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On the Levi condition for Goursat problem

By

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We consider the Goursat problem in the class of C^∞-function. First, we consider the case of constant coefficients. We give a Levi's condition which is analogous to A. Lax's theorem [2] for the hyperbolic operator. Next, we consider the case of variable coefficients. In this case we give a sufficient condition for wellposedness of Goursat problem.

Part 1, constant coefficients

§ 1. Introduction and results.

Let us consider the following differential operator.

$$(1.1)\quad P(D_t, D_x, D_y)=\sum_{j=l}^{m} C_j(D_x, D_y)D_t^{m-j}, \quad t\geqq 0, \quad x\in R^1, \quad y\in R^n,$$

$$D_t=-i\frac{\partial}{\partial t},\ D_x=-i\frac{\partial}{\partial x},\ D_y=\left(-i\frac{\partial}{\partial y_1},\ -i\frac{\partial}{\partial y_2},\dots,\ -i\frac{\partial}{\partial y_n}\right) \text{ where}$$

$C_j(\zeta, \eta)$ is a polynomial with constant coefficients of order $\leqq j$ and $\mathring{C}_l(1, 0)=1$ ($\mathring{C}_l$ is the homogeneous part of degree l of C_l).
Let us consider the following problem (we say Goursat problem).

$$(\mathrm{P})\qquad \begin{cases} Pu=0, \quad t\geqq 0, \quad x\in R^1, \quad y\in R^n \\ D_t^i u(0, x, y)=\phi_i(x, y)\in \mathscr{E}_{(x,y)}, \quad 0\leqq i\leqq m-l-1 \\ D_x^j u(t, 0, y)=\psi_j\in \mathscr{E}_{(t,y)}, \quad 0\leqq j\leqq l-1, \quad t\geqq 0 \end{cases}$$

where we impose among $\{\phi_i\}$ and $\{\psi_j\}$ the following compatibility condition;

$$(\mathrm{C})\quad D_x^j\phi_i(0, y)=D_t^i\psi(0, y), \quad 0\leqq i\leqq m-l-1, \quad 0\leqq j\leqq l-1, \quad y\in R^n.$$

We say that the Goursat problem (P) is $\mathscr{E}$-wellposed if for any data $\{\phi_i\}$, $\{\psi_j\}$ with compatibility condition (C), there exists a unique solution $u(t, x, y)\in \mathscr{E}_{(t,x,y)}$, $t\geqq 0$.

T. Nishitani [3] investigated the above Goursat problem (P). Some of his results are the following:

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Theorem 1.1. *In order that* (P) *is* $\mathscr{E}$*-wellposed it is necessary and sufficient that the following condition* (G) *is fulfilled.*

$$\text{(G)} \qquad \begin{cases} \textit{There exists a positive constant } \varepsilon>0 \textit{ such that for} \\ \textit{every } \delta \textit{ with } 0<|\delta|<\varepsilon,\ P(D_t, D_x, D_y) \textit{ is hyperbolic with} \\ \textit{respect to } (1, \delta, 0). \end{cases}$$

Theorem 1.2. *If* (P) *is* $\mathscr{E}$*-wellposed, then the principal part* P_m *of* P *is decomposed as follows:*

$$P_m(\tau, \zeta, \eta)=\mathring{C}_l(\zeta, \eta)Q_{m-l}(\tau, \zeta, \eta).$$

i.e. $\mathring{C}_j(\zeta, \eta)$ *(the principla part of* $C_j(\zeta, \eta)$*) is divisible by* $\mathring{C}_l(\zeta, \eta)$. *And moreover* Q_{m-l} *is hyperbolic with respect to* (1, 0, 0).

Theorem 1.3. *If* (P) *is* $\mathscr{E}$*-wellposed then there exists a positive constant* L, *the root* $\tau(\xi, \eta; r)$ *of* $P(\tau, \xi+ir, \eta)=0$ *has the following estimate;*

$$\operatorname{Im}\tau(\xi, \eta; r)>-K|r|, \quad (\xi, \eta)\in R^{n+1}, \quad r\in R^1, \quad |r|>L, \tag{1.2}$$

where K *is constant which is independent of* (ξ, η).

Theorem 1.3 is correspond to Hadamard's inequality for hyperbolic operator. Theorem 1.3 is due to Corollary 3.1, p. 184 in [3].

Theorem 1.4. (P) *is* $\mathscr{E}$*-wellposed then* $C_l(D_x, D_y)$ *is hyperbolic with respect to* (1, 0).

According to Theorem 1.2 and Theorem 1.4, if (P) is $\mathscr{E}$-wellposed then P_m is the following;

$$P_m(\tau, \zeta, \eta)=\prod_{j=1}^{n'}(\zeta-\lambda_j(\eta))^{\nu_j}\prod_{i=1}^{n''}(\tau-\tau_i(\zeta, \eta))^{\rho_i} \tag{1.3}$$

$$\sum_{j=1}^{n'}\nu_j=l, \quad \sum_{i=1}^{n''}\rho_i=m-l$$

where $\lambda_j(\eta)$, $\tau_i(\zeta, \eta)$ are homogeneous degree 1 and real for $\eta\in R^n$, $(\zeta, \eta)\in R^{n+1}$ respectively.
Here we assume that the multiplicity of roots are constant. Namely

$$\text{(A)} \quad \begin{cases} \lambda_j(\eta)\neq\lambda_{j'}(\eta) & \text{for } j\neq j', \quad \eta\in R^n, \quad \eta\neq 0, \\ \tau_i(\zeta, \eta)\neq\tau_{i'}(\zeta, \eta) & \text{for } i\neq i', \quad (\zeta, \eta)\in R^{n+1}, \quad (\zeta, \eta)\neq(0, 0). \end{cases}$$

Let

$$P=P_m+\sum_{k=1}^{m}P_{m-k} \tag{1.4}$$

where P_{m-k} is a homogeneous part of degree $m-k$ of P. Our result is the follouing;

Theorem 1. *Under the assumption* (1.3) *and* (A), *in order that* (P) *is* $\mathscr{E}$*-well-posed it is necessary and sufficient that* P_{m-k} *is the follouing*

$$(1.5)\qquad P_{m-k}(\tau, \zeta, \eta) = \sum_{k_1+k_2=k} q_{k_1k_2}(\tau, \zeta, \eta) \prod_{j=1}^{n'} (\zeta-\lambda_j)^{\nu_j-k_1} \prod_{i=1}^{n''} (\tau-\tau_i)^{\rho_i-k_2}. \text{[1]}$$

where $0 \leqq k_1 \leqq \max_j \nu_j = \mathring{\nu}$, $0 \leqq k_2 \leqq \max_i \rho_i = \mathring{\rho}$

$$(\zeta-\lambda_j)^{\nu_j-k_1}=1 \quad \textit{for} \quad \nu_j-k_1 \leqq 0,$$

$$(\tau-\tau_i)^{\rho_i-k_2}=1 \quad \textit{for} \quad \rho_i-k_2 \leqq 0$$

and $q_{k_1k_2}(\tau, \zeta, \eta)$ *is polynomial of* (τ, ζ, η).

Before proving this theorem we make the reduction of the operator P. First, let us put $m-l=m'$. The order of τ of P_{m-k} is at most m'. Then we can write

$$(1.6)\qquad P = C_l \prod_{i=1}^{n''} (\tau-\tau_i)^{\rho_i} + \sum_{k=1}^{m} \tilde{P}_{m-k}$$

where the order of τ of $\tilde{P}_{m-k}$ is at most $m'-1$. Let $C_{l,h}$ be the homogeneous part of degree $l-h$ of C_l,

$$(1.7)\qquad C_l = \mathring{C}_l + C_{l,1} + C_{l,2} + \cdots + C_{l,l}.$$

Because of Theorem 1.4 and A. Lax's theorem, $C_{l,h}$ is divisible by $\prod_{j=1}^{n'} (\zeta-\lambda_j)^{\nu_j-h}$. Let us write

$$(1.8)\qquad C_{l,h} = a_{(h)}(\zeta, \eta) \prod_{j=1}^{n'} (\zeta-\lambda_j)^{\nu_i-h} \quad \text{for} \quad h \leqq \mathring{\nu}.$$

Remark 1.1. The homogeneous part of degree $m-k$ of $C_l \prod_{i=1}^{n''} (\tau-\tau_i)^{\rho_i}$ is divisible by $\prod_{j=1}^{n'} (\zeta-\lambda_j)^{\nu_j-k} \prod_{i=1}^{n''} (\tau-\tau_i)^{\rho_i}$.

§ 2. The proof of necessity of Theorem 1.

At first we prove the following:

Proposition 2.1. *If* (P) is $\mathscr{E}$*-wellposed then* $\tilde{P}_{m-k}$ *is divisible by* $\prod_{j=1}^{n'} (\zeta-\lambda_j)^{\nu_j-k+1}$ *for* $k \leqq \mathring{\nu}$.

