k^{-1} \quad \text{on} \quad \Gamma\times \boldsymbol{R}.$$

This permits us to obtain an asymptotic solution $u(x, t)$ in a simpler form than in [3]. By using this simpler form of asymptotic solutions we can reduce the problem (1.3) to an integral equation on Γ_1, which is also of a simpler form. Consequently we can solve the integral equation by the Neumann series and obtain a representation (1.4) by a rearrangement of the Neumann series. This representation (1.4) is crucial for this paper.

The results of this paper and an outline of the proofs were announced in [6].

2. Remarks on the behavior of broken rays.

We generealize Lemma 3.3 and its corollary of [2] to a form containing a parameter k. Hereafter we use freely the notations and results on the broken rays of §3 of [2], and §4 of [3].

Lemma 2.1. *Let* ε *be a positive constant. For large* $k>0$ *every broken ray* $\mathscr{X}(x, \xi)$ *such that* $x\in\Gamma-S(k^{-\varepsilon})$, $\xi\in\Sigma_x^+$ *and* $\mathscr{X}(x, \xi)\cap S(k^{-\varepsilon})=\emptyset$ *satisfies*

$$^{\#}\mathscr{X}(x, \xi)\leqslant 1+C\varepsilon\log k, \tag{2.1}$$

where C *is a constant independent of* ε *and* k.

Proof. The strict convexity of $\mathcal{O}_j$, $j=1, 2$, implies

$$n(x)\cdot x'\geqslant c|x'|^2 \quad (c>0). \tag{2.2}$$

Let $x(s)$ be a representation of $\mathscr{X}(x, \xi)$ by a parameter s the length of the broken ray from x. For $x(s)\in L_j$

$$\frac{d}{ds}|x(s)'|^2=2X'_j\cdot\Xi'_j+(s-l_j)|\Xi'_j|^2,$$

which shows that $\frac{d}{ds}|x(s)'|^2$ is increasing on $(\bar{l}_j, \bar{l}_{j+1})$, and that

$$\left[\frac{d}{ds}|x(s)'|^2\right]_{s=\bar{l}_j+0}=2X'_j\cdot\Xi'_j\,.$$

Similarly we have

$$\left[\frac{d}{ds}|x(s)'|^2\right]_{s=\bar{l}_j-0}=2X'_j\cdot\Xi'_{j-1}\,.$$

Thus

$$\left[\frac{d}{ds}|x(s)'|^2\right]_{s=\bar{l}_j+0}-\left[\frac{d}{ds}|x(s)'|^2\right]_{s=\bar{l}_j-0} \tag{2.3}$$
$$=2X'_j\cdot(\Xi_j-\Xi_{j-1})=4(-n(X_j)\cdot\Xi_{j-1})n(X_j)\cdot X'_j\geqslant c|X'_j|^2.$$

First step. Suppose that $x'\cdot\xi'\geqslant 0$. On L_0, since

$$\frac{d}{ds}|x(s)'|^2\geqslant\left[\frac{d}{ds}|x(s)'|^2\right]_{s=+0}=x'\cdot\xi\geqslant 0,$$

we have

$$|X'_1|\geqslant|x'|\geqslant k^{-\varepsilon}. \tag{2.4}$$

By the monotonicity of $\frac{d}{ds}|x(s)'|^2$ and (2.3) we have on L_j, $j\geqslant 1$

$$\frac{d}{ds}|x(s)'|^2\geqslant 2c|X'_j|^2,$$

which implies that $|X'_{j+1}|^2-|X'_j|^2\geqslant 2cl_j|X'_j|^2$, namely

$$|X'_{j+1}|^2\geqslant(1+2cd)|X'_j|^2.$$

Combining this estimate with (2.4) we have

$$|X'_{j+1}|^2\geqslant(1+2cd)^j k^{-2\varepsilon}.$$

Therefore j such that $|X'_{j+1}|\leqslant$ diameter of $\mathcal{O}$ must satisfy

$$j\leqslant 2C\varepsilon\log k,$$

which shows (2.1).

Second step. Consider the case of $x'\cdot\xi'<0$. Lemma 3.3 of [2] shows that, if $\mathscr{X}(x,\xi)\cap S(k^{-\varepsilon})=\emptyset$, $|x(s)|\to\infty$ as $s\to\infty$. Then there exists j_0 such that

$$|X'_{j_0}|^2=\min_{j>0}|X'_j|^2\quad(\geqslant k^{-2\varepsilon}).$$

Note that

$$X'_{j_0-1}\cdot\Xi'_{j_0-1}\leqslant 0, \tag{2.5}$$

(2.6) $$X'_{j_0+1}\cdot \Xi_{j_0+1}>0.$$

Indeed, if (2.6) does not hold we have from (2.3)

$$\frac{d}{ds}|x(s)'|^2\leqslant -ck^{-2\varepsilon} \quad \text{on} \quad L_{j_0},$$

which implies

$$|X'_{j_0+1}|^2<|X'_{j_0}|^2.$$

This contradicts with the choice of j_0. (2.5) may be shown by the same argument. Now by using the first step we have from (2.6)

(2.7) $${}^{\#}\mathscr{X}(x,\xi)\leqslant j_0+2C\varepsilon\log k.$$

Consider a broken ray $\mathscr{X}(X_{j_0}, -\Xi_{j_0-1})$. This ray follows the reverse course of a part of $\mathscr{X}(x,\xi)$ from x to X_{j_0}. Note that $\Xi_1(X_{j_0}, -\Xi_{j_0-1})=-\Xi_{j_0-2}$, and $X'_{j_0-1}\cdot \Xi'_{j_0-1}>X'_{j_0-1}\cdot\Xi'_{j_0-2}$. Then

$$X_1(X_{j_0}, -\Xi_{j_0-1})'\cdot\Xi_1(X_{j_0}, -\Xi_{j_0-1})=X'_{j_0-1}\cdot(-\Xi_{j_0-2})>X'_{j_0-1}\cdot(-\Xi_{j_0-1})>0.$$

This implies that

$${}^{\#}\mathscr{X}(X_{j_0}, -\Xi_{j_0-1})\leqslant 1+4C\varepsilon\log k.$$

Therefore we have $j_0\leqslant 1+4C\varepsilon\log k$. Combining this with (2.7) we have (2.1).

Corollary 2.2. *If we choose $\delta>0$ sufficiently small, then for any $x\in S((1+\delta)k^{-\varepsilon})$, $\xi\in\Sigma_x^+$ such that $X_1(x,\xi)\in S((1+\delta)k^{-\varepsilon})-S(k^{-\varepsilon})$ we have*

$${}^{\#}\mathscr{X}(x,\xi)\leqslant 1+C\varepsilon\log k.$$

Proof. Suppose that

(2.8) $$\left[\frac{d}{ds}|x(s)'|^2\right]_{s=l_0-0}=X'_1\cdot\xi'\geqslant 0.$$

Then we have from (2.3) $X'_1\cdot\Xi'_1\geqslant ck^{-2\varepsilon}$. Taking account of $|X'_1|\geqslant k^{-\varepsilon}$ we have from the first step of the proof of Lemma 2.1

$${}^{\#}\mathscr{X}(x,\xi)\leqslant 1+C\varepsilon\log k.$$

When $x\in S(k^{-\varepsilon})$, $X_1\in S((1+\delta)k^{-\varepsilon})-S(k^{-\varepsilon})$, $|X'_1|^2\geqslant|x'|^2$ and the monotonicity of $\frac{d}{ds}|x(s)'|^2$ imply (2.8).

Thus the remaining case is that $\left[\frac{d}{ds}|x(s)'|^2\right]_{s=l_0-0}<0$ and $x\notin S(k^{-\varepsilon})$. By using the monotonicity of $\frac{d}{ds}|x(s)'|^2$ we have

$$\left[-\frac{d}{ds}|x(s)'|^2\right]_{s=l_0-0}\leqslant\frac{1}{d}(|x'|^2-|X'_1|^2)\leqslant\frac{2\delta}{d}k^{-2\varepsilon}.$$

From (2.3) and $|X'_1|\geqslant k^{-\varepsilon}$ we have

$$\left[\frac{d}{ds}|x(s)'|^2\right]_{s=l_0+0}=X'_1\cdot\Xi'_1\geqslant\left(c-\frac{2\delta}{d}\right)k^{-2\varepsilon}.$$

If we choose $\delta>0$ so small we have

$$X_1' \cdot \Xi_1' \geqslant \frac{c}{2} k^{-2\varepsilon}. \tag{2.9}$$

By applying the first step of the proof of Lemma 2.1 we have the assertion.

Remark. Under the assumption of Corollary, since (2.9) holds, we have $|X_2'|^2-|X_1'|^2 \geqslant 2d\frac{c}{2}k^{-2\varepsilon}$, from which it follows that

$$|X_2'| \geqslant (1+dc)^{1/2}k^{-\varepsilon} \geqslant (1+\delta)k^{-\varepsilon}$$

when δ is sufficiently small.

Corollary 2.3. *Let $x \in S((1+\delta)k^{-\varepsilon})$. Then for any broken ray $\mathscr{X}(y, \xi)$ such that $X_q(y, \xi)=x$, $y \in S((1+\delta)k^{-\varepsilon})$ we have*

$$X_j(y, \xi) \in S(k^{-\varepsilon}) \qquad \textit{for} \quad j=1, 2,\dots, q-1.$$

Proof. Suppose that $q \geqslant 2$. If $X_1 \in S((1+\delta)k^{-\varepsilon})-S(k^{-\varepsilon})$, we have $X_2 \notin S((1+\delta)k^{-\varepsilon})$ from the above remark. Thus we have $X_1 \in S(k^{-\varepsilon})$. Repeating this argument we have the assertion.

3. Construction of asymptotic solutions (I).

Hereafter we fix ε as $0<\varepsilon<1/2$. Let χ_j, $j=1, 2, 3, 4$ be real valued smooth functions defined on $\boldsymbol{R}$ such that

$$\chi_j(l)=\begin{cases} 1 & l \leqslant 1+(j-1)\delta, \\ 0 & l \geqslant 1+j\delta, \end{cases}$$

and let w_k, η_k, υ_k, θ_k be functions in $C^\infty(\Gamma_1)$ defined by

$$w_k(x(\sigma))=\chi_4(|\sigma|k^\varepsilon),$$
$$\eta_k(x(\sigma))=\chi_3(|\sigma|k^\varepsilon),$$
$$\upsilon_k(x(\sigma))=\chi_2(|\sigma|k^\varepsilon),$$
$$\theta_k(x(\sigma))=\chi_1(|\sigma|k^\varepsilon).$$

Let $h(t) \in C_0^\infty(0, d/2)$ satisfying $h(t) \geqslant 0$ and

$$\int_{\boldsymbol{R}} h(t)dt=1. \tag{3.1}$$

Let m be an oscillatory boundary data defined by

$$m(x, t; k)=e^{ik(\psi(x)-t)}f(x, t; k), \tag{3.2}$$
$$f(x, t; k)=w_k(x)h(t-j_\infty(x)),$$

where $\psi(x) \in C^\infty(S_1(\delta_0))$ satisfying Condition C of §7 of [2], and $j_\infty(x)$ is the one introduced in Lemma 5.4 of [3].

We construct an asymptotic solution for m of the problem

$$
(3.3) \qquad \begin{cases} \square u=0 & \text{in} \quad \Omega\times \boldsymbol{R} \\ u=m & \text{on} \quad \Gamma_1\times \boldsymbol{R} \\ u=0 & \text{on} \quad \Gamma_2\times \boldsymbol{R} \\ \operatorname{supp} u\subset \Omega\times\{t;\, t\geqslant 0\}\,. \end{cases}
$$

The procedure of construction is substantially same as in [3], but the treatment of the boundary condition is different.

From now on we denote $S_j(k^{-\varepsilon})$, $S_j((1+\delta)k^{-\varepsilon})$, $S(k^{-\varepsilon})$ and $S((1+\delta)k^{-\varepsilon})$ by $S_{j,k}$, $\tilde{S}_{j,k}$, S_k and $\tilde{S}_k$ respectively, and by $\omega(\delta)$ a domain surrounded by $S_j(\delta)$, $j=1$, 2 and a cylinder $\{x;\, \operatorname{dis}(x, L)=\delta\}$. First fix a large integer N and construct $u_q(x, t;\, k)$, $q=0$, 1, 2,... in the form

$$
u_q(x, t;\, k)=e^{ik(\varphi_q(x)-t)}v_q(x, t;\, k)\,,
$$

$$
v_q(x, t;\, k)=\sum_{j=0}^{N} v_{j,q}(x, t;\, k)(ik)^{-j}.
$$

Since ψ satisfies Condition C of [2] we can construct successively a sequence of phase functions φ_0, φ_1, φ_2,... following the process in pages 136 and 137 of [3]. Note that we have the following

Lemma 3.1. *It holds that*

$$
(3.4) \qquad |\varphi_{2p}(\cdot)-(\varphi_\infty(\cdot)+2pd+d_0)|_m(\omega(\delta_0))\leqslant C_m\alpha^{2p} \qquad (0<\alpha<1)\,,
$$

$$
(3.5) \qquad |\varphi_{2p+1}(\cdot)-(\tilde{\varphi}_\infty(\cdot)+(2p+1)d+d_0)|_m(\omega(\delta_0))\leqslant C_m\alpha^{2p},
$$

where φ_∞ are $\tilde{\varphi}_\infty$ are functions independent of ψ and d_0 is a constant depending smoothly on ψ.

Proof. Recall estimates (7.9) and (7.10) of [3], and remark that we have $\tilde{d}_0=d_0+d$ from their definition. Since $|\nabla\varphi_{2p}|=1$, $|\nabla\varphi_\infty|=1$ in $\omega(\delta_0)$ and $\frac{\partial\varphi_{2p}}{\partial n}>0$, $\frac{\partial\varphi_\infty}{\partial n}>0$ on $S_1(\delta_0)$, and estimate (7.9) on $S_1(\delta_0)$ implies (3.4). We have (3.5) from (7.10) of [3]. Q.E.D.