Proof. Let us give a rouch sketch of the proof of the proof of Prop. 2.1. We assume (P) to be $\mathscr{E}$-wellposed. If for some k and j_0, $P_{m-k}(\tau, \zeta, \eta)$ is not divisible

1) According to the theorem of analytic functions $\prod_{j=1}^{n'} (\zeta-\lambda_j)^{\nu_{jk_1}} \prod_{i=1}^{n''} (\tau-\tau_{ii})^{\rho_i-k_2}$ is polynomial of (τ, ζ, η).

by $(\zeta-\lambda_{j_0})^{\nu_{j_0}-k+1}$ then we can find a root $\tau(\zeta, \eta)$ of $P(\tau, \zeta, \eta)=0$ which does not satisfy the inequality (1.2) in Theorem 3.1.
Without loss of genelality, we can consider $j_0=1$. Put

(2.1)
$$\tilde{P}_{m-k}(\tau, \zeta, \eta)=(\zeta-\lambda_1(\eta))^{\nu_1-k+1}\, Q^{(k)}_{m-\nu_1-1}(\tau, \zeta, \eta) + \sum_{s=0}^{\nu_1-k} (\zeta-\lambda_1(\eta))^s\, q^{(k)}_{m-k-s}(\tau, \eta).$$

Then

(2.2)
$$P(\tau, \zeta, \eta)= C_l \prod_{i=1}^{n''} (\tau-\tau_i)^{\rho_i} + \sum_{k=1}^{\nu_1} (\zeta-\lambda_1(\eta))^{\nu_1-k+1}\, Q^{(k)}_{m-\nu_1-1}(\tau, \zeta, \eta) + \sum_{k=1}^{\nu_1} \sum_{s=0}^{\nu_1-k} (\zeta-\lambda_1(\eta))^s\, q^{(k)}_{m-k-s}(\tau, \eta) + R_{m-\nu_1-1}(\tau, \zeta, \eta).$$

Where $R_{m-\nu_1-1}(\tau, \zeta, \eta)=\sum_{k=\nu_1+1}^{m} P_{m-k}$.
Being $\lambda_1(\eta)$ homogeneous degree 1, we have

(2.3)
$$\lambda_1(\eta)=|\eta|\lambda_1(\omega), \quad \eta\in R^n, \quad \omega\in\Omega=\{\eta;\ |\eta|=1\}.$$

Put

(2.4)
$$\zeta=|\eta|\lambda_1(\omega)+ir+\mathring{\xi}$$

where r, $\mathring{\xi}$ are real and $|r|>L$ (appear in Theorem 1.3). And consider the root of (2.5).

(2.5)
$$P(\tau, |\eta|\lambda_1(\omega)+\mathring{\xi}+ir, \eta)=0$$

If we show the following two lemmas, the proof of Prop. 2.1 is complete.

Lemma 2.1. *If $q^{(k)}_{m-k-s}(\tau, \eta)\not\equiv 0$ for some (k, s) with $1\leqq k\leqq \nu_1$ and $0\leqq s<\nu_1-k$, then there exists τ, a root of (2.5), which has the following expantion in the neithborhood of $|\eta|=\infty$ for some r, $\mathring{\xi}$ and ω.*

(2.6)
$$\tau=c|\eta|^{\alpha}+c'|\eta|^{\alpha'}+c''|\eta|^{\alpha''}+\cdots$$
$$\alpha>\alpha'>\alpha''\cdots, \quad \alpha>1, \quad \operatorname{Im} c<0.$$

Lemma 2.2. *If $q^{(k)}_{m-\nu_1}(\tau, \eta)\not\equiv 0$ for some k $(1\leqq k\leqq \nu_1)$, then for some r, $\mathring{\xi}$ and ω, there exists a root of (2.5) which has the following expansion in the neithborhood of $|\eta|=\infty$.*

(2.7)
$$\tau=c|\eta|+c'|\eta|^{\alpha'}+c''|\eta|^{\alpha''}+\cdots$$
$$1>\alpha'>\alpha''>\cdots, \quad \operatorname{Im} c<0.$$

Proof of Lemmas. Dividing $P(\tau, \zeta, \eta)=0$ by C_l, we have

(2.8)
$$\prod_i (\tau-\tau_i)^{\rho_i}$$
$$+K(\zeta, \eta) \sum_{k=1}^{\nu_1} Q^{(k)}_{m-\nu_1-1}(\tau, \zeta, \eta)/\{(\zeta-\lambda_1)^{k-1} \prod_{j\neq 1} (\zeta-\lambda_j)^{\nu_j}\}$$
$$+K \sum_{k=1}^{\nu_1} \sum_{s=0}^{\nu_1-k} q^{(k)}_{m-k-s}(\tau, \eta)/\{(\zeta-\lambda_1)^{\nu_1-s} \prod_{j\neq 1} (\zeta-\lambda_j)^{\nu_j}\}$$
$$+KR_{m-\nu_1-1}(\tau, \zeta, \eta)/ \prod_j (\zeta-\lambda_j)^{\nu_j}=0.$$

where $K(\zeta, \eta)=\mathring{C}_l(\zeta, \eta)/C_l(\zeta, \eta)$.
Because of (1.7), (1.8), for $|ir+\mathring{\xi}|$ large, $K(\zeta, \eta)$ has a limit when $|\eta|\to\infty$. Let

(2.9)
$$\lim_{|\eta|\to\infty} K(|\eta|\lambda_1(\omega)+\mathring{\xi}+ir, \eta)=\mathring{K}(\mathring{\xi}+ir, \omega).$$

Moreover we have

(2.10)
$$\lim_{|\mathring{\xi}+ir|\to\infty} K(\mathring{\xi}+ir, \omega)=1.$$

For fixed $\omega\in\Omega$, let us write

(2.11)
$$q^{(k)}_{m-k-s}(\tau, \eta)= \sum_{p=0}^{p_{ks}} a_{k,s,p}\tau^p|\eta|^{m-k-s-p}$$

where $p_{ks}\leqq \min\{m'-1, m-k-s\}$ and $a_{k,s,p_{ks}}\neq 0$. Let (2.6) be the root of (2.8) and let substitute (2.6) into (2.8). The highest order of $|\eta|$ in $\prod_i (\tau-\tau_i)^{\rho_i}$ is $m'\alpha$. The order of $|\eta|$ in the second and fourth terms in (2.8) is less than $\alpha(m'-1)$. By (2.11), the highest order of $|\eta|$ in $q^{(k)}_{m-k-s}(\tau, \eta)/\{(\zeta-\lambda_1)^{\nu_1-s}\prod_{j\neq 1}(\zeta-\lambda_j)^{\nu_j}\}$ is

(2.12)
$$\alpha p_{ks}+m-k-s-p_{ks}-(1-\nu_1)$$

Let α_{ks} be the α, which is obtained by (2.12)$=m'\alpha$. Namely

(2.13)
$$\alpha_{ks}=\{m'-p_{ks}+(\nu_1-k)-s\}/(m'-p_{ks})=1+\{(\nu_1-k-s)/(m'-p_{ks})\}.$$

Notice that $\alpha_{ks}>1$ for $0\leqq s<\nu_1-k$. Let

(2.14)
$$\mathring{\alpha}=\max_{1\leqq k\leqq \nu_1, 0\leqq s<\nu_1-k} \alpha_{ks}$$

(2.15)
$$\mathring{A}=\{(k, s); \alpha_{ks}=\mathring{\alpha}\},$$

then

(2.16)
$$\mathring{\alpha}p_{ks}+m-k-s-p_{ks}-(1-\nu_1)<\mathring{\alpha}m', \qquad \text{for} \quad (k, s)\notin \mathring{A}.$$

Let $\alpha=\mathring{\alpha}$ in (2.6), the coefficient c of $|\eta|\mathring{\alpha}$ is determined by the follouing equation;

(2.17)
$$c^{m'}+K \sum_{(k,s)\in\mathring{A}} c^{p_{ks}}a_{k,s,p_{ks}}/(ir+\mathring{\xi})^{\nu_1-s} \prod_{j\neq 1} \{\lambda_1(\omega)-\lambda_j(\omega)\}^{\nu_j}=0.$$

We will show that for some r, $\mathring{\xi}$ the equation (2.17) has root c with Im $c<0$. Let

(2.18) $$\max_{(k,s)\in\mathring{A}} p_{ks}=\mathring{p}.$$

(2.19) $$\mathring{A}'=\{(k, s); (k, s)\in\mathring{A}, p_{ks}=\mathring{p}\}.$$

Differentiating (2.17) $\mathring{p}$ times by c, we have

(2.20) $$c^{m'-\mathring{p}}+\mathring{K}K_1\sum_{(k,s)\in\mathring{A}'} a_{k,s,\mathring{p}}/(ir+\mathring{\xi})^{\nu_1-s}=0,$$

where K_1 is constant independent of r and $\mathring{\xi}$ (but depends on ω). By (2.13)

(2.21) $$k=(1-\mathring{\alpha})(m'-\mathring{p})+\nu_1-s, \quad (k, s)\in\mathring{A}'.$$

Namely when $(k, s)\in\mathring{A}'$, if s is fixed then k is determined unique. Then $\sum_{(k,s)\in\mathring{A}'} a_{k,s,\mathring{p}}/(ir+\mathring{\xi})^{\nu_1-s}$ is polynomial of $1/(ir+\mathring{\xi})$ and is not identically zero.

When $m'-\mathring{p}\geqq 3$, there exits *a root of* (2.20) such that $\operatorname{Im} c<0$ for r, $\mathring{\xi}$ with $\sum_{(k,s)\in\mathring{A}'} a_{k,s,\mathring{p}}/(ir+\mathring{\xi})^{\nu_1-s}\neq 0$. In the case $m'-\mathring{p}\leqq 2$, considering (2.10), for proper r, $\mathring{\xi}$ with $|ir+\mathring{\xi}|$ large, (2.20) has a root c with $\operatorname{Im} c<0$. Because of Lemma 8.1 in Appendix, (2.17) has a root c with $\operatorname{Im} c<0$ for some $(r, \mathring{\xi})\in R^2$. This complete the proof of Lemma 2.1.