Following [3] we set

$$
T_q=2\frac{\partial}{\partial t}+2\nabla\varphi_q\cdot\nabla+\Delta\varphi_q\,.
$$

We define $v_{0,q}$, $q=0$, 1,... as follows:

$$
(3.6) \qquad T_q v_{0,q}=0 \quad \text{in} \quad \omega\times\boldsymbol{R},
$$

and

$$
(3.7) \qquad v_{0,0}(x, t;\, k)=f(x, t;\, k) \quad \text{on} \quad S_1(\delta_0)\times\boldsymbol{R},
$$

for $p \geqslant 1$

$$(3.8)\qquad \begin{cases} v_{0,2p-1}(x, t; k) = v_{0,2p-2}(x, t; k) & \text{on} \quad S_2(\delta_0) \times \boldsymbol{R}, \\ v_{0,2p}(x, t; k) = \theta_k(x) v_{0,2p-1}(x, t; k) & \text{on} \quad S_1(\delta_0) \times \boldsymbol{R}. \end{cases}$$

For $j \geqslant 1$ we define $\{v_{j,q}\}_{q=0}^{\infty}$ successively by

$$(3.9)\qquad T_q v_{j,q} = \Box v_{j-1,q} \qquad \text{in} \quad \omega \times \boldsymbol{R} \quad \text{for all} \quad q,$$

$$(3.10)\qquad v_{j,p}(x, t; k) = 0 \qquad \text{on} \quad S_1(\delta_0) \times \boldsymbol{R},$$

$$(3.11)\qquad v_{j,2p+1}(x, t; k) = v_{j,2p}(x, t; k) \quad \text{on} \quad S_2(\delta_0) \times \boldsymbol{R}.$$

In Section 3 of [3] a function $j_\infty(x)$ on $S_1(\delta_0) \cup S_2(\delta_0)$ was introduced. Now we extend it to $j(x)$ and $\tilde{j}(x)$ by the following two ways:

$$(3.12)\qquad j(x) = j_\infty(y) + l \qquad \text{for} \quad x = y + l\nabla\varphi_\infty(y), \quad y \in S_1(\delta_0),$$

$$(3.13)\qquad \tilde{j}(x) = j_\infty(y) + l \qquad \text{for} \quad x = z + l\nabla\tilde{\varphi}_\infty(z), \quad z \in S_2(\delta_0).$$

Recalling the proof of Lemma 5.3 of [3] we have

$$j(x) = \tilde{j}(x) + d \quad \text{on} \quad S_2(\delta_0),$$

$$\tilde{j}(x) = j(x) + d \quad \text{on} \quad S_1(\delta_0),$$

and Remark 3 of [3] can be written as

$$(3.14)\qquad \tilde{j}(x) = h_\infty(x) + j(X_{-1}^{\infty}(x)) - d \qquad \text{for} \quad x \in S_2(\delta_0),$$

$$(3.15)\qquad j(x) = h_\infty(x) + \tilde{j}(X_{-1}^{\infty}(x)) - d \qquad \text{for} \quad x \in S_1(\delta_0).$$

We extend $a(x)$ and $\tilde{a}(x)$, which are defined in [3] as functions on $S_1(\delta_0)$ and $S_2(\delta_0)$ respectively, to functions in $\omega(\delta_0)$ by

$$a(x) = [G_{\varphi_\infty}(y + l\nabla\varphi_\infty(y))/G_{\varphi_\infty}(y)]^{1/2} a(y),$$

$$\tilde{a}(x) = [G_{\tilde{\varphi}_\infty}(z + l\nabla\tilde{\varphi}_\infty(z))/G_{\tilde{\varphi}_\infty}(z)]^{1/2} a(z),$$

where y and z are linked to x by the relations in (3.12) and (3.13).

Lemma 3.2. *Set*

$$v_{0,\infty}(x, t; k) = v_k(y) a(x) h(t - j(x)),$$

$$\tilde{v}_{0,\infty}(x, t; k) = \lambda v_k(z) \tilde{a}(x) h(t - \tilde{j}(x)),$$

where y and z are linked to x by relations in (3.12) *and* (3.13) *respectively. Putting*

$$g_{2p+1}(x, t; k) = v_{0,2p+1}(x, t; k) - b w_k(A_0)(\lambda\tilde{\lambda})^p v_{0,\infty}(x, t - 2dp - j(A_0) - d_\infty; k),$$

$$g_{2p+2}(x, t; k) = v_{0,2p+2}(x, t; k) - b w_k(A_0)(\lambda\tilde{\lambda})^p \tilde{v}_{0,\infty}(x, t - (2p+1)d - j(A_0) - d_\infty; k),$$

where λ, $\tilde{\lambda}$, b, A_0 *are the ones in Proposition* 5.6 *of* [3], *and* d_∞ *denotes* $d_{\infty,0}$, *we have*

(3.16) $$|g_q|_m(\omega(\delta_2))\times \boldsymbol{R})\leqslant C_m q(\lambda\tilde{\lambda}\alpha)^{q/2}M_m,$$

where α *is the one in Proposition* 5.6 *of* [3] *and*

$$M_m=|f|_m(S_1(\delta_0)\times \boldsymbol{R}).$$

Proof. Let $x\in\tilde{S}_k$. If $X_{-q-1}(x, \nabla\varphi_q)\notin\tilde{S}_{1,k}$, we have from the consideration in Lemma 5.3 of [3]

$$v_{0,q}=0.$$

When $X_{-q-1}(x, \nabla\varphi_q)\in\tilde{S}_{1,k}$, by applying Corollary 2.3 we have $X_{-j}(x, \nabla\varphi_q)\in S_k$ for $1\geqslant j\geqslant q$, which implies that $v_k(X_{-j}(x, \nabla\varphi_q))=1$ for $1\leqslant j\leqslant q$ when $X_{-j}\in\Gamma_1$. Therefore the representation (5.9) in [3] is also valid. Thus we have from the proof of Proposition 5.6 of [3]

(3.17) $$|v_{0,2p}-(\lambda\tilde{\lambda})^p w(A_0)a(x)bh(t-j(x)-2pd-j(A_0)-d_\infty)|_m(\tilde{S}_{2,k}\times \boldsymbol{R})$$
$$\leqslant C_m(\lambda\tilde{\lambda}\alpha)^p M_m,$$

(3.18) $$|v_{0,2p+1}-\lambda(\lambda\tilde{\lambda})^p w(A_0)\tilde{a}(x)bh(t-\tilde{j}(x)-(2p+1)d-j(A_0)-d_\infty)|_m(\tilde{S}_{1,k}\times \boldsymbol{R})$$
$$\leqslant C_m(\lambda\tilde{\lambda}\alpha)^p M_m.$$

Let $x\in\omega(\delta_2)$ and $q=2p$. Denote by y the point in $S_1(\delta_2)$ such that

$$x=y+l\nabla\varphi_{2p}(y).$$

Then

$$v_{0,2p}(x, t; k)=[G_{\varphi_{2p}}(x)/G_{\varphi_{2p}}(y)]^{1/2}v_k(y)v_{0,2p}(y, t-|x-y|; k).$$

By combining Lemma 3.1 and (3.17) we have the assertion for $q=2p$. For $q=2p+1$ a proof is done by the same way. Q.E.D.

Remark 3.1. Since $a(x)$, $\tilde{a}(x)$, $j(x)$ are determined only by $\mathcal{O}$, $v_{0,\infty}$, $\tilde{v}_{0,\infty}$ are independent of ψ and w_k.

Remark 3.2. Set

$$T_\infty=2\frac{\partial}{\partial t}+2\nabla\varphi_\infty\cdot\nabla+\Delta\varphi_\infty,$$

$$\tilde{T}_\infty=2\frac{\partial}{\partial t}+2\nabla\tilde{\varphi}_\infty\cdot\nabla+\Delta\tilde{\varphi}_\infty.$$

Then we have

(3.19) $$T_\infty v_{0,\infty}(x, t; k)=0 \quad\text{in}\quad \omega(\delta_0)\times\boldsymbol{R},$$

(3.20) $$\tilde{T}_\infty\tilde{v}_{0,\infty}(x, t; k)=0 \quad\text{in}\quad \omega(\delta_0)\times\boldsymbol{R},$$

and

$$v_{0,\infty}(x, t; k)=\tilde{v}_{0,\infty}(x, t-d; k) \quad \text{on} \quad S_{2,k}\times \boldsymbol{R}, \tag{3.21}$$

$$\tilde{v}_{0,\infty}(x, t; k)=\lambda\tilde{\lambda}v_{0,\infty}(x, t-d; k) \quad \text{on} \quad S_{1,k}\times \boldsymbol{R}. \tag{3.22}$$

Though these are obvious from the process of the definitions of a, $\tilde{a}$ and j, $\tilde{j}$, we give some explanation. It is evident from the formula (5.2) of [3] and the way of extention of $a(x)$ and $j(x)$ that $v_{0,\infty}$ satisfies

$$\begin{cases} T_\infty v=0 & \text{in} \quad \omega(\delta_0)\times \boldsymbol{R}, \\ v=v_k(x)a(x)h(t-j(x)) & \text{on} \quad S_1(\delta_0)\times \boldsymbol{R}. \end{cases}$$

Then by formula (5.3) of [3] we have for $x\in S_2(\delta_0)$

$$v_{0,\infty}(x, t; k)=\Lambda_\infty(x)a(X_{-1}^\infty(x))h((t-h_\infty(x))-j(X_{-1}(x))$$

by Remark 2 of page 156 of [3] and (3.14)

$$=\lambda\tilde{a}(x)h(t-\tilde{j}(x)-d)=\tilde{v}_{0,\infty}(x, t-d; k).$$

Let $v_{j,\infty}$ and $\tilde{v}_{j,\infty}$ be functions satisfying

$$\begin{cases} T_\infty v_{j,\infty}=\Box v_{j-1,\infty} & \text{in} \quad \omega(\delta_2)\times \boldsymbol{R}, \\ v_{j,\infty}=0 & \text{on} \quad S_1(\delta_2)\times \boldsymbol{R}, \end{cases} \tag{3.23}$$

$$\begin{cases} \tilde{T}_\infty \tilde{v}_{j,\infty}=\Box \tilde{v}_{j-1}, & \text{in} \quad \omega(\delta_2)\times \boldsymbol{R}, \\ \tilde{v}_{j,\infty}=v_{j,\infty} & \text{on} \quad S_2(\delta_2)\times \boldsymbol{R}. \end{cases} \tag{3.24}$$

Lemma 3.3. *For $j\geqslant 1$, we have*

$$v_{j,\infty}(x, t; k)=\sum_{l=0}^{j} a_{j,l}(x; k)h^{(l)}(t-j(x)), \tag{3.25}$$

$$\tilde{v}_{j,\infty}(x, t; k)=\sum_{l=0}^{j} \tilde{a}_{j,l}(x; k)h^{(l)}(t-\tilde{j}(x)), \tag{3.26}$$

where $a_{j,l}$ and $\tilde{a}_{j,l}$ are functions independent of ψ. Especially on $S_{1,k}$ we have

$$\tilde{v}_{j,\infty}(x, t; k)=\sum_{l=0}^{j} a_{j,l}^0(x)h^{(l)}(t-j(x)) \tag{3.27}$$

where $a_{j,l}^0(x)$ is a function independent of k.

Proof. First consider the case of $j=1$. Note that $|\nabla j(x)|=1$. Then

$$\Box v_{0,\infty}(x, t; k)=-h'(t-j(x))\{2\nabla j\cdot\nabla(v_k a)+\Delta j\cdot(v_k a)\} -h(t-j(x))\Delta(v_k a).$$

Putting

$$v(s)=v_{1,\infty}(y+s\nabla\varphi_\infty(y), t+s; k), \quad y\in S_1(\delta_2).$$

From the definition of $j(x)$ we have $j(y+s\nabla\varphi_\infty(y))=j(y)+s$. Then it follows that

$$2\frac{d}{ds}v(s)+(\Delta\varphi_\infty)(y+s\nabla\varphi_\infty(y))v(s) \tag{3.28}$$

$$=(T_\infty v_{1,\infty})(y+s\nabla\varphi_\infty(y),\ t+s)$$
$$=-h'(t-j(y))b_1(y+s\nabla\varphi_\infty(y))-h(t-j(y))b_0(y+s\nabla\varphi_\infty(y))$$

where $b_1(x)=(2\nabla j\cdot\nabla(v_k a)+\Delta j\cdot v_k a)(x)$, $b_0(x)=\Delta(v_k a)(x)$. By the integration of (3.28) we have

$$v(s)=-\sum_{m=0}^{1} h^{(m)}(t-j(y))\int_0^s b_m(y+l\nabla\varphi_\infty(y))\left[\frac{G_{\varphi_\infty}(y+s\nabla\varphi_\infty(y))}{G_{\varphi_\infty}(y+l\nabla\varphi_\infty(y))}\right]^{1/2} dl$$
$$=\sum_{m=0}^{1} h^{(m)}(t+s-j(y+s\nabla\varphi_\infty(y))a_{1,m}(y+s\nabla\varphi_\infty(y))\,.$$

Indeed,

$$v_{1,\infty}(x,\ t)=\sum_{m=0}^{1} h^{(m)}(t-j(x))a_{1,m}(x;\ k)\,.$$

Thus (3.25) is proved for $j=1$. Repeating this argument we have (3.25) for $j\geqslant 2$. For (3.26) the proof is done by the same way.