Next let us prove Lemma 2.2. Because of Lemma 2.1, $q^{(k)}_{m-k-s}\equiv 0$ for $k+s\neq\nu_1$. Then (2.8) becomes the following;

(2.8′) $$\prod_i(\tau-\tau_i)^{\rho_i}$$
$$+K(\zeta, \eta)\sum_{k=1}^{\nu_1} Q^{(k)}_{m-\nu_1-1}(\tau, \zeta, \eta)/(\zeta-\lambda_1)^{k-1}\prod_{j\neq 1}(\zeta-\lambda_j)^{\nu_j}$$
$$+K\sum_{k=1}^{\nu_1} q^{(k)}_{m-\nu_1}(\tau, \eta)/(\zeta-\lambda_1)^k\prod_{j\neq 1}(\zeta-\lambda_j)^{\nu_j}$$
$$+KR_{m-\nu_1-1}(\tau, \zeta, \eta)/\prod_j(\zeta-\lambda_j)^{\nu_j}=0.$$

Let (2.7) be a root of (2.8′). Substituting (2.7) into (2.8′), the highest order of $|\eta|$ is m'. Consider the coefficient of $|\eta|^{m'}$. Because of (2.22),

(2.22) $$\tau_j(|\eta|\lambda_1(\omega), \eta)=|\eta|\tau_j(\lambda_1(\omega), \omega),$$

the coefficient c of $|\eta|$ in (2.7) is determined by the following:

(2.23) $$\prod_{j=1}^{n''}(c-\mathring{\tau}_j)^{\rho_j}$$
$$+\mathring{K}\sum_{k=1}^{\nu_1} q^{(k)}_{m-\nu_1}(c, \omega)/(ir+\mathring{\xi})^k\prod_{j\neq 1}\{\lambda_1(\omega)-\lambda_j(\omega)\}^{\nu_j}=0$$

where $\mathring{\tau}_j=\tau_j(\lambda_1(\omega), \omega)$. For proper $(r, \mathring{\xi})\in R^2$, there exists a root c with $\operatorname{Im} c\neq 0$. If we replace ω by $-\omega$, c becomes $-c$. Then for some $(r, \mathring{\xi})\in R^2$ and $\omega\in\Omega$, (2.23) has a root c with $\operatorname{Im} c<0$. Thus we complete the proof of Lemma 2.2.

Next, we prove the following:

Proposition 2.2. *Let us consider* $\tilde{P}_{m-k}(\tau, \zeta, \eta)$ *be a polynomial of* $\tau-\tau_i$ $(=\tilde{\tau})$. *If* (P) *is* $\mathscr{E}$*-wellposed then the coefficient of* $\tilde{\tau}^{\rho_i-s}$ *is divisible by* $(\zeta-\lambda_j(\eta))^{\nu_j-k+s}$ *for* $1\leqq j\leqq n'$ *and* $1\leqq i\leqq n''$, *where* $\rho_i\geqq s\geqq 1$ *for* $1\leqq k\leqq \nu_j$ *and* $\rho_i\geqq s>k-\nu_i$ *for* $\nu_j<k<\nu_j+\rho_i$.

Proof. Without loss of generality we can consider $i=1$, $j=1$. And let us write ν and ρ instead of ν_1 and ρ_1 respectively. In the case $1\leqq k\leqq \nu$, by Proposition 2.1 and the theory of analytic function we can write

$$\tilde{P}_{m-k}=(\zeta-\lambda_1(\lambda))^{\nu-k+1}\tilde{\tau}^{\rho-1}q_{k,1}(\tilde{\tau}, \zeta, \eta) + \sum_{s=2}^{\rho}\tilde{\tau}^{\rho-s}(\zeta-\lambda_1(\eta))^{\omega(k,s)}q_{k,s}(\zeta, \eta), \tag{2.24}$$

$$q_{k,s}(\lambda_1(\eta), \eta)\not\equiv 0, \quad \omega(k, s)\geqq \nu-k+1,$$

where $\omega(k, s)$ is not negaive integer and

$$\begin{cases} \nu-k+1+\rho-1+\text{ order } q_{k,1}=m-k, \\ \rho-s+\omega(k, s)+\text{ order } q_{k,s}=m-k. \end{cases} \tag{2.25}$$

In the case $\nu<k<\nu+\rho$, we can write

$$\tilde{P}_{m-k}=\tilde{\tau}^{\rho-(k-\nu)}q_{k,k-\nu}(\tilde{\tau}, \zeta, \eta) + \sum_{s=k-\nu+1}^{\rho}\tilde{\tau}^{\rho-s}(\zeta-\lambda_1)^{\omega(k,s)}q_{k,s}(\zeta, \eta), \tag{2.26}$$

where $q_{k,s}(\lambda_1(\eta), \eta)\not\equiv 0$, and

$$\begin{cases} \rho-(k-\nu)+\text{ order } q_{k,k-\nu}=m-k, \\ \rho-s+\omega(k, s)+\text{ order } q_{k,s}=m-k. \end{cases} \tag{2.27}$$

We are going to prove

$$\omega(k, s)\geqq \nu-(k-s). \tag{2.28}$$

Let

$$P(\tilde{\tau}+\tau_1, \zeta, \eta)=\tilde{P}_m+\sum_{k=1}^{\nu}\tilde{P}_{m-k}+\sum_{k=\nu+1}^{\nu+\rho-1}\tilde{P}_{m-k}+R_{m-(\nu+\rho)}, \tag{2.29}$$

where $\tilde{P}_m=C_l\prod_{i=1}^{n''}(\tau-\tau_i)^{\rho_i}=C_l\tilde{\tau}^{\rho}\prod_{i\neq 1}(\tilde{\tau}+\tau_1-\tau_i)^{\rho_i}$. Substituting (2.24) and (2.26) into (2.29), we have

$$P=\tilde{P}_m+\sum_{k=1}^{\nu}(\zeta-\lambda_1(\eta))^{\nu-k+1}\tilde{\tau}^{\rho-1}q_{k,1}(\tilde{\tau}, \zeta, \eta) + \sum_{k=1}^{\nu}\sum_{s=2}^{\rho}\tilde{\tau}^{\rho-s}(\zeta-\lambda_1(\eta))^{\omega(k,s)}q_{k,s}(\zeta, \eta) \tag{2.30}$$

$$+\sum_{k=\nu+1}^{\nu+\rho-1} \tilde{\tau}^{\rho-(k-\nu)} q_{k,k-\nu}(\tilde{\tau}, \zeta, \eta)$$

$$+\sum_{k=\nu+1}^{\nu+\rho-1} \sum_{s=k-\nu+1}^{\rho} \tilde{\tau}^{\rho-s}(\zeta-\lambda_1(\eta))^{\omega(k,s)} q_{k,s}(\zeta, \eta)+R_{m-(\nu+\rho)}.$$

Let there exist $(\mathring{k}, \mathring{s})$ and $\mathring{\omega} \in \Omega$ such that

(2.31) $$q_{\mathring{k}\mathring{s}}(\lambda_1(\mathring{\omega}), \mathring{\omega}) \neq 0, \quad \omega(\mathring{k}, \mathring{s})<\nu-(\mathring{k}-\mathring{s}).$$

Putting

(2.32) $$\eta=\eta'\mathring{\omega}, \quad \eta' \in R^1$$

(2.33) $$\zeta=\lambda_1(\eta)+ir+\mathring{\xi} \equiv \eta'\lambda_1(\mathring{\omega})+ir+\mathring{\xi},$$

and consider the root $\tilde{\tau}$ of (2.34),

(2.34) $$P(\tilde{\tau}+\tau_1, \zeta, \eta)=0.$$

If we show the following lemma, the proof of Proposition 2.2 is complete.

Lemma 2.3. *When* (2.32) *and* (2.33) *hold,* (2.34) *has a root* $\tilde{\tau}$ *which has the following expansion in the neithborhood of* $\eta'=\infty$ *for some* r, $\mathring{\xi}$,

(2.35) $$\tilde{\tau}=c\eta'^{\alpha}+c'\eta'^{\alpha'}+c''\eta'^{\alpha''}+\cdots$$

$$\alpha>\alpha'>\alpha''>\cdots, \quad 0<\alpha<1, \quad \operatorname{Im} c<0.$$

Proof of Lemma 2.3. Dividing $P(\tilde{\tau}+\tau_1, \zeta, \eta)=0$ by $C_l(\zeta, \eta)$, we have

(2.36) $$\tilde{\tau}^{\rho} \prod_{i\neq 1} (\tilde{\tau}-\tilde{\tau}_i)^{\rho_i} \{K(\zeta, \eta)/\prod_j (\zeta-\lambda_j)^{\nu_j}\} \times$$

$$\times \{\sum_{k=1}^{\nu} (\zeta-\lambda_1(\eta))^{\nu-k-1} \tilde{\tau}^{\rho-1} q_{k,1}(\tilde{\tau}, \zeta, \eta)$$

$$+\sum_{k=1}^{\nu} \sum_{s=2}^{\rho} \tilde{\tau}^{\rho-s}(\zeta-\lambda_1(\eta))^{\omega(k,s)} q_{k,s}(\zeta, \eta)$$

$$+\sum_{k=\nu+1}^{\nu+\rho-1} \tilde{\tau}^{\rho-(k-\nu)} q_{k,k-\nu}(\tilde{\tau}, \zeta, \eta)$$

$$+\sum_{k=\nu+1}^{\nu+\rho-1} \sum_{s=k-\nu+1}^{\rho} \tilde{\tau}^{\rho-s}(\zeta-\lambda_1(\eta))^{\omega(k,s)} q_{k,s}(\zeta, \eta)+R_{m-(\nu+\rho)}\}=0,$$

where $\tilde{\tau}_i=\tau_i-\tau_1$. Substituting (2.32), (2.33) and (2.35) into (2.36), we consider the highest order of η' in the each term of (2.36). The order of η' of $\tilde{\tau}^{\rho} \prod_{i\neq 1} (\tilde{\tau}-\tilde{\tau}_i)^{\rho_i}$ is $\alpha\rho+m'-\rho$. The order of η' of the secound, the fourth and the last terms of (2.36) are less than $\alpha\rho+m'-\rho$, and moreover are not equal to $\alpha\rho+m'-\rho$. The order of η' of the third and the fifth terms are $\alpha(\rho-s)+\text{order}\, q_{k,s}-(1-\nu)$. By (2.25) and (2.27)

(2.37) $$\alpha(\rho-s)+\text{order}\ q_{k,s}-(l-\nu)$$

$$=\alpha(\rho-s)+m-k-\omega(k,s)-\rho+s-l+\nu$$

$$=\alpha\rho+m'-\rho-\alpha s+s+\nu-k-\omega(k,s).$$

When $\omega(k,s)\geqq s+\nu-k$, (2.37) $<\alpha\rho+m'-\rho$. Then, in this case, $c=0$. When $\omega(k,s)<s+\nu-k$, let $\alpha(k,s)$ be α which satisfy the following:

$$\alpha\rho+m'-\rho=\alpha\rho+m'-\rho-\alpha s+s+\nu-k-\omega(k,s). \tag{3.38}$$