Since $y=X^{\infty}_{-2}(x)\in S_{1,k}$ for $x\in S_{1,k}$, it follows that $v_k(x)=1$ near y. Then in (3.28) we may regard $v_k=1$ for all s namely

$$\frac{d}{ds}v(s)=-h'(t-j(y))\{2\nabla j\cdot\nabla a+\Delta j\cdot a\}_{x=y+s\nabla\varphi_\infty(y)}$$
$$-h(t-j(y))(\Delta a)(y+s\nabla\varphi_\infty(y))\,,$$

from which we have (3.27). Q. E. D.

By using a representation formula (6.6) of [3] for solutions of the transport equations and Lemmas 3.1 and 3.2 we have the following lemma by induction in j.

Lemma 3.4. *For $j\geqslant 1$, it holds that*

$$|bw_k(A_0)(\lambda\tilde\lambda)^p v_{j,\infty}(x,t-2pd-j(A_0)-d_\infty;\ k)-v_{j,2p}(x,\ t;\ k)|_m(\omega(\delta_2)\times \boldsymbol{R}) \leqslant C_{m,j}p(\lambda\tilde\lambda\alpha)^p M_{m+2j}\,, \tag{3.29}$$

$$|bw_k(A_0)(\lambda\tilde\lambda)^p \tilde v_{j,\infty}(x,t-(2p+1)d-j(A_0)-d_\infty;k)-v_{j,2p+1}(x,t;k)|_m(\omega(\delta_2)\times \boldsymbol{R}) \leqslant C_{m,j}p(\lambda\tilde\lambda\alpha)^p M_{m+2j}\,. \tag{3.30}$$

Since the transport equations (3.6) and (3.9) are satisfied we have for all q

$$\Box u_q(x,\ t;\ k)=e^{ik(\varphi_q(x)-t)}(ik)^{-N}\Box v_{N,q}\,. \tag{3.31}$$

Similarly, by setting

$$u_\infty(x,\ t;\ k)=e^{ik(\varphi_\infty(x)-t)}\sum_{j=0}^{N} v_{j,\infty}(x,\ t;\ k)(ik)^{-j},$$
$$\tilde u_\infty(x,\ t;\ k)=e^{ik(\tilde\varphi_\infty(x)-t)}\sum_{j=0}^{N} \tilde v_{j,\infty}(x,\ t;\ k)(ik)^{-j},$$

we have

(3.32) $$\square u_\infty(x, t; k) = e^{ik(\varphi_\infty(x)-t)}(ik)^{-N}\square v_{N,\infty},$$

(3.33) $$\square \tilde{u}_\infty(x, t; k) = e^{ik(\tilde{\varphi}_\infty(x)-t)}(ik)^{-N}\square \tilde{v}_{N,\infty}.$$

Now combining Lemmas 3.1, 3.2 and .34 we have

Lemma 3.5. *It holds that*

(3.34) $$|e^{ik(d_0-j(A_0)-d_\infty)}bw_k(A_0)(\lambda\tilde{\lambda})^p u_\infty(x, t-2pd-j(A_0)-d_\infty; k) \\ -u_{2p}(x, t; k)|_m(\omega(\delta_2)\times \boldsymbol{R}) \leqslant C_m k^m(\lambda\tilde{\lambda}\alpha)^p \sum_{j=0}^{N} M_{m+2j}k^{-j},$$

(3.35) $$|e^{ik(d_0-j(A_0)-d_\infty)}bw_k(A_0)(\lambda\tilde{\lambda})^p \tilde{u}_\infty(x, t-(2p+1)d-j(A_0)-d_\infty; k) \\ -u_{2p+1}(x, t; k)|_m(\omega(\delta_2)\times \boldsymbol{R}) \leqslant C_m k^m(\lambda\tilde{\lambda}\alpha)^p \sum_{j=0}^{N} M_{m+2j}k^{-j},$$

(3.36) $$|e^{ik(d_0-j(A_0)-d_\infty)}bw_k(A_0)(\lambda\tilde{\lambda})^p \square u_\infty(x, t-2pd-j(A_0)-d_\infty; k) \\ -\square u_{2p}(x, t; k)|_m(\omega(\delta_2)\times \boldsymbol{R}) \leqslant C_m k^{-N+m+1}(\lambda\tilde{\lambda}\alpha)^p M_{2N+m},$$

(3.37) $$|e^{ik(d_0-j(A_0)-d_\infty)}bw_k(A_0)(\lambda\tilde{\lambda})^p \square \tilde{u}_\infty(x, t-(2p+1)d-j(A_0)-d_\infty; k) \\ -\square u_{2p+1}(x, t; k)|_m(\omega(\delta_2)\times \boldsymbol{R}) \leqslant C_m k^{-N+m+1}(\lambda\tilde{\lambda}\alpha)^p M_{2N+m}.$$

Note that φ_q can be extended into a neighborhood in $\boldsymbol{R}^3$ of $\omega(\delta_0)$ verifying $|\nabla\varphi_q| =1$. Denote one of such neighborhoods by $\tilde{\omega}$. Then $v_{j,q}$ are also extended into $\tilde{\omega}$ verifying the transport equations. Similarly we extend φ_∞, $\tilde{\varphi}_\infty$, $v_{j,\infty}$, $\tilde{v}_{j,\infty}$ by the same way. Thus we may suppose that the relations and estimates (3.29)∼(3.37) hold in $\tilde{\omega}$.

Let $\chi(x)\in C_0^\infty(\boldsymbol{R}^3)$ such that its support is contained in $\tilde{\omega}$ and $\chi=1$ on $\omega(\delta_0)$. Evidently we have from (3.36) replaced $\omega(\delta_2)$ by $\tilde{\omega}$

(3.38) $$|e^{ik(d_0-j(A_0)-d_\infty)}bw_k(A_0)(\lambda\tilde{\lambda})^p \chi(x)\square u_\infty(x, t-2pd-j(A_0)-d_\infty; k) \\ -\chi(x)\square u_{2p}(x, t; k)|_m(\boldsymbol{R}^3\times \boldsymbol{R}) \leqslant C_m k^{-N+m+1}(\lambda\tilde{\lambda}\alpha)^p M_{2N+m}.$$

Denote by $u'_{2p}(x, t; k)$ the solution of

$$\begin{cases} \square w = -\chi(x)\square u_{2p} & \text{in} \quad \boldsymbol{R}^3\times \boldsymbol{R} \\ \operatorname{supp} w \subset \boldsymbol{R}^3\times\{t;\ t\geqslant 0\}, \end{cases}$$

and by $u'_\infty(x, t; k)$ the solution of

$$\begin{cases} \square w = -\chi(x)\square u_\infty & \text{in} \quad \boldsymbol{R}^3\times \boldsymbol{R} \\ \operatorname{supp} w \subset \boldsymbol{R}^3\times\{t;\ t\geqslant 0\}. \end{cases}$$

Since $\operatorname{supp}\chi\square u_{2p}\subset\tilde{\omega}\times(2pd-R_0, 2p+R_0)$, we have from the Huygens principle

(3.39) $$\operatorname{supp} u'_{2p}\subset\{(x, t);\ t-2pd-R_0\leqslant|x|\leqslant t+2pd+R_0,\ t\geqslant 2pd-R_0\}.$$

From (3.31) it follows

$$|u'_{2p}|_m(\boldsymbol{R}^3 \times \boldsymbol{R}) \leqslant C_m k^{-N+m+3}(\lambda\tilde{\lambda}\alpha)^p M_{2N+m}. \tag{3.40}$$

From (3.38) we have

$$|e^{ik(d_0-j(A_0)-d_\infty)} b w_k(A_0)(\lambda\tilde{\lambda})^p u'_\infty(x,\, t-2pd-j(A_0)-d_\infty;\, k) \\ -u'_{2p}(x,\, t;\, k)|_m(\boldsymbol{R}^3 \times \boldsymbol{R}) \leqslant C_m k^{-N+m+3}(\lambda\tilde{\lambda}\alpha)^p M_{2N+m}. \tag{3.41}$$

Evidently we have the same type estimate for $q=2p+1$, namely

$$|e^{ik(d_0-j(A_0)-d_\infty)} b w_k(A_0)(\lambda\tilde{\lambda})^p \tilde{u}'_\infty(x,\, t-(2p+1)d-j(A_0)-d_\infty;\, k) \\ -u'_{2p+1}(x,\, t;\, k)|_m(\boldsymbol{R}^3 \times \boldsymbol{R}) \leqslant C_m k^{-N+m+3}(\lambda\tilde{\lambda}\alpha)^p M_{2N+m}. \tag{3.42}$$

4. Construction of asymptotic solutions (II).

Let m be an oscillatory boundary data of the form

$$m(x,\, t;\, k)=e^{ik(\psi(x)-t)} f(x,\, t;\, k), \quad f \in C_0^\infty(\Gamma_1 \times (T,\, T+d/2)) \tag{4.1}$$

where $\psi \in C^\infty(\Gamma_1)$ is a function satisfying Condition C of or $\theta(x,\, \eta,\, \beta)$ of Lemma 7.1 of [2].

Lemma 4.1. *For a positive integer N we have a function $u(x,\, t;\, k)=u'(x,\, t;\, k)+u''(x,\, t;\, k)$ satisfying*

$$\Box u=0 \quad in \quad \Omega \times \boldsymbol{R}, \tag{4.2}$$

$$\operatorname{supp} u' \subset \bigcup_{(x,t) \in \operatorname{supp} f} \mathscr{L}(x,\, t;\, \nabla\varphi) \tag{4.3}$$

$$|u'|_m(\Omega_R,\, t) \leqslant C_{m,R} e^{-c_0 t} |\nabla\psi|_{m+N'} k^{m+1} \\ \times \sum_{j=0}^{N} k^{-j}(t-T)^j |e^{c_0 T} f(\cdot\,,\cdot)|_{m+2j}(\Gamma_1 \times \boldsymbol{R}), \tag{4.4}$$

$$|u''|_m(\Omega,\, t) \leqslant C_m e^{-c_0 t} |\nabla\psi|_{m+N'} k^{-N+m+1} \\ \times (t-T)^N |e^{c_0 T} f(\cdot\,,\cdot)|_{N'+m}(\Gamma_1 \times \boldsymbol{R}), \tag{4.5}$$

$$|u-m|_m(\Gamma,\, t) \leqslant C_m e^{-c_0 t} |\nabla\psi|_{m+N'} k^{-N+m} \\ \times (t-T)^N |e^{c_0 T} f(\cdot\,,\cdot)|_{N'+m}(\Gamma_1 \times \boldsymbol{R}). \tag{4.6}$$

Proof. We follow the process of the proof of Proposition 8.1 of [2] except an argument on estimations of the amplitude function of $w^{(N)}$ in §8. Namely, instead of the estimate in §8 we use a precise asymptotic formula proved in sections 5 and 6 of [3].

Corollary 4.2. *Suppose that m of* (4.1) *verifies*

$$^{\#}\mathscr{X}(x,\, \nabla\varphi(x)) \leqslant \log k \qquad for\ all \quad x \in \operatorname{Proj}_x \operatorname{supp} f, \tag{4.7}$$

where $\varphi(x)$ denotes the one in the definition of Condition C or $\theta+\frac{2}{3}\rho^{3/2}$. Then we *have a function* $u(x, t; k)$ *satisfying* (4.2), (4.3) *and*

$$|u'|_m(\Omega_R, t) \leqslant C_{m,R} e^{-c_0 t} |\nabla \varphi|_{m+N'} k^{m+1} \times \sum_{j=0}^{N} (k^{-1}(\log k + R))^j |e^{c_0 T} f|_m(\Gamma_1 \times \boldsymbol{R}), \tag{4.8}$$

$$|u''|_m(\Omega, t) \leqslant C_m e^{-c_0 t} |\nabla \varphi|_{m+N'} (k^{-1}(\log k + R))^{-N+m} \times |e^{c_0 T} f|_{N'+m}(\Gamma_1 \times \boldsymbol{R}), \tag{4.9}$$

$$|u - m|_m(\Gamma, t) \leqslant C_m e^{-c_0 t} |\nabla \varphi|_{m+N'} (k^{-1}(\log k + R))^{-N+m} \times |e^{c_0 T} f|_{N'+m}(\Gamma_1 \times \boldsymbol{R}). \tag{4.10}$$

Moreover

$$\operatorname{supp} u|_{\Gamma \times \boldsymbol{R}} \subset \Gamma \times [T, T + d \log k + 2\rho_0], \quad (\rho_0 = \textit{diameter of } \mathcal{O}). \tag{4.11}$$

Proof. We have from (4.3) and (4.7)

$$\operatorname{supp} u' \subset \{(x, t);\ t \leqslant T + d \log k + \rho_0 + |x|\},$$

which implies

$$t - T \leqslant d \log k + R + \rho_0 \quad \text{on} \quad \operatorname{supp} u' \cap (\Omega_R \times \boldsymbol{R}).$$

Thus (4.9) follows from (4.4). Recalling the process of the construction of u'', which corresponds to u' in the previous section, we have (4.9) and (4.10). Q. E. D.