Namely

$$\alpha(k,s)=\frac{s+\nu-k-\omega(k,s)}{s}=1-\frac{k+\omega(k,s)-\nu}{s} \tag{2.39}$$

We have

$$0<\alpha(k,s)<1 \qquad \text{for} \quad \omega(k,s)<s+\nu-k. \tag{2.40}$$

The first inequality of (2.40) is obvious. Let us prove $\alpha(k,s)<1$. In the case $\nu\geqq k\geqq 1$, by Proposition 2.1 $\omega(k,s)\geqq\nu-k+1$, then $\alpha(k,s)<1$. In the case $\nu<k$, because of $\omega(k,s)\geqq 0$, obviously we have $k+\omega(k,s)>\nu$. Then $\alpha(k,s)<1$. Let

$$\mathring{\alpha}=\max_{(k,s)}\alpha(k,s) \tag{2.41}$$

and let

$$\Gamma=\{(k,s);\ \mathring{\alpha}=\alpha(k,s)\}. \tag{2.42}$$

We have (2.37)$<\mathring{\alpha}\rho+m'-\rho$ for $(k,s)\notin\Gamma$. Let $\alpha=\mathring{\alpha}$ in (2.35), coefficient c of $\eta'^{\mathring{\alpha}}$ is determined by the following;

$$\begin{aligned} &c^{\rho}\prod_{i\neq 1}(-\tilde{\tau}_i)^{\rho_i}\\ &\quad+\mathring{K}\sum_{(k,s)\in\Gamma}c^{\rho-s}(ir+\mathring{\xi})^{\omega(k,s)}q_{k,s}(\lambda_1(\mathring{\omega}),\mathring{\omega})/\{(ir+\mathring{\xi})^{\nu}\times\\ &\quad\times\prod_{j\neq 1}(\lambda_1(\mathring{\omega})-\lambda_j(\mathring{\omega}))^{\nu_j}\}=0, \end{aligned} \tag{2.43}$$

where $\tilde{\tau}_i=\tilde{\tau}_i(\lambda_1(\mathring{\omega}),\mathring{\omega})$ and $\mathring{K}\approx 1$ for $|ir+\mathring{\xi}|$ large. We want to show that (2.43) has a root c with $\mathrm{Im}\,c<0$ for some $(r,\mathring{\xi})\in R^2$. For $(k,s)\in\Gamma$, $\mathring{\alpha}=1-\dfrac{k+\omega(k,s)-\nu}{s}$, then

$$\omega(k,s)-\nu=s(1-\mathring{\alpha})-k, \quad (k,s)\in\Gamma. \tag{2.44}$$

Let

$$\bar{s}=\min_{(k,s)\in\Gamma}s \tag{2.45}$$

By (2.36), $\bar{s}\geqq 2$. Differentiating (2.43) $\rho-\bar{s}$ times by c, we have

$$c^{\bar{s}}+\mathring{K}\sum_{(k,s)\in\Gamma}K(k,\bar{s})(ir+\mathring{\xi})^{\omega(k,\bar{s})-\nu}=0, \tag{2.46}$$

where $K(k, \bar{s})$ is constant which depends on k, $\bar{s}$ and $\mathring{\omega}$ but independent of r, $\mathring{\xi}$. By (2.44), (2.46) becomes (2.46′).

$$c^{\bar{s}}+\mathring{K}\sum_{(k,\bar{s})\in\Gamma}K(k, \bar{s})(ir+\mathring{\xi})^{\bar{s}(1-\mathring{\alpha})-k}=0. \tag{2.46′}$$

In the case $\bar{s}\geqq 3$ or $\bar{s}=2$ and $\bar{s}(1-\mathring{\alpha})-k\neq 0$, (2.46′) has a root c with $\operatorname{Im} c<0$ for some suitable $(r, \mathring{\xi})\in R^2$. Let us consider the case $\bar{s}=2$ and $\bar{s}(1-\mathring{\alpha})-k=0$. Namely $k=2(1-\mathring{\alpha})$. Because of that $2(1-\mathring{\alpha})$ is positive integer and $0<\mathring{\alpha}<1$, we have $\mathring{\alpha}=\frac{1}{2}$ and $k=1$. In this case if we replace η' by $-\eta'$ in (2.32), $K(k, \bar{s})$ (in (2.46′)) becomes $-K(k, \bar{s})$. Then (2.46′) has a root c with $\operatorname{Im} c<0$ if necessary replacing η' by $-\eta'$. By the Lemma 8.1 in the appendix, (2.43) has a root c with $\operatorname{Im} c<0$ for some $(r, \mathring{\xi})\in R^2$ if necessary replacing η' by $-\eta'$.

The proof of necessity of Theorem 1. Paying attention to the multiplicity of the roots τ_i, we put

$$\prod_i(\tau-\tau_i)^{\rho_i}=\{(\tau-\tau_1)(\tau-\tau_2)\cdots(\tau-\tau_{n_1})\}^{\sigma_1}\{(\tau-\tau_{n_1+1})\cdots(\tau-\tau_{n_2})\}^{\sigma_2}$$
$$\cdots\{(\tau-\tau_{n_{s-1}+1})\cdots(\tau-\tau_{n_s})\}^{\sigma_s}.$$

where $n_s=n''$,

$$\rho_1=\rho_2=\cdots=\rho_{n_1}=\sigma_1>\rho_{n_1+1}=\cdots=\rho_{n_2}=\sigma_2>\cdots>\rho_{n_{s-1}+1}=\cdots=\rho_{n_s}=\sigma_s>0.$$

And let us write

$$\begin{aligned}\tilde{P}_{m-k}(\tau, \zeta, \eta)&\\ &=\prod_i(\tau-\tau_i)^{\rho_i-1}q_1(\tau, \zeta, \eta)+\prod_i(\tau-\tau_i)^{\rho_i-2}q_2(\tau, \zeta, \eta)\\ &\quad+\prod_i(\tau-\tau_i)^{\rho_i-3}q_3(\tau, \zeta, \eta)+\cdots\\ &\quad+(\tau-\tau_1)\cdots(\tau-\tau_{n_1})q_{\sigma_1-1}(\tau, \zeta, \eta)+q_{\sigma_1}(\tau, \zeta, \eta).\end{aligned} \tag{2.47}$$

where $\prod_i(\tau-\tau_i)^{\rho_i-2}q_2$ is not divisible by $\prod_i(\tau-\tau_i)^{\rho_i-1}$ and the order of τ of $\prod_i(\tau-\tau_i)^{\rho_i-2}q_2$ is less than the order of $\prod_i(\tau-\tau_i)^{\rho_i-1}$, $\prod_i(\tau-\tau_i)^{\rho_i-3}q_3$ is not divisible by $\prod_i(\tau-\tau_i)^{\rho_i-2}$ and it's order of τ is less than the order of $\prod_i(\tau-\tau_i)^{\rho_i-2}$,..., q_{σ_1} is not divisible by $(\tau-\tau_1)(\tau-\tau_2)\cdots(\tau-\tau_{n_1})$ and the order of τ of q_{σ_1} is less than n_1. The order of τ of q_{σ_1} is at most n_1-1.

$$\tilde{P}_{m-k}(\tau_i, \zeta, \eta)=q_{\sigma_1}(\tau_i, \zeta, \eta), \quad i=1, 2,\dots, n_1. \tag{2.48}$$

by Prop. 2.2, $\tilde{P}_{m-k}(\tau_i, \zeta, \eta)$ is divisible by $\prod_j(\zeta-\lambda_j)^{\nu_j-k+\sigma_1}$. Then $q_{\sigma_1}(\tau_i, \zeta, \eta)\equiv 0 \bmod \prod_j(\zeta-\lambda_j)^{\nu_j-k+\sigma_1}$, $i=1, 2,\dots, n_1$. By the Lemma 8.2 in the appendix,

$$q_{\sigma_1}(\tau, \zeta, \eta)\equiv 0 \bmod \prod_j(\zeta-\lambda_j)^{\nu_i-k+\sigma_1}.$$

Let us use the induction. Assuming that q_h be divisible by $\prod_j (\zeta-\lambda_j)^{\nu_j-k+h}$ for $h=\sigma_1, \sigma_1-1,\dots, \sigma'+1$. We want to show that $q_{\sigma'}$ is divisible by $\prod_j (\zeta-\lambda_j)^{\nu_j-k+\sigma'}$. Let

(2.49) $$\prod_j (\tau-\tau_j)^{\rho_j-\sigma'+1}/\prod_j (\tau-\tau_j)^{\rho_j-\sigma'}=(\tau-\tau_1)(\tau-\tau_2)\cdots(\tau-\tau_{r'}).$$

The order of τ of $q_{\sigma'}$ is at most $r'-1$. By the Prop 2.2 and the assumption of the induction we have

(2.50) $$\left(\frac{\partial}{\partial\tau}\right)^{\rho_i-\sigma'}\tilde{P}_{m-k}(\tau, \zeta, \eta)\,|_{\tau=\tau_i}$$
$$\equiv(\rho_i-\sigma')!\prod_{j\neq i}(\tau_i-\tau_j)^{\rho_j-\sigma'}q_{\sigma'}(\tau_i, \zeta, \eta)\equiv 0$$
$$\bmod \prod_j (\zeta-\lambda_j)^{\nu_j-k+\sigma'},\quad i=1, 2,\dots, r'.$$

Then by the Lemma 8.2 in the appendix

$$q_{\sigma'}(\tau, \zeta, \eta)\equiv 0 \bmod \prod_j (\tau-\tau_j)^{\nu_j-k+\sigma'}.$$

Thus we complete the proof of the necessity of the Theorem 1.

§ 3. The proof of sufficiency of Theorem 1.

We prove the following (cf. Theorem 1.1)

Proposition 3.1. *P_{m-k} has the form* (1.5), *then there exists $\varepsilon^0>0$ such that $P(\tau, \zeta, \eta)$ is hyperbolic with respect to* (1, ε, 0) *for any ε with* $0<|\varepsilon|<\varepsilon^0$.

Proof.

(1.3) $$P_m(\tau, \zeta, \eta)=\prod_{h=1}^{n''}(\tau-\tau_h(\zeta, \eta))^{\rho_h}\prod_{j=1}^{n'}(\zeta-\lambda_j(\eta))^{\nu_j}.$$

Then

(3.1) $$P_m(\tau, \varepsilon\tau+\xi, \eta)=\prod_h (\tau-\tau_h(\varepsilon\tau+\xi, \eta))^{\rho_h}\prod_j (\varepsilon\tau+\xi-\lambda_j(\eta))^{\nu_j}.$$

At first we study the root τ of $P_m(\tau, \varepsilon\tau+\xi, \eta)=0$. Namely consider

(3.2) $$\tau-\tau_h(\varepsilon\tau+\xi, \eta)=0.$$

$\tau_h(\zeta, \eta)$ is analytic in $(\zeta, \eta)\in C^{n+1}$, $\eta\neq 0$, and is homogeneous degree 1 with respect to (ζ, η). Then by the theorem of the implicit function, (3.2) is written by the following (for small ε[2]).

(3.3) $$\tau=\mathring{\tau}_h(\xi, \eta; \varepsilon).$$

2) (3.3) is valid for $|\varepsilon|<\left\{\sup_{\substack{(\xi,\eta)\in R^{n+1}\\ \eta\neq 0}}\left|\frac{\partial}{\partial\zeta}\tau_h(\xi, \eta)\right|\right\}^{-1}$.

And $\mathring{\tau}_h(\xi, \eta; \varepsilon)$ is real for $(\xi, \eta) \in R^{n+1}$. So (3.1) becomes

$$(3.4) \quad P_m(\tau, \varepsilon\tau+\xi, \eta)=C(\varepsilon)\prod_h (\tau-\mathring{\tau}_h(\xi, \eta; \varepsilon))^{\rho_h} \prod_j (\tau-\mathring{\lambda}_j(\xi, \eta; \varepsilon))^{\nu_j},$$

where

$$(3.5) \qquad \mathring{\lambda}_j(\xi, \eta; \varepsilon)=\frac{1}{\varepsilon}(\lambda_j(\eta)-\xi).$$

The root τ of $P_m(\tau, \varepsilon\tau+\xi, \eta)=0$ are

$$(3.6) \qquad \begin{cases} \mathring{\tau}_h(\xi, \eta; \varepsilon), & h=1, 2,\dots, n'', \\ \mathring{\lambda}_j(\xi, \eta; \varepsilon), & j=1, 2,\dots, n'. \end{cases}$$

Let us consider the multiplicity of the roots (3.6). First, by the assumption (A), we have

$$(3.7) \quad \mathring{\tau}_h(\xi, \eta; \varepsilon)\neq \mathring{\tau}_{h'}(\xi, \eta; \varepsilon) \qquad \text{for} \quad h\neq h', \quad (\xi, \eta)\in R^{n+1}, \quad (\xi, \eta)\neq(0, 0)$$

Secondary,

$$\mathring{\lambda}_j-\mathring{\lambda}_{j'}=\frac{1}{\varepsilon}(\lambda_j(\eta)-\lambda_{j'}(\eta)),$$

then by (A), we have

$$(3.8) \qquad \mathring{\lambda}_j(\xi, \eta; \varepsilon)\neq \mathring{\lambda}_{j'}(\xi, \eta; \varepsilon) \qquad \text{for} \quad j\neq j', \quad \eta\neq 0, \quad (\xi, \eta)\in R^{n+1}$$

and

$$(3.8') \qquad \mathring{\lambda}_j(\xi, 0; \varepsilon)=\mathring{\lambda}_{j'}(\xi, 0; \varepsilon)=\frac{-\xi}{\varepsilon}.$$

Finally let us consider the case

$$(3.9) \qquad \mathring{\lambda}_j(\xi, \eta; \varepsilon)=\mathring{\tau}_h(\xi, \eta; \varepsilon).$$

If (3.9) hold, by (3.5) we have

$$(3.10) \qquad \xi=\lambda_j(\eta)-\varepsilon\tau_h(\lambda_j(\eta), \eta).$$

Conversely if (3.10) is valid, we have (3.9). Let

$$(3.11) \qquad \xi_{hj}(\omega)=\lambda_j(\omega)-\varepsilon\tau_h(\lambda_j(\omega), \omega), \quad \omega\in\Omega.$$

Then (3.10) becomes (3.10′).

$$(3.10') \qquad \xi=\xi_{hj}(\omega)\,|\eta|.$$

We remark that $\xi_{hj}(\omega)$ is real and for small ε, $\xi_{hj}(\omega)=\xi_{pq}(\omega)$ if and only if $h=p$ and $j=q$. Hereafter we study for fixed $\omega\in\Omega$. So let us write $\eta=\eta'\omega$, $\eta'\in R^1_+$. By the above consideration we have the following;

$$(3.12) \qquad \mathring{\lambda}_j(\xi, \eta'\omega; \varepsilon)\neq \mathring{\tau}_h(\xi, \eta'\omega; \varepsilon) \qquad \text{for} \quad \xi\neq\xi_{hj}(\omega)\eta',$$

(3.13) $$\mathring{\lambda}_j(\xi, \eta'\omega; \varepsilon) = \mathring{\tau}_h(\xi, \eta'\omega; \varepsilon) \qquad \text{for} \quad \xi = \xi_{hj}(\omega)\eta'.$$

Let

(3.14) $$\mathring{\lambda}_j(\xi, \eta'\omega; \varepsilon) - \mathring{\tau}_h(\xi, \eta'\omega; \varepsilon) = (\xi - \xi_{hj}\eta')^{p(h,j)} Q_{hj}(\xi, \eta').$$

Where $p(h, j)$ is positive integer[3)] $Q_{hj}(\xi, \eta')$ is homogeneous function of degree $1-p(h, j)$ and $Q_{hj}(\xi, \eta') \neq 0$ for $|(\xi, \eta')| = 1$. Then there exist positive constant m_1, M_2 such that

(3.15) $$m_1 < |Q_{hj}(\xi, \eta')| < M_2, \qquad \text{for} \quad |(\xi, \eta')| = 1,$$
$$h = 1, 2, \ldots, n'', \quad j = 1, 2, \ldots, n'.$$

The root τ of $P_m(\tau, \varepsilon\tau + \xi, \eta'\omega) = 0$ are real for $(\xi, \eta') \in R^2_+$. Where $R^2_+ = \{(\xi, \eta'); (\xi, \eta') \in R^2, \eta' \geqq 0\}$. Then using Rouche's theorem, we are going to prove that if P_{m-k} has the form (1.5) then the roots τ of $P_m(\tau, \varepsilon\tau + \xi, \eta'\omega) + \sum_k P_{m-k}(\tau, \varepsilon\tau + \xi, \eta'\omega) = 0$ are near the roots τ of $P_m(\tau, \varepsilon\tau + \xi, \eta'\omega) = 0$. More precisely

$$|\operatorname{Im} \tau(\xi, \eta')| < \text{constant (independent of } (\xi, \eta') \in R^2_+).$$

To avoide complication we introduce new notation. We arrange $\{\xi_{hj}\}$ in order of size. Let $\max_{h,j} \xi_{hj} = \overset{(1)}{\xi}$, the next be $\overset{(2)}{\xi}, \ldots$, the last be $\overset{(\beta)}{\xi}$. Where $\overset{(1)}{\xi} > \overset{(2)}{\xi} > \cdots > \overset{(\beta)}{\xi} = \min_{h,j} \xi_{hj}$, $\{\xi_{hj}\} = \{\overset{(i)}{\xi}\}$ and $n'n'' = \beta$. In (3.14), let us write p_s instead of $p(h, j)$ if $\xi_{hj} = \overset{(s)}{\xi}$.

We separate R^2_+ into some parts and in each part we use Rouche's theorem. Let

$$B_0 = \{(\xi, \eta'); |(\xi, \eta')| \leqq M_1, \eta' \geqq 0\},$$
$$D_0 = \{(\xi, \eta'); |(\xi, \eta')| \geqq M_1, 0 \leqq \eta' \leqq a_0\},$$
$$D_0^- = \{(\xi, \eta'); |(\xi, \eta')| \geqq M_1, \eta' \geqq a_0, \xi \geqq b_1\eta'\},$$
$$D_0^+ = \{(\xi, \eta'); |(\xi, \eta')| \geqq M_1, \eta' \geqq a_0, \xi \leqq b_{\beta+1}\eta'\}$$

for $i = 1, 2, \ldots, \beta$

$$D_i = \{(\xi, \eta'); |(\xi, \eta')| \geqq M_1, \eta' \geqq 0, |\xi - \overset{(i)}{\xi}\eta'| \leqq a_i|(\xi, \eta')|^{(p_i-1)/p_i}\}$$
$$D_i^+ = \{(\xi, \eta'); |(\xi, \eta')| \geqq M_1, \eta' \geqq 0, \overset{(i)}{\xi}\eta' + a_i|(\xi, \eta')|^{(p_i-1)/p_i} \leqq \xi \leqq b_i\eta'\},$$
$$D_i^- = \{(\xi, \eta'); |(\xi, \eta') \geqq M_1, \eta' \geqq 0, b_{i+1}\eta' \leqq \xi \leqq \overset{(i)}{\xi}\eta' - a_i|(\xi, \eta')|^{(p_i-1)/p_i}\},$$
$$|(\xi, \eta')| = \{|\xi|^2 + |\eta'|^2\}^{1/2}.$$

Where

(3.17) $$b_1 > \overset{(1)}{\xi} > b_2 > \overset{(2)}{\xi} > \cdots > \overset{(\beta)}{\xi} > b_{\beta+1}, \quad p_i \geqq 1,$$

---

3) This follow from the fact that $\mathring{\lambda}_j(\xi, \eta; \varepsilon) - \mathring{\tau}_h(\xi, \eta; \varepsilon)$ is analytic for $\eta \neq 0$.

and $M_1>0$, $a_i>0$, the size of M_1 and a_i are defined later. And obviously we have $B_0 \cup D_0 \cup D_0^- \cup D_0^+ \bigcup_{i=1}^{\beta} (D_i \cup D_i^+ \cup D_i^-) = R_+^2$.