Set

$$m_p(x, t; k) = e^{ik(\varphi_{2p}(x) - t)} f_p(x, t; k), \tag{4.12}$$

$$f_p(x, t; k) = (1 - \theta_k(x)) \sum_{j=0}^{N} v_{j,2p-1}(x, t; k)(ik)^{-j},$$

$$m_\infty(x, t; k) = e^{ik(\varphi_\infty(x) - t)} f_\infty(x, t; k), \tag{4.13}$$

$$f_\infty(x, t; k) = (1 - \theta_k(x)) \sum_{j=0}^{N} v_{j,\infty}(x, t; k)(ik)^{-j}.$$

Then (3.17) and (3.29) imply that

$$|b w_k(A_0)(\lambda\tilde{\lambda})^p f_\infty(x, t - 2pd - j(A_0) - d_\infty; k) e^{ik(d_\infty - j(A_0) - d_0)} - f_p(x, t; k)|_m(\Gamma_1 \times \boldsymbol{R}) \leqslant C_{m,N} p^N (\lambda\tilde{\lambda}\alpha)^p \sum_{j=0}^{N} k^{-j} M_{2j+m}. \tag{4.14}$$

Note that we have from Corollary 2.2

$$^{\#}\mathscr{X}(x, \nabla \varphi_{2p}(x)),\ {}^{\#}\mathscr{X}(x, \nabla \varphi_\infty(x)) \leqslant \log k \qquad \text{for all} \quad x \in \operatorname{supp}(1 - \theta_k). \tag{4.15}$$

By applying Corollary 4.2 to m_p and m_∞, and we get z_p and z_∞ verifying (4.2), (4.8) $\sim$(4.11), where $T=0$ for z_∞ and $T=2pd$ for z_p. Remark that since $e^{-c_0 2d} = \lambda\tilde{\lambda}$ it holds that

$$|e^{2pdc_0}f_p|_m(\Gamma_1\times \boldsymbol{R})\leqslant C_m M_m \qquad \text{for all } p.$$

Since the process of the proof of Lemma 4.1 indicates the continuity of a correspondance from $\{\psi, f\}$ to u, we have from (4.14) and Lemma 3.1 the following

Lemma 4.3. *It holds that*

$$(4.16)\quad |bw_k(A_0)e^{ik(d_0-j(A_0)-d_\infty)}(\lambda\tilde{\lambda})^p z_\infty(x, t-2pd-j(A_0)-d_\infty; k)-z_p(x, t; k)|_m(\Omega_R, t)$$
$$\leqslant C_{m,R}(\lambda\tilde{\lambda}\alpha)^p e^{-c_0(t-2pd)}k^{m+1}\sum_{j=0}^{N} k^{-j}(\log k)^j M_{2j+m},$$

$$(4.17)\qquad \square z_p=0 \quad in \quad \Omega\times \boldsymbol{R}$$

$$(4.18)\qquad \operatorname{supp} z_p|_{\Gamma\times\boldsymbol{R}}\subset\Gamma\times[2pd,\ 2pd+\ d\log k+\rho_0].$$

Set

$$(4.19)\qquad r_p(x, t; k)=u_{2p}(x, t; k)+u'_{2p}(x, t; k)-u_{2p+1}(x, t; k)$$
$$-u'_{2p+1}(x, t; k)-z_p(x, t; k),$$

$$(4.20)\qquad r(x, t; k)=\sum_{p=0}^{\infty} r_p(x, t; k)$$

and

$$(4.21)\qquad r_\infty(x, t; k)=e^{ik(d_0-j(A_0)-d_\infty)}\{u_\infty(x, t; k)-u'_\infty(x, t; k)$$
$$-(\tilde{u}_\infty(x, t-d; k)-\tilde{u}'_\infty(x, t-d; k))-z_\infty(x, t; k)\}.$$

We have from (3.34), (3.35), (3.41), (3.42) and (4.16)

Lemma 4.4. *It holds that*

$$(4.22)\qquad |bw_k(A_0)(\lambda\tilde{\lambda})^p r_\infty(x, t-2dp-j(A_0)-d_\infty; k)-r_p(x, t; k)|_m(\Omega_R\times\boldsymbol{R})$$
$$\leqslant C_{m,R}(\lambda\tilde{\lambda}\alpha)^p k^{m+1}\sum_{j=0}^{N} k^{-j}M_{2j+m}.$$

Next we consider the behavior of $r(x, t; k)$ on the boundary. Taking account of (3.8) we have on Γ_1

$$(4.23)\qquad r(x, t; k)-m(x, t; k)=\sum_{p=0}^{\infty}\{e^{ik(\varphi_{2p}-t)}\theta_k(x)\sum_{j=1}^{N} v_{j,2p+1}(ik)^{-j}$$
$$+(m_p(x, t; k)-z_p(x, t; k))-u'_{2p}(x, t; k)+u'_{2p+1}(x, t; k)\}.$$

From (3.8), (3.11) we have on Γ_2

$$(4.24)\qquad r(x, t; k)=\sum_{p=0}^{\infty}\{-u'_{2p}(x, t; k)+u'_{2p+1}(x, t; k)-z_p(x, t; k)\}.$$

Summing up the argument up to now we have

Proposition 4.5. *For an oscillatory boundary data* (3.2) *we have a function* $r(x, t; k)$ *defined by* (4.20) *verifying*

(4.25) $$\square r(x, t; k)=0 \quad in \quad \Omega\times \boldsymbol{R}$$

and (4.22), (4.23) *and* (4.24).

5. Laplace transform of asymptotic solutions.

We consider the Laplace transform of $r(x, t; k)$ with respect to t, that is,

(5.1) $$\hat{r}(x, \mu; k)=\int_{-\infty}^{\infty} e^{-\mu t} r(x, t; k)dt.$$

First we restrict μ in $\{\mu; \operatorname{Re}\mu>0\}$. Evidently the integral of the right hand side of (5.1) converges absolutely. Therefore $\hat{r}(x, \mu; k)$ is an $H^{\infty}(\Omega)$-valued holomorphic function. It follows from (4.25) that

(5.2) $$(\mu^2-\Delta)\hat{r}(x, \mu; k)=0 \quad \text{in} \quad \Omega.$$

Set

$$s(x, t; k)=r(x, t; k)-bw_k(A_0)\sum_{p=0}^{\infty}(\lambda\tilde{\lambda})^p r_{\infty}(x, t-2pd-j(A_0)-d_{\infty}; k)$$

and we have from (3.39) and (4.22)

(5.3) $$|s|_m(\Omega_R, t)\leqslant C_{m,R}(\log k)e^{-(c_0+c_1)(t-R-C\varepsilon\log k)}k^{m+1}\sum_{j=0}^{N} k^{-j}M_{2j+m}.$$

Thus it follows

(5.4) $$|\hat{s}(x, \mu; k)|_m(\Omega_R)\leqslant C_{m,R}e^{(c_0+c_1)(C\varepsilon\log k+R)}k^{m+1}\sum_{j=0}^{N} k^{-j}M_{2j+m}$$

$$\text{for} \quad \mu\in\mathscr{D}_{\varepsilon'}=\{\mu; \operatorname{Re}\mu\geqslant -c_0-c_1+\varepsilon'\}.$$

On the other hand

$$\int e^{-\mu t}\sum_{p=0}^{\infty}(\lambda\tilde{\lambda})^p r_{\infty}(x, t-2pd-j(A_0)-d_{\infty}; k)dt$$
$$=\sum_{p=0}^{\infty}(\lambda\tilde{\lambda})^p e^{-\mu(2pd+j(A_0)+d_{\infty})}\hat{r}_{\infty}(x, \mu; k)$$
$$=\mathscr{P}(\mu)^{-1}e^{-\mu(j(A_0)+d_{\infty})}\hat{r}_{\infty}(x, \mu; k),$$

where

$$\mathscr{P}(\mu)=1-\lambda\tilde{\lambda}e^{-2d\mu}.$$

Lemma 5.1. *The Laplace transform of* $r(x, t; k)$ *is of the form*

(5.5) $$\hat{r}(x, \mu; k)=bw_k(A_0)\mathscr{P}(\mu)^{-1}e^{-\mu(j(A_0)+d_{\infty})}\hat{r}_{\infty}(x, \mu; k)+\hat{s}(x, \mu; k)$$

where $\hat{r}_{\infty}(x, \mu; k)$ *is a* $C^{\infty}(\bar{\Omega})$*-valued entire function and* $\hat{s}(x, \mu; k)$ *is* $C^{\infty}(\bar{\Omega})$*-*

valued holomorphic function in $\mathscr{D}=\{\mu;\ \mathrm{Re}\,\mu>-c_0-c_1\}$ *verifying an estimates* (5.4) *for any* $R>0$ *and* $\varepsilon'>0$.

Proof. Beside the fact that r is entire, Lemma is already proved. The estimations on the support of $r_\infty(x, t; k)$, namely (3.39) for u'_∞ and $\tilde{u}'_\infty$, (4.18) for z_∞ imply that for any $x\in\Omega_R$ the support in t of r_∞ is contained in a fixed bounded set, from which we have the entireness of r_∞. Q.E.D.

Next we consider the form of $\hat{r}$ on the boundary.

Lemma 5.2. *On* Γ_1 *we have*

$$\hat{r}(x, \mu; k)=\hat{m}(x, \mu; k)+bw_k(A_0)e^{-\mu(j(A_0)+d_\infty)}\frac{2}{\mathscr{P}(\mu)}\hat{r}_\infty(x, \mu; k)+\hat{s}_1(x, \mu; k), \tag{5.6}$$

where $\hat{s}_1(x, \mu; k)$ *is* $C^\infty(\Gamma_1)$*-valued holomorphic in* $\mathscr{D}$ *and satisfies estimates*

$$|\hat{s}_1(x, \mu; k)|\leqslant C_{\varepsilon'}(\theta_k(x)\log k+k^{-N}k^{\varepsilon N'})M_{N'} \quad \text{for} \quad \mu\in\mathscr{D}_{\varepsilon'}, \tag{5.7}$$

$$|\hat{s}_1(x, \mu; k)|\leqslant C(\theta_k(x)k^{-1}+k^{-N}k^{\varepsilon N'})M_{N'} \quad \text{for} \quad \mathrm{Re}\,\mu\geqslant -c_0-2d(\log k)^{-1}. \tag{5.8}$$

Proof. On Γ_1 we see from the definition

$$r_\infty(x, t; k)=e^{ik(d_0-j(A_0)-d_\infty)}\{e^{ik(\varphi_\infty(x)-t)}\theta_k(x)$$
$$\cdot\sum_{j=0}^{N}(ik)^{-j}\tilde{v}_{j,\infty}(x, t-d; k)-u'_\infty(x, t; k)+\tilde{u}'_\infty(x, t; k)\}.$$

Set

$$s_1(x, t; k)=r(x, t; k)-m(x, t; k)$$
$$-bw_k(A_0)\sum_{p=0}^{\infty}(\lambda\tilde{\lambda})^p r_\infty(x, t-2pd-j(A_0)-d_\infty; k).$$

Then it follows from (4.23) that

$$\begin{aligned}
s_1(x, t; k)=&\sum_{p=0}^{\infty}[\theta_k(x)\{e^{ikt}\sum_{j=0}^{N}(ik)^{-j}(e^{ik(\varphi_{2q}(x))}v_{j,2p+1}(x, t; k)\\
&-e^{ik(\varphi_\infty(x)+2pd+d_0)}(\lambda\tilde{\lambda})^p\tilde{v}_{j,\infty}(x, t-(2p+1)d; k))\}\\
&+\{(m_p-z_p)(x, t; k)-bw_k(A_0)(\lambda\tilde{\lambda})^p(m_\infty-z_\infty)(x, t-2pd-d_\infty; k)\}\\
&+\{u'_{2p}(x, t; k)-bw_k(A_0)(\lambda\tilde{\lambda})^p u'_\infty(x, t-2pd-d_\infty; k)\}\\
&+\{u'_{2p+1}(x, t; k)-bw_k(A_0)(\lambda\tilde{\lambda})^p\tilde{u}'_\infty(x, t-(2p+1)d-d_\infty; k)\}\\
=&I_1+I_2+I_3+I_4.
\end{aligned}$$

First consider I_1. Set

$$\begin{aligned}
I_{1,j,p}=&\theta_k(x)e^{ik\varphi_{2p}(x)}v_{j,2p+1}(x, t; k)\\
&-e^{ik(\varphi_\infty(x)+2pd+d_0)}(\lambda\tilde{\lambda})^p\tilde{v}_{j,\infty}(x, t-(2p+1)d-d_\infty; k)\\
=&\theta_k(x)(e^{ik\varphi_{2p}}-e^{ik(\varphi_\infty+2pd+d_0)})(\lambda\tilde{\lambda})^p\tilde{v}_{j,\infty}(x, t-(2p+1)d-d_\infty; k)
\end{aligned}$$

$$+\theta_k(x)e^{ik\varphi_{2p}}\{v_{j,2p+1}(x, t; k)-(\lambda\tilde{\lambda})^p\tilde{v}_{j,\infty}(x, t-(2p+1)d-d_\infty; k)\}$$
$$=II_{j,p}+III_{j,p}.$$

By using (3.4) we have

$$|II_{j,p}|_m(\Gamma_1\times \boldsymbol{R})\leqslant C_m \max(k\alpha^{2p}, 2)(\lambda\tilde{\lambda})^p M_{m+2j}\theta_k(x).$$

Then, by setting

$$\sum_{p=0}^{\infty} II_{j,p}=\sum_{p=0}^{\log k} II_{j,p}+\sum_{p=\log k}^{\infty} II_{j,p}=II_j^{(1)}+II_j^{(2)}$$

we have

$$|II_j^{(1)}|\leqslant C\theta_k(x)e^{-c_0t}\log k \quad \text{and} \quad \operatorname{supp} II_j^{(1)}\subset(0, 2d\log k+\rho_0).$$

Then it follows that

$$|\widehat{II}_j^{(1)}(x, \mu; k)|\leqslant C\theta_k(x)\log k\int_0^{2d\log k+\rho_0} e^{-c_0t}e^{-\operatorname{Re}\mu t}dt$$
$$\leqslant C\theta_k(x)e^{-2d\log k(-\operatorname{Re}\mu-c_0)}\log k.$$

Therefore we have

$$|\widehat{II}_j^{(1)}(x, \mu; k)|\leqslant C\theta_k(x)\log k \quad \text{if} \quad \operatorname{Re}\mu\geqslant -c_0-2d(\log k)^{-1}.$$

It is evident that

$$|\widehat{II}_j^{(2)}(x, \mu; k)|\leqslant C\theta_k(x) \qquad \text{for} \quad \operatorname{Re}\mu\leqslant -c_0-c_1/2.$$

On I_l, $l=3, 4$, estimates (3.41) and (3.42) imply

$$|\hat{I}_l(x, \mu; k)|\leqslant Ck^{-N}k^{\varepsilon N'} \qquad \text{for all} \quad \mu\in\mathscr{D},$$

and on I_2 the process of the construction of z_p assures

$$|\hat{I}_2(x, \mu; k)|\leqslant Ck^{-N}k^{\varepsilon N'} \qquad \text{for all} \quad \mu\in\mathscr{D}.$$

Combining these estimates we have (5.8).