Let q_{m-k} be the homogeneous polynomial of degree $m-k$ and has the form

$$q_{m-k}(\tau, \zeta, \eta) = \gamma(\tau, \zeta, \eta) \prod_h (\tau - \tau_h)^{\rho_h - k_1} \prod_j (\zeta - \lambda_j)^{\nu_j - k_2}, \tag{3.18}$$

where $\gamma(\tau, \zeta, \eta)$ is the homogeneous polynomial. Put

$$S(\tau) = q_{m-k}(\tau, \varepsilon\tau + \xi, \eta) / P_m(\tau, \varepsilon\tau + \xi, \eta). \tag{3.19}$$

First we consider the case $(\xi, \eta') \in D_1^+$. Without loss of generality we can consider $\overset{(1)}{\xi} = \xi_{11}$. Let us consider the value of S on the circle with center $\mathring{\lambda}_1$ and radius R in the τ-plane. Namely

$$\tau = \mathring{\lambda}_1 + Re^{i\theta} \tag{3.20}$$

$$\begin{aligned} S(\mathring{\lambda}_1 + Re^{i\theta}) &= \{(Re^{i\theta})^{\nu_1 - k_2} (\mathring{\lambda}_1 - \mathring{\tau}_1 + Re^{i\theta})^{\rho_1 - k_1} \tilde{q}_{m-k-(\nu_1 - k_2 + \rho_1 - k_1)}\} / \\ &\quad \{(Re^{i\theta})^{\nu_1} (\mathring{\lambda}_1 - \mathring{\tau}_1 + Re^{i\theta})^{\rho_1} \prod_{j \neq 1} (\mathring{\lambda}_1 - \mathring{\lambda}_j + Re^{i\theta})^{\nu_j} \prod_{i \neq 1} (\mathring{\lambda}_1 - \mathring{\tau}_i + Re^{i\theta})^{\rho_i}\} \end{aligned} \tag{3.21}$$

where the order of $\tilde{q}_p$ is at most p. We have

$$\begin{gathered} \mathring{\lambda}_1(\xi, \eta) - \lambda_j(\xi, \eta) \sim \text{const}.\eta' \sim \text{const}. |(\xi, \eta')| \\ \eta = \eta'\omega, \quad (\xi, \eta') \in D_1^+. \end{gathered} \tag{3.22}$$

By (3.14) and (3.15) we have

$$\begin{aligned} |\mathring{\lambda}_1(\xi, \eta) - \mathring{\tau}_h(\xi, \eta)| &= |(\xi - \xi_{h1}\eta')^{p(h,1)} Q_{h1}(\xi, \eta')| \\ &> \{(\overset{(1)}{\xi} - \xi_{h1})\eta'\}^{p(h,1)} m_1 |(\xi, \eta')|^{1-p(h,1)} \sim \text{const}. |(\xi, \eta')|, \end{aligned} \tag{3.23}$$

$$\begin{aligned} |\mathring{\lambda}_1 - \mathring{\tau}_1| &= |(\xi - \overset{(1)}{\xi}\eta')^{p_1} Q_{11}(\xi, \eta)| \\ &\geqq \{a_1 |(\xi, \eta')|^{(p_1 - 1)/p_1}\}^{p_1} m_1 / |(\xi, \eta')|^{p_1 - 1} = (a_1)^{p_1} m_1. \end{aligned} \tag{3.24}$$

We require

$$(a_1)^{p_1} m_1 > 2R. \tag{3.25}$$

When $\rho_1 \geqq k_1$ and $\nu_1 \geqq k_2$ we have

$$|S(\mathring{\lambda}_1 + Re^{i\theta})| < \text{const}. R^{-(k_1 + k_2)}. \tag{3.26}$$

In the another case, namely $\rho_1 < k_1$ or $\nu_1 < k_2$, we have

$$|S(\mathring{\lambda}_1 + Re^{i\theta})| < \text{const}. |(\xi, \eta')|^{-1}. \tag{3.27}$$

Then if we take R and M_1 large, $|S(\mathring{\lambda}_1 + Re^{i\theta})|$ becomes small. In the nearly same way $|S(\mathring{\tau}_1+Re^{i\theta})|$, $|S(\mathring{\lambda}_j+Re^{i\theta})|$ $(j=2, 3,\dots, n')$, $|S(\mathring{\tau}_h+Re^{i\theta})|$ $(h=2, 3,\dots, n'')$ become small if we take R and M_1 large.

When (ξ, η') is in D_i^+ $(i\neq 0)$ or D_i^-, we requir

$$(a_i)^{p_i}m_1>2R. \tag{3.25'}$$

When (ξ, η') is in D_0^- or D_0^+, we reiquire

$$\text{const. } a_0>R. \tag{3.25''}$$

In these case $|S(\mathring{\tau}_h+Re^{i\theta})|$ and $|S(\mathring{\lambda}_j+Re^{i\theta})|$ become small if we take R and M_1 (and a_i) large.

Next, (ξ, η') is in $D_i i=1, 2,\dots, \beta$, we require

$$R'>2M_2(a_i)^{p_i} \tag{3.28}$$

In this case (for example in D_1 and $\overset{(1)}{\xi}=\xi_{11}$) $\mathring{\tau}_1$ is in the circle with center $\mathring{\lambda}_1$ and radius R'. The estimate S on the circle with center λ_j $j=1, 2,\dots, n'$ or $\mathring{\tau}_i$ $i=2, 3,\dots, n''$ and radius R' are obtained in the nearly same way as the above case. In the case (ξ, η') is in D_0, we require

$$R'>\frac{2a_0M}{\varepsilon}, \quad \text{where} \quad M_3=\sup_{i,j,\omega\in\Omega}|\lambda_i(\omega)-\lambda_j(\omega)|\,. \tag{3.28'}$$

When $(\xi, \eta')\in B_0$, obviously $P(\tau, \varepsilon\tau+\xi, \eta)=0$ has a root with $|\tau(\xi, \eta)|<R''$. After all by Rouche's theorme we conclude that if we take M_1, R, R', R'' large with (3.25), (3.25'), (3.25''), (3.28) and (3.28'), then $P(\tau, \varepsilon\tau+\xi, \eta)=0$ has a root with $|\text{Im}\,\tau(\xi, \eta)|<\max\{R, R', R''\}$. Thus we complete the proof of Prop. 3.1.

Part 2, variable coefficients

§4. Introduction and results.

Here we show a sufficient conditin of the C^∞-Goursat problem with variable coefficients.

Let us consider the operator L.

$$L=PQ-R \tag{4.1}$$

P, Q and R are the following. First, we explain about P.

$$P=\sum_{i+j\leqq m}a_{ij}(t, x, y;D_y)D_t^iD_x^j \tag{4.2}$$

where $a_{ij}(t, x, y; D_y)$ is a pseudo differential operator of order $m-(i+j)$. We assume

$$a_{ij}(t, x, y;\eta)\in S_{1,0}^{m-(i+j)} \tag{4.3}$$

(t, x) is considered as parameter and $(t, x)\to a_{ij}(t, x, y; \eta)\in S_{1,0}^{m-(i+j)}$ is smooth for $(t, x)\in R_+^1\times R^1$. Let P_m be a principal part of P. i.e.

(4.4) $$P_m(\tau;\ \xi,\ \eta)=\sum_{j+k\leqq m}\mathring{a}_{jk}(t,\ x,\ y;\ \eta)\tau^j\xi^k$$

where $\mathring{a}_{jk}$ is of homogeneous of degree $m-(j+k)$ in η. Let

(4.5) $$P_m(\tau,\ \xi,\ \eta)=\prod_{j=1}^{n''}(\tau-\tau_j(t,\ x,\ y;\ \xi,\ \eta))^{\rho_j}.$$

Here we assume

(A–1) The root τ of $P_m(\tau,\ \xi,\ \eta)=0$ is real and it's multiplicity is constant. Moreover there exists a positive constant δ (which is independent of $(t,\ x,\ y)$ and $(\xi,\ \eta)$, but depends on $(T,\ X)$) such that

(4.6) $$|\tau_j(t,\ x,\ y;\ \xi,\ \eta)-\tau_h(t,\ x,\ y;\ \xi,\ \eta)|\geqq\delta|(\xi,\ \eta)| \quad \text{for} \quad j\neq k,$$
$$(t,\ x,\ y)\in[0,\ T]\times[-X,\ X]\times R^n,\quad T,\ X>0,\quad (\xi,\ \eta)\in R^{n+1}\setminus\{0\}.$$

(A–2) P is hyperbolic[4] with repect to the dirction t. Namely the lower order terms of P satisfy the Levi conditions.

Next we explain about Q.

(4.7) $$Q=\sum_{j=0}^{l}b_j(t,\ x,\ y;\ D_y)D_x^j$$

where t is considered as paremeter. $b_j(t,\ x,\ y;\ D_y)$ is a pseudo differential operator of order $l-j$. We assume

(4.8) $$b_j(t,\ x,\ y;\ \eta)\in S^{l-j}_{1,0}$$

$(t,\ x)$ is considered as parameter and $(t,\ x)\to b_j(t,\ x,\ y)\in S^{l-j}_{1,0}$ is smooth for $(t,\ x)\in R_1^+\times R^1$. Let Q_l be a principal part of Q.

(4.9) $$Q_l(\lambda;\ \eta)=\sum_{j=0}^{l}\mathring{b}_j(t,\ x,\ y;\ \eta)\lambda^j$$

where $\mathring{b}_j$ is of homogeneous of degree $l-j$ in η. Let

(4.10) $$Q_l(\lambda;\ \eta)=\prod_{j=1}^{n'}(\lambda-\lambda_j(t,\ x,\ y;\ \eta))^{\nu_j}.$$

(A–3) The root λ of $Q_l(\lambda;\ \eta)=0$ is real and it's multiplicity is constant. Moreover there exists a positive constant δ' such that

(4.11) $$|\lambda_j(t,\ x,\ y;\ \eta)-\lambda_k(t,\ x,\ y;\ \eta)|\geqq\delta'|\eta|$$
$$(t,\ x,\ y)\in[0,\ T]\times[-X,\ X]\times R^n,\quad \eta\in R^n\setminus\{0\}.$$

(A–4) Q is hyperbolic with respect to the direction x. Namely the lower order terms of Q satisfy the Levi conditions (refer to A–4′)

Let us write

(4.12) $$D_x-\lambda_j(t,\ x,\ y;\ D_y)=\partial_j.$$

4) About the definition "hyperbolic" refer to (A–4′).

$$(4.13)\quad \begin{cases} \partial_1\partial_2\cdots\partial_{q_\nu}=\Gamma(q_\nu) \\ \partial_1\partial_2\cdots\partial_{q_{\nu-1}}=\Gamma(q_{\nu-1}) \\ \cdots\cdots \\ \partial_1\partial_2\cdots\partial_{q_1}=\Gamma(q_1) \end{cases}$$

where $1\leqq q_1\leqq q_2\leqq\cdots\leqq q_\nu$ and

$$(4.14)\quad \begin{cases} \lambda_1\lambda_2,\dots,\lambda_{q_1} \text{ are } \nu\text{-tuple roots} \\ \lambda_{q_1+1},\dots,\lambda_{q_2} \text{ are } (\nu-1)\text{-tuple roots} \\ \cdots\cdots \\ \lambda_{q_{\nu-1}-1},\dots,\lambda_{q_\nu} \text{ are simple roots.} \end{cases}$$

The assumption (A–4) is equivalent to (A–4′) (Levi condition (in this paper) means that Q has the form of (4.15)).

(A–4′) Q is the following:

$$(4.15)\quad \begin{aligned} Q=&\Gamma(q_\nu)\Gamma(q_{\nu-1})\cdots\Gamma(q_1)+A(q_\nu-1)\Gamma(q_{\nu-1})\Gamma(q_{\nu-2})\cdots\Gamma(q_1) \\ &+A(q_\nu+q_{\nu-1}-2)\Gamma(q_{\nu-2})\cdots\Gamma(q_1)+\cdots \\ &+A(q_\nu+q_{\nu-1}+\cdots+q_2-(\nu-1))\Gamma(q_1)+A(l-\nu) \end{aligned}$$

where $A(k)\equiv A(k;\, t,\, x,\, y,\, D_x,\, D_y)$ and it is the pseudo differential operater with respect to y and differential operater with respet to x, of total order k.
Finally we explain about R.

(A–5) R is the following

$$(4.16)\quad \begin{aligned} R=&B(m-r)\Gamma(q_\nu)\Gamma(q_{\nu-1})\cdots\Gamma(q_1) \\ &+B(m-r+q_\nu-1)\Gamma(q_{\nu-1})\Gamma(q_{\nu-2})\cdots\Gamma(q_1) \\ &+B(m-r+q_\nu+q_{\nu-1}-2)\Gamma(q_{\nu-2})\Gamma(q_{\nu-3})\cdots\Gamma(q_1)+\cdots \\ &+B(m-r+q_\nu+q_{\nu-1}+\cdots+q_2-(\nu-1))\Gamma(q_1)+B(m-r+l-\nu), \end{aligned}$$

where $B(k)$ is differential operator with respect to t and x, pseudo differential operator with respect to y, and it's total order is at most k. Moreover the order of D_t in $B(k)$ is at most $m-r$. And r is the multiplicity of the root τ of $P_m=0$. Namely $r=\max_j \rho_j$.

Let us consider the following problem:

$$(4.17)\quad \begin{cases} Lu=(PQ-R)u=f\in\mathscr{E}_t(\tilde{H}^\infty_{x,y}), \\ D_t^i|_{t=0}=\phi_i(x,\, y)\in\tilde{H}^\infty_{x,y}, \quad 0\leqq i\leqq m-1, \\ D_x^j|_{x=0}=\psi_j(t,\, y)\in\mathscr{E}_t(H^\infty_y), \quad 0\leqq j\leqq l-1, \\ D_x^j\phi_i(0,\, y)=D_t^i\psi_j(0,\, y), \quad 0\leqq i\leqq m-1, \quad 0\leqq j\leqq l-1, \end{cases}$$

where $\tilde{H}^{\infty}_{x,y}=\{f\in C^{\infty}_{x,y};\int_{R^n}\int_{|x|<X}|D^{\alpha}_x D^{\beta}_y f|^2 dxdy<\infty$ for $\forall\alpha$, $\forall\beta$, $\forall X>0\}$

Theorem 2. *If we assume* (A–1)~(A–5) *then Goursat problem* (4.17) *has a unique solution in* $\mathscr{E}_t(\tilde{H}^{\infty}_{x,y})$.

We prove this theorem by the induction. For this we need the domain of dependence.

§5. Domain of dependence and estimate.

Let

$$\tau_{\max}=\max_{t\in[0,T],|x|\leqq X,y\in R^n,|\xi|=1}|\tau_i(t,x,y;\xi,0)| \tag{5.1}$$

$$\mathscr{D}(t_0,x_0)=\{(t,x,y);\ |x-x_0|<\tau_{\max}(t_0-t),\ t\geqq 0\} \tag{5.2}$$

$$\Omega(t_0,X_0)=\bigcup_{|x_0|<X_0}\mathscr{D}(t_0,x_0),\quad X_0>0. \tag{5.3}$$

Take a point (t_0, X_0) and fix it. Putting

$$\Omega(t_0,X_0)\equiv\Omega. \tag{5.4}$$

And denote $\Omega(s)$ the intesection Ω and the hyperplane $t=s$. Namely

$$\Omega(s)=\Omega\cap\{(s,x,y)\}. \tag{5.5}$$

Proposition 5.1.

$$\begin{aligned} &Pv=f\in\mathscr{E}_t(\tilde{H}^{\infty}_{xy},) \\ &D^i_t v|_{t=0}=\phi_i(x,y)\in\tilde{H}^{\infty}_{x,y},\quad 0\leqq i\leqq m-1. \end{aligned} \tag{5.6}$$

Under the assumption (A–1) *and* (A–2), *the solution of the Cauchy problem* (5.6) *has the following estimate*;

$$\begin{aligned} &\sum_{i=0}^{m-r+p}\|D^i_t v\|_{k+m-r+p-i,D(t)} \\ &\leqq C_1(k,p)\{\sum_{i=0}^{m-1}\|\phi_i\|_{k+m-1+p-i,\Omega(0)} \\ &\quad+\int_0^t\sum_{i=0}^{p}\|D^i_s f(s)\|_{k+p-i,\Omega(s)}ds\}\qquad \textit{for } \forall p,\ \forall k. \end{aligned} \tag{5.7}$$

where $\|f\|^2_{k,\Omega(t)}=\sum_{j+|\alpha|\leqq k}\int_{\Omega(t)}|D^j_x D^{\alpha}_y|^2 dxdy$, *and* $C_1(k,p)$ *is a constant depending on* k, p *and* $\Omega(t)$ *but independent of* f *and* $\{\phi_i\}$.

This propoition is proved by the following tow lemmas.

Lemma 5.1. *Let us consider* (5.6). *We assume* (*A*–1), (*A*–2) *and moreover* $f\in\mathscr{E}_t(H^{\infty}_{x,y})$, $\phi_i\in H^{\infty}_{x,y}$ *then the solution of* (5.6) *has the following estimate*;

$$(5.7') \qquad \sum_{i=0}^{m-r+p} \|D_t^i v\|_{k+m-r+p-i}$$
$$\leqq C_1(k, p)\{\sum_{i=0}^{m-1} \|\phi_i\|_{k+m-1+p-i} + \int_0^t \sum_{i=0}^{p} \|D_s^i f(s)\|_{k+p-i} ds\}.$$

Lemma 5.2. *In the Cauchy problem* (5.6), *the domain of dependence of a point* (t_0, x_0, y) *is* $\mathscr{D}(t_0, x_0)$. *Namely if* $f \equiv 0$ *in* $\mathscr{D}(t_0, x_0)$ *and* $\phi_i \equiv 0$ *at* $\mathscr{D}(t_0, x_0) \cap \{t=0\}$, *then* $v \equiv 0$ *in* $\mathscr{D}(t_0, x_0)$.

Next, let us consider the solution of $Qu = v$.

Proposition 5.2.

$$(5.8) \qquad Qu = v \in \mathscr{E}_t(\tilde{H}^\infty_{x,y})$$
$$D_x^j u|_{x=0} = \psi_j(t, y) \in \mathscr{E}_t(H^\infty_y), \quad 0 \leqq j \leqq l-1.$$

Under the assumption (*A*–3) *and* (A–4), *the solution of the Cauchy problem* (5.8) *has the following estimate*;

$$(5.9) \qquad \sum_{h=0}^{p'} \|D_t^h\{\Gamma(q_{\nu-i}) \cdots \Gamma(q_1)u\}\|_{q'(i)+k+p'-h, \Omega(t)}$$
$$\leqq C_2(k, p')\{\sum_{h=0}^{p'} \sum_{j=0}^{l-1} \|D_t^h \psi_j(t, y)\|_{y, k+p'+l-1-j-h}$$
$$+ \sum_{h=0}^{p'} \|D_t^h v\|_{k+p'-h, \Omega(t)}\} \quad 0 \leqq i \leqq \nu$$

where $q'(i) = q_\nu + q_{\nu-1} + \cdots + q_{\nu-i+1} - i = l - (q_1 + q_2 + \cdots + q_{\nu-i}) - i$, $\|\psi\|^2_{y,k} = \sum_{|\alpha| \leqq k} \int |D_y^\alpha \psi|^2 dy$, $C_2(k, p')$ *is constant depending on* k, p' *and* $\Omega(t)$, *but independent of* v *and* $\{\psi_j\}$. *Especially when* $i = \nu$, (5.9) *is the following*;

$$(5.10) \qquad \sum_{h=0}^{p'} \|D_t^h u\|_{l-\nu+k+p'-h, \Omega(t)}$$
$$\leqq C_2(k, p)\{\sum_{h=0}^{p'} \sum_{j=0}^{l-1} \|D_t^h \psi_j(t, y)\|_{y, k+p'+l-1-j-h}$$
$$+ \sum_{h=0}^{p'} \|D_t^h v\|_{k+p'-h, \Omega(t)}\}$$

The proof of Proposition 5.2 is in §7.