If we use an estimate $|II_{j,p}|\leqslant\theta_k(x)k(\alpha\lambda\tilde{\lambda})^p$ for all p (5.7) follows immediately.

Q. E. D.

Next consider $\hat{r}$ on Γ_2.

Lemma 5.3. *On Γ_2 we have*

$$\hat{r}(x, \mu; k)=bw_k(A_0)e^{-\mu(j(A_0)+d_\infty)}\frac{1}{\mathscr{P}(\mu)}\hat{r}_\infty(x, \mu; k)+\hat{s}_2(x, \mu; k), \tag{5.9}$$

where $\hat{s}_2$ is a $C^\infty(\Gamma_2)$-valued holomorphic function in $\mathscr{D}_{\varepsilon'}$ and satisfies an estimate for $\mu\in\mathscr{D}_{\varepsilon'}$ $(\varepsilon'>0)$

$$|\hat{r}_\infty(x, \mu; k)|_m+|\hat{s}_2(x, \mu; k)|_m\leqslant C_{\varepsilon'}k^{-N+m+\varepsilon N'}M_{N'+m} \quad on \quad \Gamma_2. \tag{5.10}$$

Proof. Set

$$s_2(x, t; k)=r(x, t; k)-bw_k(A_0)\sum_{p=0}^{\infty}(\lambda\tilde{\lambda})^p r_\infty(x, t-2pd-j(A_0)-d_\infty; k).$$

Then we have

$$\begin{aligned}s_1(x, t; k)=\sum_{p=0}^{\infty}\{&-z_p(x, t; k)+bw_k(A_0)(\lambda\tilde{\lambda})^p z_\infty(x, t-2pd-j(A_0)-d_\infty; k)\\&+u'_{2p}(x, t; k)-bw_k(A_0)(\lambda\tilde{\lambda})^p u'_\infty(x, t-2pd-j(A_0)-d_\infty; k)\\&+u'_{2p+1}(x, t; k)-bw_k(A_0)(\lambda\tilde{\lambda})^p \tilde{u}'_\infty(x, t-2pd-j(A_0)-d_\infty; k)\}.\end{aligned}$$

Thus estimate (5.10) on $\hat{s}_2$ is done by the same way as for I_l, $l=2, 3, 4$ in the previous lemma. For $\hat{r}_\infty$. recall that $r_\infty=z_\infty$ on Γ_2, and we have the desired estimate.

Set

$$\Omega^{(2)}=\boldsymbol{R}^3-\mathcal{O}_2.$$

Denote by $U^{(2)}(\mu)g$ for $\operatorname{Re}\mu>0$ and $g\in C^\infty(\Gamma_2)$ the solution of

$$\begin{cases}(\mu^2-\Delta)u=0 & \text{in } \Omega^{(2)},\\ u=g & \text{on } \Gamma_2,\\ u\in L^2(\Omega^{(2)}).\end{cases}$$

Then $U^{(2)}(\mu)$ can be prolonged analytically into

$$\{\mu;\ \operatorname{Re}\geqslant-\beta,\ |\mu|\geqslant C_\beta\}\quad(=\mathscr{D}_\beta^{(2)})$$

for any $\beta>0$, where C_β is a constant depending on β. Moreover,

$$|u|_m(\Omega_R^{(2)})\leqslant C_{m,R,\beta}\|g\|_{H^m(\Gamma_2)}\qquad\text{for}\quad \mu\in\mathscr{D}_\beta^{(2)}.\text{[1)]}$$

Set

$$e(x, \mu; k)=(e^{-(\mu+ik)j(x)}h(\mu+ik))^{-1}(\hat{r}(x, \mu; k)-U^{(2)}(\mu)r(\cdot, \mu; k)|_{\Gamma_2}).$$

Now we shall show the following

Proposition 5.4. *Let ψ be a function verifying Condition C. Then there exists $e(x, \mu; k)$ of the form*

$$(5.11)\qquad e(x, \mu; k)=\frac{1}{\mathscr{P}(\mu)}bw_k(A_0)e^{ikd_0}e^{-(\mu+ik)(j(A_0)+d_\infty)}e_1(x, \mu; k)+e_2(x, \mu; k)$$

verifying the following:

(i) *e_1 and e_2 are $C^\infty(\bar{\Omega})$-valued holomorphic functions defined in $\mathscr{D}$ and e_1 is independent of ψ, and they satisfy estimates*

1) See for example, Ikawa M., Mixed problems for the wave equation, III, Exponential decay of solutions, Publ. Res. Inst. Math. Sci. Kyoto Univ. 14 (1978), 71–110.

(5.12) $$|e_l(x,\ \mu;\ k)|_m(\Omega_R)\leqslant C_{m,R}k^{m+1}\sum_{j=0}^{N}k^{-j}M_{2j+m}$$

for $\mu\in\mathscr{D}_{(k)}=\{\mu;\ |\mathrm{Im}\,\mu+ik|\leqslant|k|^{-1/2},\ -c_0-\log k\leqslant\mathrm{Re}\,\mu\leqslant 1\}$, *and* $e_l(\cdot,\ \mu;\ k)\in L^2(\Omega)$ *for* $\mathrm{Re}\,\mu>0$,

(ii) $(\mu^2-\Delta)e(x,\ \mu;\ k)=0$ *in* Ω,

(iii) $e(x,\ \mu;\ k)=0$ *on* Γ_2,

(iv) *on* Γ_1 *e is of the form*

$$e(x,\ \mu;\ k)=e^{ik\psi(x)}w_k(x)+\Big[\frac{1}{\mathscr{P}(\mu)}bw_k(A_0)e^{ikd_0}e^{-(\mu+ik)(j(A_0)+d_\infty)}$$
$$\times\{\theta_k(x)e^{ik\varphi_\infty(x)}\sum_{j=1}^{N}\sum_{l=0}^{2j}a_{j,l}^0(x)(ik)^{-j}(\mu+ik)^l+b_1(x,\ \mu;\ k)\}\Big]+b_2(x,\ \mu;\ k),$$

where b_1 *and* b_2 *are* $C^\infty(\Gamma_1)$*-valued holomorphic function in* $\mathscr{D}$ *and* b_1 *is independent of* ψ. *Moreover they satisfy estiamtes*

(5.13) $$|b_1(\cdot,\ \mu;\ k)|_m(\Gamma_1)\leqslant C_m k^{-N+m}M_{2N'+m}$$

(5.14) $$|b_2(x,\ \mu;\ k)|\leqslant k^{-1}\theta_k(x)\log k+Ck^{-(1-\varepsilon)N}.$$

Proof. Form the definition of $e(x,\ \mu;\ k)$ (ii) and (III) follow immediately. Note that we have

$$|\hat{h}(\mu+ik)|\geqslant c>0 \qquad \text{for all} \quad \mu\in\mathscr{D}_{(k)},$$

where c is a constant independent of k. Set

$$\tilde{e}_1(x,\ \mu;\ k)=(\hat{r}_\infty(x,\ \mu;\ k)-U^{(2)}(\mu)(\hat{r}_\infty(\cdot,\ \mu;\ k)|_{\Gamma_2})(x))(e^{-(\mu+ik)j(x)}\hat{h}(\mu+ik))^{-1},$$
$$e_2(x,\ \mu;\ k)=(\hat{s}(x,\ \mu;\ k)-U^{(2)}(\mu)(\hat{s}_2(\cdot,\ \mu;\ k)(x))(e^{-(\mu+ik)j(x)}\hat{h}(\mu+ik))^{-1},$$

From the definition (4.21) of r_∞ and (5.5) we have

$$\tilde{e}_2=e^{ikd_0}e^{-(\mu+ik)(j(A_0)+d_\infty)}e_2,\quad e_2 \text{ is independent of } \psi.$$

Lemmas 5.2 and 5.3 imply (5.12). Now we show (iv). From the definition we have

$$\hat{r}_\infty(x,\ \mu;\ k)=e^{ik(d_0-j(A_0)-d_\infty)}\{\hat{u}_\infty(x,\ \mu;\ k)-\hat{u}'_\infty(x,\ \mu;\ k)$$
$$-e^{-\mu d}(\hat{\hat{u}}_\infty(x,\ \mu;\ k)-\hat{\hat{u}}'_\infty(x,\ \mu;\ k))-\hat{z}_\infty(x,\ \mu;\ k)\},$$

and by using (3.21), (3.22), (3.23), (3.24) and Lemma 3.3

$$\hat{u}_\infty(x,\ \mu;\ k)-e^{-\mu d}\hat{\hat{u}}_\infty(x,\ \mu;\ k)$$
$$=\theta_k(x)e^{ik\varphi_\infty(x)}\sum_{j=1}^{N}(ik)^{-j}\sum_{l=0}^{2j}a_{j,l}(x;\ k)(\mu+ik)^l e^{-(\mu+ik)j(x)}\hat{h}(\mu+ik).$$

Thus by putting

$$b_1(x,\ \mu;\ k)=(-\hat{u}'_\infty(x,\ \mu;\ k)+\hat{\hat{u}}'_\infty(x,\ \mu;\ k)-U^{(2)}(\mu)(\hat{r}_\infty(\cdot,\ \mu;\ k)|_{\Gamma_2})(x))$$
$$\cdot(e^{-(\mu+ik)j(x)}\hat{h}(\mu+ik))^{-1},$$

$$b_2(x, \mu; k) = (\hat{s}_1(x, \mu; k) - U^{(2)}(\mu)(\hat{s}_2(\cdot, \mu; k))(x)) \cdot (e^{-(\mu+ik)j(x)}\hat{h}(\mu+ik))^{-1},$$

we have (5.13) and (5.13) from (3.32), (3.33) and the definition $u'_\infty(x, t; k)$ and $\tilde{u}'_\infty(x, t; k)$, and (5.10). The representation is immediately derived from the definition of e.

6. Definition of $U_0(\mu)$.

Let $g(x) \in C_0^\infty(S_1((1+3\delta)k^{-\varepsilon}))$. Then we have by the Fourier's inversion formula

$$g(x(\sigma)) = (2\pi)^{-2} \iint e^{i(\sigma-\sigma')\cdot\xi} g(x(\sigma'))d\sigma' d\xi$$

$$= \int w_k(x(\sigma))e^{ik\sigma\cdot\xi}\hat{g}(k\xi)k^2 d\xi,$$

where

$$\hat{g}(\xi) = (2\pi)^{-2} \int_{\boldsymbol{R}} e^{i\sigma\cdot\xi} g(x(\sigma))d\sigma.$$

Define $\psi(x, \xi) \in C^\infty(S_1(\delta_0))$ by

$$\psi(x(\sigma), \xi) = \sigma \cdot \xi.$$

When $|\xi| < 1-\delta$ $(\delta > 0)$, $\psi(x, \xi)$ satisfies Condition C and if $|\xi| > \delta_0$

$$^{\#}\mathscr{X}(x, \nabla\varphi(x, \xi)) \leqslant K$$

holds for some fixed K. By using §7 of [2] and $U^{(2)}(\mu)$ we have immediately the following lemma

Lemma 6.1. *For $|\xi| \geqslant \delta_0$ there exists $e(x, \mu; k, \xi)$ verifying*

$$(6.1)\qquad \begin{cases} (\mu^2 - \Delta)e(x, \mu; k, \xi) = 0 & in \quad \Omega, \\ e(x, \mu; k, \xi) = 0 & on \quad \Gamma_2, \\ e(x, \mu; k, \xi) = e^{ik\cdot\xi}w_k(\sigma) + b_2(x, \mu; k, \xi) & on \quad \Gamma_1, \end{cases}$$

where e_3 is homomorphic in $\operatorname{Re}\mu > -c_0 - c_1$ and $e \in L^2(\Omega)$ for $\operatorname{Re}\mu > 0$, and

$$(6.2)\qquad |e(\cdot, \mu; k, \xi)|_m(\Omega_R) \leqslant C_{m,R}k^{m+1},$$

$$(6.3)\qquad |b_2(\cdot, \mu; k, \xi)|_m(\Gamma_1) \leqslant C_m k^{-N+m}.$$

Apply Proposition 5.4 for $\psi(x) = \psi(x, \xi)$, $|\xi| \leqslant \delta_0$. Denote $e(x, \mu; k)$, d_0, d_∞, b, A_0 in Proposition in 5.4 for $\psi(x, \xi)$ by $e(x, \mu; k, \xi)$, $d_0(\xi)$, $d_\infty(\xi)$, $b(\xi)$, $A_0(\xi)$ respectively. Set

$$(U_1(\mu; k)g)(x) = \int_{\boldsymbol{R}^2} e(x, \mu; k, \xi)\hat{g}(k\xi)k^2 d\xi.$$

Taking account of the independency of e_1 on ψ we have from (5.11) that

$$U_1(\mu;\ k)g=\frac{e_1(x,\ \mu;\ k)}{\mathscr{P}(\mu)}F_0(\mu;\ k)g+U_1(\mu;\ k)g,$$

where

$$F_0(\mu;\ k)g=\int_{|\xi|<\delta_0}w_k(A_0(\xi))b(\xi)e^{ikd_0(\xi)}e^{-(\mu+ik)(j(A_0(\xi))+d_\infty(\xi))}\hat{g}(k\xi)k^2d\xi,$$

$$U_1(\mu;\ k)g=\int_{\boldsymbol{R}^2}b_2(x,\ \mu;\ k,\ \xi)\hat{g}(k\xi)k^2d\xi.$$

From Proposition 5.4 and Lemma 6.1 we have

Lemma 6.2. *$U_1(\mu;\ k)$ is $\mathscr{L}(C_0^\infty(S_1((1+3\delta)k^{-\varepsilon})),\ C^\infty(\bar{\Omega}))$-valued holomorphic function defined in $\mathscr{D}-\{\mu;\ \mathscr{P}(\mu)=0\}$, and satisfies*

$$\begin{cases}(\mu^2-\Delta)U_1(\mu;\ k)g=0 & in\quad \Omega,\\ U_1(\mu;\ k)g=0 & on\quad \Gamma_2.\end{cases}$$

Moreover we have

$$U_1(\mu;\ k)g\in L^2(\Omega)\qquad for\quad \mathrm{Re}\,\mu>0.$$

Now consider the boundary valued of U_1g on Γ_1. From (iv) of Proposition 5.4 and (6.1) it follows that for $x\in\Gamma_1$

$$\begin{aligned}U_1(\mu;\ k)g&=\int_{\boldsymbol{R}^2}e^{ik\psi(x,\xi)}w_k(x)\hat{g}(k\xi)k^2d\xi\\&\quad+\frac{1}{\mathscr{P}(\mu)}\{\theta_k(x)a(x,\ \mu;\ k)+b_1(x,\ \mu;\ k)\}F_0(\mu;\ k)g\\&\quad+\int_{\boldsymbol{R}^2}b_2(x,\ \mu;\ k,\ \xi)\hat{g}(k\xi)k^2d\xi\\&=g(x)+\frac{1}{\mathscr{P}(\mu)}\tilde{E}_1(\mu;\ k)g+\tilde{E}_2(\mu;\ k)g.\end{aligned}$$

Lemma 6.3. *For $\mu\in\mathscr{D}_{(k)}$ we have*

$$\|\tilde{E}_2(\mu;\ k)g\|_{L^2(\Gamma_1)}\leqslant Ck^{-\varepsilon/2}\|g\|_{L^2(\Gamma_1)}. \tag{6.4}$$

Proof. From (5.14) and (6.3) we have

$$\begin{aligned}\int_{\Gamma_1}|\tilde{E}_2g(x)|^2dx&\leqslant 2\int_{\Gamma_1}\Big(\int_{|\xi|<\delta_0}\theta_k(x)\frac{\log k}{k}|\hat{g}(k\xi)|k^2d\xi\Big)^2dx\\&\quad+2\int_{\Gamma_1}\Big(\int_{\boldsymbol{R}^2}k^{-N}k^{\varepsilon N}|\hat{g}(k\xi)|k^2d\xi\Big)^2dx\\&\leqslant 2\int_{\Gamma_1}(\log k)^2\theta_k(x)^2dx\int_{\boldsymbol{R}^2}|\hat{g}(k\xi)|^2k^2d\xi\cdot\int_{|\xi|<\delta_0}d\xi\\&\quad+C(k^{-N+1+\varepsilon N})^2\int_{|\xi|<\delta_0}k^2|\hat{g}(k\xi)|^2d\xi\\&\leqslant 2(\log k)^2k^{-\varepsilon}\|g\|^2_{L^2}+Ck^{-N/2}\|g\|^2_{L^2}.\end{aligned}$$

Q. E. D.

With the aid of Lemma 2.4 we introduce $U_2(\mu;\, k)$ as a slight modification of $U_2(\mu)$ in §9 of [2], namely

Lemma 6.4. *There exists an operator $U_2(\mu;\, k)$ which is $\mathscr{L}(C^\infty(\Gamma_1),\, C^\infty(\bar{\Omega}))$-valued entire function verifying*

$$U_2(\mu;\, k)g \in L^2(\Omega) \quad \text{for} \quad \operatorname{Re}\mu > 0$$

$$(\mu^2 - \Delta)U_2(\mu;\, k)g = 0 \quad \text{in} \quad \Omega \qquad \text{for all} \quad \mu, \tag{6.5}$$

$$U_2(\mu;\, k)g = 0 \qquad \text{on} \quad \Gamma_2 \qquad \text{for all} \quad \mu, \tag{6.6}$$

$$\|U_2(\mu;\, k)g|_{\Gamma_1}\|_{L^2(\Gamma_1)} \leqslant Ck\|g\|_{L^2(\Gamma_1)}, \tag{6.7}$$

$$|(1-\upsilon_k)(U_2(\mu;\, k)g - g)|_m(\Gamma_1) \leqslant C_m k^{-N+m} e^{(-c_0 - \operatorname{Re}\mu)\log k}\|g\|_{L^2(\Gamma_1)}. \tag{6.8}$$

Now define an operator $U_0(\mu;\, k) \in \mathscr{L}(C^\infty(\Gamma_1),\, C^\infty(\bar{\Omega}))$ by

$$U_0(\mu;\, k)g = U_1(\mu;\, k)(\eta_k g - \upsilon_k U_2(\mu;\, k)(1-\eta_k)g\,|_{\Gamma_1}) + U_2(\mu;\, k)(1-\eta_k)g. \tag{6.9}$$

Let us set

$$M(\mu;\, k)g = U_0(\mu;\, k)g\,|_{\Gamma_1},$$

$$B_1(\mu;\, k)g = \eta_k(x)g,$$

$$B_2(\mu;\, k)g = -\upsilon_k(x)U_2(\mu;\, k)(1-\eta_k)g\,|_{\Gamma_1}.$$

Then we have

$$Mg = g + \mathscr{P}(\mu)^{-1}\tilde{E}_1(B_1 + B_2)g + \tilde{E}_2(B_1 + B_2)g + E_3 g, \tag{6.10}$$

where

$$E_3(\mu;\, k)g = (1-\upsilon_k)U_2(\mu;\, k)(1-\eta_k)g\,|_{\Gamma_1}.$$

It follows immediately from (6.8)

$$\|E_3 g\|_{H^m(\Gamma_1)} \leqslant C_m k^{-N+m}\|g\|_{L^2(\Gamma_1)}. \tag{6.11}$$

We set

$$E_2(\mu;\, k) = \tilde{E}_2(\mu;\, k)(B_1(\mu;\, k) + B_2(\mu;\, k)),$$

$$F_j(\mu;\, k) = F_0(\mu;\, k)B_j(\mu;\, k), \quad j = 1,\, 2,$$

$$F(\mu;\, k) = F_1(\mu;\, k) + F_2(\mu;\, k),$$

$$H_1(\mu;\, k) = \mathscr{P}(\mu)^{-1}\theta_k(x)a(x,\, \mu;\, k)F(\mu;\, k),$$

$$H_2(\mu;\, k) = \mathscr{P}(\mu)^{-1}b_1(x,\, \mu;\, k),$$

$$H(\mu;\, k) = H_1(\mu;\, k) + H_2(\mu;\, k).$$

Remark that from the definitions it follows that

$$(B_1 + B_2)H_1 = H_1, \tag{6.12}$$

(6.13) $$(B_1+B_2)\tilde{E}_2=\tilde{E}_2.$$

If we use the above notations (6.10) may be written as

(6.14) $$M=I+H_1+H_2+E_2+E_3.$$

7. Explicit representation of $U(\mu)$.

Lemma 7.1. *Suppose that A and B are bounded operators in a Hilbert space X such that*

$$\mathscr{A}=A+A^2+A^3+\cdots,\quad \mathscr{B}=B+B^1+B^2+\cdots$$

converge in the opreator norm and $\|(\mathscr{B}\mathscr{A})^j\|\leqslant C\eta^j$ $(0<\eta<1)$ holds. Set

$$\mathscr{C}_1=\mathscr{A}+\mathscr{A}\mathscr{B}+\mathscr{A}\mathscr{B}\mathscr{A}+\mathscr{A}\mathscr{B}\mathscr{A}\mathscr{B}+\cdots,$$

$$\mathscr{C}_2=\mathscr{B}+\mathscr{B}\mathscr{A}+\mathscr{B}\mathscr{A}\mathscr{B}+\mathscr{B}\mathscr{A}\mathscr{B}\mathscr{A}+\cdots,$$

$$\mathscr{C}=\mathscr{C}_1+\mathscr{C}_2.$$

Then we have

$$(A+B)\mathscr{C}=\mathscr{C}-(A+B).$$

Proof. By using

$$A\mathscr{A}=\mathscr{A}-A,\quad B\mathscr{B}=\mathscr{B}-B$$

we have

$$\begin{aligned}A\mathscr{C}_1&=(\mathscr{A}-A)+(\mathscr{A}-A)\mathscr{B}+(\mathscr{A}-A)\mathscr{B}\mathscr{A}+\cdots\\&=\mathscr{C}_1-A-A\mathscr{C}_2.\end{aligned}$$

Similarly we have

$$B\mathscr{C}_2=\mathscr{C}_2-B-B\mathscr{C}_1.$$

Thus

$$\begin{aligned}(A+B)(\mathscr{C}_1+\mathscr{C}_2)&=A\mathscr{C}_1+B\mathscr{C}_2+B\mathscr{C}_1+A\mathscr{C}_2\\&=\mathscr{C}_1+\mathscr{C}_2-(A+B).\end{aligned}$$ Q.E.D.

Now apply the above lemma to the operators in the previous section.

Lemma 7.2. *There exists a bounded operator $\mathscr{E}(\mu;k)$ in $L^2(\Gamma_1)$ such that*

(7.1) $$(E_2+E_3)\mathscr{E}=\mathscr{E}-(E_2+E_3).$$

$\mathscr{E}(\mu,k)$ is holomorphic in $\mathrm{Re}\,\mu>-c_0-\log k$ and satisfies

(7.2) $$\|\mathscr{E}-\tilde{\mathscr{E}}_2(B_1+B_2)\|\leqslant Ck^{-N},$$

where

$$\tilde{\mathscr{E}}_2=\tilde{E}_2+\tilde{E}_2^2+\tilde{E}_2^3+\cdots.$$

Proof. From (6.13) we have

$$E_2^j=\tilde{E}_2^j(B_1+B_2),$$

then (6.4) assures the convergence of $\mathscr{E}_2$ and

$$\mathscr{E}_2=\tilde{\mathscr{E}}_2(B_1+B_2).$$

It is evident from (6.11) that

$$\mathscr{E}_3=E_3+E_3^2+E_3^3+\cdots$$

converges and

$$\|\mathscr{E}_3\|_{\mathscr{L}(L^2(\Gamma_1))}\leqslant Ck^{-N}.$$

Thus by applying Lemma 7.1 we have immediately (7.1) and (7.2). Q.E.D.

Set

$$\begin{aligned}\gamma_1(\mu;\,k)&=F_0(B_1+B_2)(\theta_k(x)a(x,\,\mu;\,k)+b_1(x,\,\mu;\,k))\\&=F_0\theta_k(x)a(x,\,\mu;\,k)+F_0(B_1+B_2)b_1(x,\,\mu;\,k)\\&=\gamma_{11}(\mu;\,k)+\gamma_{12}(\mu;\,k).\end{aligned}$$

Then we have

$$\begin{aligned}\mathscr{H}&=H+H^2+H^3+\cdots\\&=\frac{\alpha}{\mathscr{P}(\mu)}\left(\sum_{j=1}^{\infty}\left(\frac{\gamma_1}{\mathscr{P}(\mu)}\right)^{j-1}\right)F=\frac{\alpha}{\mathscr{P}(\mu)-\gamma_1}F.\end{aligned}\tag{7.3}$$

Lemma 7.3. *There exists an $\mathscr{L}(L^2(\Gamma_1))$-valued holomorphic function $\mathscr{M}(\mu;\,k)$ in $\operatorname{Re}\mu>-c_0-\log k$ satisfying*

$$(H+E_2+E_3)\mathscr{M}=\mathscr{M}-(H+E_2+E_3).\tag{7.4}$$

Here, $\mathscr{M}$ is of the form

$$(I+\mathscr{M})=I+\mathscr{E}+\frac{\alpha+\mathscr{E}\alpha}{\mathscr{P}(\mu)-\gamma}F(I+\mathscr{E}),\tag{7.5}$$

where

$$\alpha(x,\,\mu;\,k)=\theta_k(x)a(x,\,\mu;\,k)+b_1(x,\,\mu;\,k),$$

γ is a complex valued holomorphic function.

Proof. Set

$$\begin{aligned}\gamma_2&=F_0(B_1+B_2)\mathscr{E}\alpha(x,\,\mu;\,k)\\&=F_0(B_1+B_2)(\tilde{\mathscr{E}}_2(B_1+B_2)\theta_k(x)a+(\mathscr{E}-\tilde{\mathscr{E}}_2(B_1+B_2))\theta_ka+\mathscr{E}b_1)\\&=F_0\tilde{\mathscr{E}}_2\theta_ka(x,\,\mu;\,k)+\{F(\mathscr{E}-\tilde{\mathscr{E}}_2(B_1+B_2))\theta_ka+F\mathscr{E}b_1\}\\&=\gamma_{20}+\gamma_{21}.\end{aligned}$$

Since we have $\|\tilde{\mathscr{E}}_2\| \leqslant Ck^{-\varepsilon/2}$ from (6.4) we have

$$|\gamma_{20}| \leqslant Ck^{-\varepsilon/2}. \tag{7.6}$$

From (7.2) and (5.13) we have

$$|\gamma_{21}| \leqslant Ck^{-N+2}. \tag{7.7}$$

Now from (7.3) and the definition of γ_2 it follows that

$$(\mathscr{E}\mathscr{H})^j = (\mathscr{P}(\mu) - \gamma_1)^j \gamma_2^{j-2} \mathscr{E}\alpha F_0(B_1 + B_2).$$

Then

$$\mathscr{N} = \sum_{j=1}^{\infty} (\mathscr{E}\mathscr{H})^j$$

converges in the operator norm and it has the form

$$\mathscr{N} = \frac{1}{\mathscr{P}(\mu) - \gamma_1 - \gamma_2} \mathscr{E}\alpha F_0(B_1 + B_2).$$

Put $\gamma = \gamma_1 + \gamma_2$, and we have

$$\gamma = F\alpha + F\mathscr{E}\alpha = F(I + \mathscr{E})\alpha. \tag{7.8}$$

Then we can apply Lemma 7.1 and

$$\mathscr{M} = \mathscr{H} + \mathscr{H}\mathscr{E} + \mathscr{H}\mathscr{E}\mathscr{H} + \cdots + \mathscr{E} + \mathscr{E}\mathscr{H} + \mathscr{E}\mathscr{H}\mathscr{E} + \cdots$$

converges and it satisfies (7.4). Since we can rewrite it as

$$\mathscr{M} = \mathscr{H}(I + \mathscr{E}) + \mathscr{H}(\mathscr{E}\mathscr{H} + (\mathscr{E}\mathscr{H})^2 + \cdots)(I + \mathscr{E}) + \mathscr{E} + (\mathscr{E}\mathscr{H} + (\mathscr{E}\mathscr{L})^2 + \cdots)(I + \mathscr{E}),$$

we have

$$\begin{aligned} I + \mathscr{M} &= I + \mathscr{E} + \frac{\alpha}{\mathscr{P}(\mu) - \gamma_1} F(I + \mathscr{E}) \\ &\quad + \frac{\alpha}{\mathscr{P}(\mu) - \gamma_1} \frac{\gamma_2}{\mathscr{P}(\mu) - \gamma} F(I + \mathscr{E}) + \frac{\mathscr{E}\alpha}{\mathscr{P}(\mu) - \gamma} F(I + \mathscr{E}) \\ &= (I + \mathscr{E}) + \frac{\alpha + \mathscr{E}\alpha}{\mathscr{P}(\mu) - \gamma} F(I + \mathscr{E}). \end{aligned} \tag{7.9}$$

Q.E.D.

Proposition 7.4. *In $\mathscr{D}_{(k)} \cap \{\mu;\ \mathrm{Re}\,\mu > 0\}$ $U(\mu)$ is represented as*

$$U(\mu) = \frac{\beta(x, \mu; k)}{\mathscr{P}(\mu) - \gamma(\mu; k)} F(\mu; k)(I + \mathscr{E}(\mu; k)) + \tilde{U}(\mu; k), \tag{7.10}$$

where $\beta(\cdot, \mu; k)$ is $C^\infty(\bar{\Omega})$-valued holomorphic function in $\mathscr{D}_{(k)}$ and $\tilde{U}(\mu; k)$ is $\mathscr{L}(C^\infty(\Gamma_1), C^\infty(\bar{\Omega}))$-valued holomorphic function in $\mathscr{D}_{(k)}$.

Proof. Set

$$U(\mu; k) = U_0(\mu; k)(I + \mathscr{M}(\mu; k)). \tag{7.11}$$

From the definition (6.9) of U_0 and Lemmas 6.2 and 6.4 we have

$$U(\mu;\, k)g \in L^2(\Omega) \qquad \text{for} \quad \operatorname{Re}\mu>0,$$

$$(\mu^2-\Delta)U(\mu;\, k)g=0 \quad \text{in} \quad \Omega,$$

$$U(\mu;\, k)g=0 \quad \text{on} \quad \Gamma_2 .$$

On the other hand on Γ_1 we have

$$U(\mu;\, k)g=(I+H+E_2+E_3)(I+\mathscr{M})g=g.$$

Thus from the uniqueness of the solution it follows that

$$U(\mu;\, k)=U(\mu) \quad \text{in} \quad \mathscr{D}_{(k)}\cap\{\mu;\, \operatorname{Re}\mu>0\}. \tag{7.12}$$

Now substitute (6.9) and (7.5) into (7.11) and use (6.4), and we have

$$\begin{aligned} U(\mu;\, k)=&\frac{e_1}{\mathscr{P}(\mu)}F_0(B_1+B_2)\Big(I+\mathscr{E}+\frac{(I+\mathscr{E})\alpha}{\mathscr{P}(\mu)-\gamma}F(I+\mathscr{E})\Big)\\ &+\tilde{U}_1(\mu;\, k)(B_1+B_2)\Big(I+\mathscr{E}+\frac{(I+\mathscr{E})\alpha}{\mathscr{P}(\mu)-\gamma}F(I+\mathscr{E})\Big)\\ &+U_2(\mu;\, k)(1-\eta_k)\Big(I+\mathscr{E}+\frac{(I+\mathscr{E})\alpha}{\mathscr{P}(\mu)-\gamma}F(I+\mathscr{E})\Big)\\ =&\frac{1}{\mathscr{P}(\mu)-\gamma}\beta(x,\, \mu;\, k)F(I+\mathscr{E})+\tilde{U}(\mu;\, k), \end{aligned}$$

where

$$\beta=e_1+U_1(B_1+B_2)(I+\mathscr{E})\alpha+U_2(1-\eta_k)(I+\mathscr{E})\alpha,$$

$$\tilde{U}=U_1(B_1+B_2)(I+\mathscr{E})+U_2(1-\eta_k)(I+\mathscr{E}).$$

From these formulas we have the required assertion.

8. Proof of Theorems 2 and 3.

First we consider an asymptotic form of γ for $k\to\infty$. From the definition of F_0 and $\alpha(x,\, \mu;\, k)$ we have

$$\gamma_{11}(\mu,\, k)=\iint e^{ikd_0(\xi)}e^{-ik\xi\cdot\sigma}e^{ik\varphi_\infty(x(\sigma))}p(\sigma,\, \xi,\, k,\, \mu)k^2d\xi d\sigma, \tag{8.1}$$

where

$$p(\sigma,\, \xi,\, k,\, \mu)=b(\xi)w_k(A_0(\xi))e^{-(\mu+ik)(j(A_0(\xi))+d(\xi))}\theta_k(x(\sigma))\sum_{j=1}^{N}\tilde{v}_{j,\infty}(x(\sigma))(ik)^{-j}. \tag{8.2}$$

Then if we restrict μ in

$$\mathscr{C}_k=\{\mu;\, |\mu+ik|\leqslant C\}, \quad C=c_0+c_1+2\pi/d,$$

it holds that

(8.3) $$|\partial_\sigma^\beta \partial_\xi^\alpha p(\sigma, \xi, k, \mu)| \leqslant Ck^{(|\alpha|+|\beta|)\varepsilon} \quad \text{for all} \quad \alpha, \beta \in N^2.$$

Set

$$\Phi(\sigma, \xi) = d_0(\xi) - \sigma \cdot \xi + (\varphi_\infty(x(\sigma)).$$

Lemma 8.1. *It holds that*

(8.4) $$d_0(\xi) \leqslant 0.$$

Proof. By the definition of $d_0(\xi)$ we have

$$d_0(\xi) = \lim_{p\to\infty} (\inf\{(|a_1 - x^{(2p-1)}| + |x^{(2p-1)} + x^{(2p-2)}| + \cdots + |x^{(1)} - x^{(0)}| - 2pd\})$$

where inferium is taken over $x^{(2p-1)}, x^{(2p-3)}, \ldots, x^{(1)} \in \Gamma_2$, $x^{(2p-2)}, x^{(2p-4)}, \ldots, x^{(2)} \in \Gamma_1$ and $x^{(0)} \in \mathscr{C}_{\varphi_0(\cdot,\xi)}(a_1)$. Since $\mathrm{dis}(a_2, \mathscr{C}_{\varphi_0(\cdot,\xi)}(a_1)) < d$ for $\xi \neq 0$, by choosing $x^{(2p-1)} = x^{(2p-3)} = \cdots = x^{(1)} = a_2$, $x^{(2p-2)} = x^{(2p-4)} = \cdots = x^{(2)} = a_1$ we see $\inf\{|a_1 - x^{(2p-1)}| + \cdots + |x^{(1)} - x^{(0)} - 2pd\} < 0$ for $\xi \neq 0$. This implies (8.4). Q. E. D.

Evidently $d_0(\xi) = 0$ for $\xi = 0$. Therefore we have from (8.4)

(8.5) $$[\partial^2_{\xi_i \xi_j} d_0(\xi)|_{\xi=0}]_{i,j=1,2} \leqslant 0.$$

From Remark 2 of §3 and Remark 1 of §5 we have

(8.6) $$[\partial^2_{\sigma_j \sigma_j} \varphi_\infty(x(\sigma))|_{\sigma=0}]_{i,j=1,2} \geqslant K > 0.$$

It is easy to check that $\sigma = \xi = 0$ is a stationary point of Φ. Since

$$\det \begin{bmatrix} \Phi_{\sigma\sigma} & \Phi_{\sigma\xi} \\ \Phi_{\xi\sigma} & \Phi_{\xi\xi} \end{bmatrix}_{\sigma=\xi=0} = \det \begin{bmatrix} \partial_\sigma^2 d_0 & -I \\ -I & \partial_\xi^2 \varphi_\infty \end{bmatrix}_{\sigma=\xi=0} \leqslant -1,$$

$\sigma = \xi = 0$ is a unique stationary point and it is non-degenerate. Thus we can apply the stationary phase method to an oscillatory integral (8.1). Because of $\Phi(0, 0) = 0$ we have

$$\begin{aligned} &|\gamma_{11} - \eta \sum_{|\nu|<2l} c_\nu (D^\nu_{\sigma,\xi} p(\sigma, \xi, k, \mu))_{\sigma=\xi=0} k^{-|\nu|/2}| \\ &\leqslant C_l k^{-l} \sum_{|\nu|<2l+5} \int |D^\nu_{\sigma,\xi} p(\sigma, \xi, k, \mu)| k^2 d\sigma d\xi \\ &\leqslant Ck^{-l+\varepsilon(2l+5)}, \end{aligned}$$

where η is a constant determined by Φ. Since $\partial_\sigma^\beta \theta_k(x(\sigma))_{\sigma=0} = 0$ for $\beta \neq 0$ we see that

$$D^\nu_{\sigma,\xi} p(\sigma, \xi, k, \mu)_{\sigma=\xi=0} = \sum_{j=0}^{N} (\sum_{h=0}^{\nu} c^\nu_{j,h}(\mu + ik)h)(ik)^{-j}$$

and we know $c^\nu_{j,h} = 0$ for $|\nu|$ odd. Thus we have

Lemma 8.2. *It holds that*

(8.7) $$\gamma_{11}(\mu, k) \sim \sum_{j=1}^{\infty} (\sum_{h=0}^{2j} c^{(1)}_{j,h}(\mu + ik)^h) k^{-j} \quad \text{for} \quad k \longrightarrow \infty,$$

where $c^{(1)}_{j,h}$ are constants.

In order to obtain an asymptotic expansion of γ_{20} we have to go back to the definition of $\tilde{\mathscr{E}}_2$. Denoting by $u_q(x, t; k, \xi)$ the one constructed following the process of §3 for

$$m(x, t; k)=e^{ik(\sigma\cdot\xi-t)}w_k(x)h(t-j(x)).$$

Set

$$I_p(x)=\int\int e^{ik(\varphi_{2p+1}(x,\xi)-t)}e^{-ik\sigma\cdot\xi+ik\varphi_\infty(x(\sigma))}h_p k^2 d\sigma d\xi, \tag{8.8}$$

$$h_p=\sum_{j=1}^{N} v_{j,2p+1}(x, t; k, \xi)(ik)^{-j}\theta_k(x(\sigma))a(x(\sigma), \mu, k).$$

Set

$$\Phi_p(x, \xi, \sigma)=\varphi_{2p+1}(x, \xi)-\sigma\cdot\xi+\varphi_\infty(x(\sigma)).$$

Lemma 8.3. *Let (ξ_p, σ_p) be a stationary point of Φ_p. Then*

$$\Phi_p(x, \xi_p, \sigma_p)=\tilde{\varphi}_\infty(x)-(2p+1)d. \tag{8.9}$$

Proof. Note that from the definition of $\varphi_0(x, \xi)$ we have

$$\sigma\cdot\xi=\varphi_0(x(\sigma), \xi).$$

As in Lemma 8.1 we have

$$\varphi_{2p+1}(x, \xi)-\sigma\cdot\xi=\inf\{|x-x^{(2p+1)}|+\cdots+|x^{(1)}-x^{(0)}|\}$$

where infimum is taken over $x^{(2p+1)}, x^{(2p-1)},\dots, x^{(1)}\in\Gamma_2$, $x^{(2p)}, x^{(2p-2)},\dots, x^{(2)}\in\Gamma_1$ and $x\in\mathscr{C}_{\varphi_0(\cdot,\xi)}(x(\sigma))$. Denote by $x_0^{(0)}, x_0^{(1)},\dots, x_0^{(2p+1)}$ the ponts which give the value of the infimum. By the argument in Lemma 4.1 of [5]

$$\partial_{\xi_j}(\varphi_{2p+1}(x, \xi)-\sigma\cdot\xi)=\lim|x_0^{(0)}-x_\varepsilon^{(0)}|/\Delta\xi_j$$

where $x_\varepsilon^{(0)}=\mathscr{C}_{\varphi_0(\cdot,\xi+\Delta\xi_j)}(x(\sigma))\cap$ line passing $x_0^{(0)}$ and $x_0^{(1)}$. Then if $x(\sigma)\neq x_0^{(0)}$ it follows that $\partial_\xi(\varphi_{2p+1}(x, \xi)-\sigma\cdot\xi)\neq 0$. Since we have from $(\nabla\varphi_0)(x(\sigma), \xi)-(\nabla\varphi_\infty)(x(\sigma))\neq 0$ that $\partial_\sigma(\varphi_0(x(\sigma), \xi)-\varphi_\infty(x(\sigma)))\neq 0$, if (ξ_p, σ_p) is a critical point of Φ_p it holds that

$$\nabla\varphi_0(x(\sigma_p), \xi_p)=\nabla\varphi_\infty(x(\sigma_p)), \tag{8.10}$$

and $x(\sigma_p)=x_0^{(0)}$ gives

$$\begin{aligned}&\varphi_{2p+1}(x, \xi_p)-\varphi_0(x(\sigma_p), \xi_p)\\&\quad=\inf\{|x-x^{(2p+1)}|+|x^{(2p+1)}-x^{(2p)}|+\cdots+|x^{(1)}-x(\sigma_p)|\}.\end{aligned} \tag{8.11}$$

By taking account of (2.3) of [4] we have from (8.10) and (8.11)

$$\varphi_{2p+1}(x, \xi_p)-\varphi_0(x(\sigma_p), \xi_p)=(\varphi_\infty(x)+(2p+1)d)-\varphi_\infty(x(\sigma_p)).$$

Thus we have (8.9). Q.E.D.

Lemma 8.4. *It holds that*

$$\gamma_{20}(\mu, k) \sim \sum_{j=1}^{\infty} \Big(\sum_{h=0}^{2j} c_{j,h}^{(2)}(\mu+ik)^h\Big)k^{-j}. \tag{8.12}$$

Proof. By the same argument as in Lemma 8.1 we have

$$\varphi_{2p+1}(a_1, \xi)-\varphi_0(a_1, \xi)\leqslant(2p+2)d$$

and

$$\varphi_{2p+1}(a_1, 0)-\varphi_0(a_1, 0)=(2p+2)d.$$

Therefore we have

$$[\partial^2_{\xi_i\xi_j}(\varphi_{2p+1}(a_1, \xi)-\varphi_0(a_1, \xi))]\leqslant 0.$$

Then we have from (8.4) and the above inequality

$$\det\begin{bmatrix} (\Phi_p)_{\sigma\sigma} & (\Phi_p)_{\sigma\xi} \\ (\Phi_p)_{\sigma\xi} & (\Phi_p)_{\xi\xi}\end{bmatrix}\leqslant -1.$$

Note that $\{\nabla\varphi_{2p+1}\}_{p=1}^{\infty}$ is a bounded set in $C^\infty(\overline{\omega(\delta)})$ and $\nabla\varphi_{2p+1}\to\nabla\tilde{\varphi}_\infty$ as $p\to\infty$ in $C^\infty(\overline{\omega(\delta)})$. Therefore Φ_p can be transformed into a quadratic form uniformly in p. Thus by applying a stationary phase method to (8.8) we have

$$\begin{aligned}|I_p(x)-e^{ik(2p+1)d}e^{ik\tilde{\varphi}_\infty(x)}\eta_p\sum_{|\nu|<2l}c_{p,\nu}\\ \cdot(D^\nu_{\sigma,\xi}h_p(x, t, \sigma, \xi, \mu; k))_{\substack{\sigma=\sigma_p\\ \xi=\xi_p}})k^{-|\nu|1/2}|\leqslant Ck^{-l}.\end{aligned}$$

For $x\in S_1((1+\delta)k^{-\varepsilon})$ we have $X^\infty_{-2j}(X)\in S_1(k^{-\varepsilon})$, $|\Xi^\infty_{2j}(x)'|\leqslant C\alpha^{2j}$ for $j\geqslant 1$. Thus $\theta_k=1$ near $x(\sigma_p)$. By using $\tilde{\varphi}_\infty(x)=\varphi_\infty(x)+d$ on Γ_1, we have for $x\in\Gamma_1$

$$|I_p(x)-e^{ik(2p+2)d}e^{ik\varphi_\infty(x)}\eta_p\sum_{|\nu|<2l}c_{p,\nu}D^\nu_{\sigma,\xi}\Big(\sum_{j=1}^{N}v_{j,2p+1}(ik)^{-j}a\Big)k^{-|\nu|/2}|\leqslant Ck^{-l}.$$

Since $|\nabla\varphi_{2p+1}-\nabla\tilde{\varphi}_\infty|_m\leqslant C_m\alpha^{2p}$, $|\eta_p-\eta_\infty|+|c_{p,\nu}-c_{\infty,\nu}|\leqslant C\alpha^{2p}$, $|\xi_p|+|\sigma_p|\leqslant C\alpha^{2p}$ we have

$$(E_2\theta_k a)(x)\sim e^{ik\varphi_\infty(x)}\sum_{j=1}^{\infty}\Big(\sum_{h=0}^{2j}c_{j,h}(x)(\mu+ik)^h\Big)k^{-j}.$$

Recalling the definition of $\tilde{\mathscr{E}}_2$ we have from the above expansion the required expansion (8.12). Q.E.D.

By combining (7.8), (8.7) and (8.12) we have

$$|\gamma(\mu, k)-\sum_{j=1}^{N-1}\Big(\sum_{h=0}^{2j}c_{j,h}(\mu+ik)^h\Big)k^{-j}|\leqslant C_N k^{-N}. \tag{8.13}$$

Proposition 8.5. *For an integer l we set $k=\pi l/d$. When $|l|$ is large, an equation in μ*

$$\mathscr{P}(\mu)-\gamma(\mu, k)=0$$

has exactly one zero $\mu_{(-l)}$ *in* $D_l=\{\mu;\ |\mu-\mu_{-l}|\leqslant C(1+|l|)^{-1/2}\}$, $\mu_{-l}=-c_0+i\frac{\pi}{d}(-l)$. *Moreover we have an asymptotic expansion of* $\mu_{(-1)}$

$$(8.14)\qquad |\mu_{(-l)}-(\mu_{-l}+\zeta_1 l^{-1}+\zeta_2 l^{-2}+\cdots+\zeta_N l^{-N})|\leqslant C_N|l|^{-N},$$

where ζ_j, $j=1, 2,\ldots, N$, *are complex constants.*

Proof. Note that $\mathscr{P}(\mu_{-l})=0$ and $\left|\frac{\partial}{\partial\mu}(\mathscr{P}-\gamma(\cdot, k))\right|\geqslant 2d-C|l|^{-1}$, $\frac{\partial}{\partial\mu}(\mathscr{P}(\mu)-\gamma(\mu, k))$ is bounded in $\operatorname{Re}\mu\geqslant -c_0-c_1$. By applying the implicit function theorem we see the unique existence of zero in D_l. From (8.13) we have (8.14). Q. E. D.

Let $u(x)\neq 0$ be an outgoing solution of

$$(8.15)\qquad \begin{cases} (\mu_{(-l)}^2-\Delta)u=0 & \text{in}\quad \Omega \\ u=0 & \text{on}\quad \Gamma. \end{cases}$$

Since we have $u(x)\in C^\infty(\bar{\Omega})$ from the regularity theorem for Δ, $u(x)$ can be extended into $\mathcal{O}$ so that it is also in $C^\infty(\boldsymbol{R}^3)$. Denote by $\tilde{u}(x)$ the extended one. Set

$$(8.16)\qquad (\mu_{(-l)}^2-\Delta)\tilde{u}=f(x)\quad \text{in}\quad \boldsymbol{R}^3.$$

Then from (8.15) $f(x)\in C^\infty(\boldsymbol{R}^3)$ and

$$(8.17)\qquad \operatorname{supp} f\subset\bar{\mathcal{O}}.$$

Let $g(x, \mu)$ be an outgoing solution of

$$(8.18)\qquad (\mu^2-\Delta)u=f\quad \text{in}\quad \boldsymbol{R}^3.$$

Note that (8.18) can be solved for all $\mu\in\boldsymbol{C}$. From the uniqueness of the outgoing solutions of (8.18) we have

$$(8.19)\qquad g(x, \mu_{(-l)})=\tilde{u}(x)\quad \text{in}\quad \boldsymbol{R}^3.$$

Set

$$v_2(x, \mu)=U_2(\mu)[g(\cdot, \mu)]_{\Gamma_2},$$
$$h(x, \mu)=g(x, \mu)|_{\Gamma_1}-v_2(x, \mu)|_{\Gamma_1}.$$

We have from (8.19)

$$(8.20)\qquad v_2(x, \mu_{(-l)})=0\quad \text{in}\quad \boldsymbol{R}^3-\bar{\mathcal{O}}_2,$$

$$(8.21)\qquad h(x, \mu_{(-l)})=0\quad \text{on}\quad \Gamma_1.$$

Set

$$v(x, \mu)=v_1(x, \mu)+v_2(x, \mu),$$
$$v_1(x, \mu)=U(\mu;\ k)h(\cdot, \mu)=(\beta(\mathscr{P}-\gamma)^{-1}F(I+\mathscr{E})+\tilde{U})h(\cdot, \mu).$$

Evidently v is outgoing and satisfies

$$\begin{cases} (\mu^2-\Delta)v=0 & \text{in} \quad \Omega \\ v=g & \text{on} \quad \Gamma, \end{cases}$$

for $0<|\mu-\mu_{(-l)}|<C(\log k)^{-1}$. Applying once more the uniqueness of the outgoing solutions we have

$$v(x, \mu)=g(x, \mu) \quad \text{in} \quad \Omega \qquad \text{for} \quad 0<|\mu-\mu_{(-l)}|<C(\log k)^{-1}.$$

Since h is $C^\infty(\Gamma_1)$-valued holomorphic function in $\operatorname{Re}\mu>-c_0-c_1$ (8.21) implies the existence of

$$\lim_{\mu\to\mu_{(-l)}} (\mathscr{P}(\mu)-\gamma(\mu, k))^{-1}h(x, \mu)=h_0(x)\in C^\infty(\Gamma_1). \tag{8.22}$$

Thus $(\mathscr{P}(\mu)-\gamma(\mu, k))^{-1}h(x, \mu)$ is holomorphic at $\mu=\mu_{(-l)}$. Then $v_1(x, \mu)$ is holomorphic at $\mu=\mu_{(-l)}$. Therefore $\lim_{\mu\to\mu_{(-l)}} v(x, \mu)$ exists and it satisfies

$$v(x, \mu_{(-l)})=[\beta(x, \mu; k)F(\mu, k)(I+\mathscr{E}(\mu; k))h_0(x)]_{\mu=\mu_{(-l)}}.$$

Since

$$v(x, \mu_{(-l)})=g(x, \mu_{(-l)})=u(x) \quad \text{in} \quad \Omega,$$

recalling the fact $F(I+\mathscr{E})h_0\in \boldsymbol{C}$, we have

$$u(x)=c\beta(x, \mu_{(-l)}, \pi l/d), \quad c\in \boldsymbol{C}.$$

This shows that

$$\dim\{u\colon \text{outgoing solution of (8.15)}\}=1.$$

9. Derivation of Theorem 1.

By using Theorem 5.1 of Chapter V of [7] we have the assertions (a) and (b) of Theorem 1 from Theorems 2 and 3. Then it suffices to show (c). By Theorem 5.4 of Chapter V of [7] we have for $\sigma\in \boldsymbol{R}$

$$\mathscr{S}(\sigma)=I+\mathscr{K}(\sigma), \quad (\mathscr{K}(\sigma)f)(\omega)=\left(\frac{i\sigma}{\pi}\right)\int_{|\theta|=1} K(\omega, \theta; \sigma)f(\theta)d\theta,$$

where $K(\omega, \theta; \sigma)=s(-\theta, \omega; \sigma)$,

$$v_-(r\theta, \omega; \sigma)\sim\frac{e^{i\sigma r}}{r}s(\theta, \omega; \sigma) \quad \text{as} \quad r\longrightarrow\infty,$$

v_- is the incoming solution of

$$\begin{cases} (\sigma^2+\Delta)v=0 & \text{in} \quad \Omega, \\ v=e^{-i\sigma x\cdot\omega} & \text{on} \quad \Gamma. \end{cases} \tag{9.1}$$

Note that

$$v_-(x, \omega, \sigma)=\overline{v_+(x, -\omega, \sigma)},$$

where v_+ is the outgoing solution of (9.1). Thus we have

$$v_+(r, -\omega, \sigma) \sim \frac{e^{-i\sigma r}}{r} K(-\theta, \omega, \sigma).$$

Setting $z=\sigma+iv$ we see that v_+ is analytic in z for $\mathrm{Im}\, z \leqslant 0$ and

$$v_+(x, \omega, z) = U(iz)(e^{-izx\cdot\omega}|_\Gamma)(x).$$

Taking account of (4.20) and (4.21)′ of page 127 of [7] we see from (1.4) that $K(\theta, \omega, \sigma)$ is prolonged analytically into $\{z;\ U(\mu)$ is holomorphic at $\mu=iz\}$, and has a pole of order 1 at $z=i\mu_{(l)}$. Since we have

$$\mathscr{S}(z) = (\mathscr{S}(\bar{z})^*)^{-1} = (I+\mathscr{K}(\bar{z})^*)^{-1},$$

we have from the argument of §4 of Chapter 9 of [11],

$$\mathscr{S}(z)f = \sum_{m=1}^{M} \frac{n_m}{z-i^{-1}\mu_{(l)}}(f, \psi_m) + \mathscr{H}(z)f \quad \text{near} \quad z=i^{-1}\mu_{(l)}.$$

On the other hand Corollary 3.2 of Chapter III of [7] says that

$$\begin{aligned} &\dim[\text{null space of } (\mu I - B)] = \dim[\text{null space of } \mathscr{S}^*(i\bar{\mu})] \\ &= \dim\{\text{eigenvector of } \mathscr{K}(i\bar{\mu})^* \text{ for eigenvalue } -1\} \\ &= \dim\{\mu\text{-outgoing solution of } (\mu^2 - \Delta)u=0 \text{ in } \Omega,\ u=0 \text{ on } \Gamma\}. \end{aligned}$$

Therefore we have $M=1$ from Theorem 3. This proves (c).

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