§6. Proof of the Theorem 2.

Let

$$(6.1) \qquad Qu = v.$$

Then $Lu = PQu - Ru = f$ is equivalent to (6.2).

$$(6.2)\qquad \begin{cases} Qu=v \\ Pv=Ru+f. \end{cases}$$

Let us rewrite

$$(6.3)\qquad D_t^i(Qu)\,|_{t=0}=\sum_{k=0}^{i} C_{ik}(x,\ y;\ D_x,\ D_y)\phi_k(x,\ y)\equiv\tilde{\phi}_i(x,\ y),$$

where C_{ik} is differential operator with respect to x, peudo differential operator with respect to y and it's total order is at most l. Now, let v_1 be a solution of

$$(6.4)\qquad Pv_1=f,\quad D_t^i v_1\,|_{t=0}=\tilde{\phi}_i(x,\ y),\quad 0\leqq i\leqq m-1.$$

And u_1 be a solution of

$$(6.5)\qquad Qu_1=v_1,\quad D_x^j u_1\,|_{x=0}=\psi_j(t,\ y),\quad 0\leqq j\leqq l-1.$$

In general, for $\rho\geqq 2$, v_ρ be the solution of

$$(6.6)\qquad Pv_\rho=Ru_{\rho-1},\quad D_t^i v_\rho\,|_{t=0}=0,\quad 0\leqq i\leqq m-1.$$

And u_ρ be the solution of

$$(6.7)\qquad Qu_\rho=v_\rho,\quad D_t^j u_\rho|_{x=0}=0,\quad 0\leqq j\leqq l-1.$$

We want to prove that the serise $u_1+u_2+\cdots$ converge. Take k and p in (5.7) and fix them. By Prop. 5.1, we have

$$(6.8)\qquad \sum_{i=0}^{m-r+p}\|D_t^i v_1\|_{k+m-r+p-i,\Omega(t)}$$

$$\leqq C_1\{\sum_{i=0}^{m-1}\|\tilde{\phi}_i\|_{k+m-1+p-i,\Omega(0)}+\int_0^t\sum_{i=0}^{p}\|D_s^i f(s)\|_{k+p-i,\Omega(s)}ds\}.$$

By Prop. 5.2, we have the estimate of u_1. In (5.9), let k be the same in (6.8) and $p'=m-r+p$. Then

$$(6.9)\qquad \sum_{h=0}^{m-r+p}\|D_t^h\{\Gamma(q_{v-i})\cdots\Gamma(q_1)u_1\}\|_{q'(i)+k+m-r+p-k,\Omega(t)}$$

$$\leqq C_2\{\sum_{h=0}^{m-r+p}\sum_{j=1}^{l-1}\|D_t^h\psi_j(t,\ y)\|_{y,k+m-r+p+l-1-j-h}$$

$$+\sum_{h=0}^{m-r+p}\|D_t^h v_1\|_{k+m-r+p-h,\Omega(t)}\},\quad 0\leqq i\leqq v.$$

Let

$$(6.10)\qquad \sum_{i=0}^{m-1}\|\tilde{\phi}_i\|_{k+m-1+p-i,\Omega(0)}=M_1,$$

$$(6.11)\qquad \sup_{0\leqq s\leqq T}\{\sum_{i=1}^{p}\|D_s^i f(s)\|_{k+p-i,\Omega(s)}\}=K,$$

$$(6.12)\qquad \sum_{h=0}^{m-r+p}\sum_{j=0}^{l-1}\|D_t^h\psi_j(t,\ y)\|_{y,k+m-r+p+l-1-j-h}=M_2.$$

Because of (6.8)~(6.12), we have

$$(6.13)\qquad \sum_{h=0}^{m-r+p} \|D_t^h\{\Gamma(q_{\nu-i})\cdots\Gamma(q_1)u_1\}\|_{q'(i)+k+m-r+p-k,\Omega(t)}$$

$$\leqq C_2M_2+C_2C_1M_1+C_1C_2\int_0^t Kds=C_2M_2+C_2C_1M_1+C_1C_2Kt.$$

By the assumption (A–5), and (6.13) we have

$$(6.14)\qquad \sum_{h=0}^{p} \|D_t^h(Ru_1)\|_{k+p-h,\Omega(t)}$$

$$\leqq C_3'\sum_{i=0}^{\nu}\sum_{h=0}^{m-r+p} \|D_t^h\{\Gamma(q_{\nu-i})\cdots\Gamma(q_1)\}\|_{k+p-h+m-r+q'(i),\Omega(t)}\,.$$

Putting $C_3=(\nu+1)\times C_3'$, by (6.13) and (6.14) we have

$$(6.15)\qquad \sum_{h=0}^{p} \|D_t^h(Ru_1)\|_{k+p-h,\Omega(t)}\leqq C_2C_3M_2+C_1C_2C_3M_1+C_1C_2C_3Kt.$$

In general, by induction we have

Proposition 6.1. *The solution* u_ρ *of the problem* (6.7) *has the following estimate*;

$$(6.16)\qquad \sum_{h=0}^{m-r+p} \|D_t^h\{\Gamma(q_{\nu-i})\cdots\Gamma(q_1)u_\rho\}\|_{q'(i)+k+m-r+p-h,\Omega(t)}$$

$$\leqq(C_1C_2C_3)^{\rho-1}\left\{(C_2M_2+C_1C_2M_1)\frac{t^{\rho-1}}{(\rho-1)!}+C_1C_2K\frac{t^\rho}{\rho!}\right\}.$$

Especially when $i=\nu$, (6.16) *is the following*;

$$(6.17)\qquad \sum_{h=0}^{m-r+p} \|D_t^h u_\rho\|_{l-\nu+m-r+k+p-h,\Omega(t)}\leqq(C_1C_2C_3)^{\rho-1}\left\{M\frac{t^{\rho-1}}{(\rho-1)!}+\tilde{K}\frac{t^\rho}{\rho!}\right\}$$

where $M=C_2M_2+C_1C_2M_1$, $\tilde{K}=C_1C_2K$.

Therefore $\sum_{\rho=1}^{\infty} D_t^h u_\rho$ $(0\leqq h\leqq m-r+p)$ is convergent in $H^{l-\nu+m-r+k+p-h}(\Omega(t))$. Putting

$$(6.18)\qquad u=\sum_{\rho=1}^{\infty} u_\rho$$

then $D_t^h u\in H^{l-\nu+m-r+k+p-h}(\Omega(t))$, $0\leqq h\leqq m-r+p$. Where k and p are arbitrary then by Sobolev's lemma $u\in C^\infty(\Omega(t))$. It is obvious that this u is the solution of the Goursat problem (4.17).

Uniqueness of the solution. Let $u^{(1)}$ and $u^{(2)}$ be the solution of (4.17). And let $w=u^{(1)}-u^{(2)}$. Then w satisfies

$$(6.19)\qquad Lw=(PQ-R)w=0,$$

$$D_t^i w|_{t=0}=0\quad 0\leqq i\leqq m-1,$$

$$D_x^j w|_{x=0}=0 \quad 0\leqq j\leqq l-1.$$

By Prop. 5.1 we have

$$\sum_{h=0}^{m-r}\|D_t^h Qw\|_{m-r-h,\Omega(t)}\leqq C_1\int_0^t\|Rw\|_{\Omega(s)}ds. \tag{6.20}$$

In Prop. 5.2, Putting $p'=m-r$ and $k=0$, we have

$$\sum_{h=0}^{m-r}\|D_t^h\{\Gamma(q_{\nu-i})\cdots\Gamma(q_1)w\}\|_{q'(i)+m-r-h,\Omega(t)} \tag{6.21}$$

$$\leqq C_2\sum_{h=0}^{m-r}\|D_t^h Qw\|_{m-r-h,\Omega(t)}, \quad 0\leqq i\leqq\nu.$$

By the assumption (A–5),

$$\|Rw\|_{\Omega(s)}\leqq C_3\sum_{i=0}^{\nu}\sum_{h=0}^{m-r}\|D_t^h\{\Gamma(q_{\nu-i})\cdots\Gamma(q_1)\}w\|_{q'(i)+m-r-h,\Omega(s)}. \tag{6.22}$$

Let

$$\sum_{i=0}^{\nu}\sum_{h=0}^{m-r}\|D_t^h\{\Gamma(q_{\nu-i})\cdots\Gamma(q_1)w\}\|_{q'(i)+m-r-h,\Omega(t)}\equiv M_3(t), \tag{6.23}$$

then, by (6.21)~(6.23) we have

$$M_3(t)\leqq(\nu+1)C_2C_1\int_0^t\|Rw\|_{\Omega(s)}ds \tag{6.24}$$

$$\leqq(\nu+1)C_1C_2C_3\int_0^t M_3(s)ds.$$

Let $\tilde{M}_3=\sup_{0\leqq t\leqq T}M_3(t)$ and $(\nu+1)C_1C_2C_3=C$, we have

$$M_3(t)\leqq C\int_0^t M_3(s)ds\leqq C\tilde{M}_3 t. \tag{6.25}$$

Then $M_3(t)\leqq C\tilde{M}_3 t$. By (6.25) we have

$$M_3(t)\leqq C\int_0^t C\tilde{M}_3 s ds=C^2\tilde{M}_3\frac{t^2}{2!}. \tag{6.26}$$

In general for arbitraly $j\geqq 1$, we have

$$M_3(t)\leqq C^j\tilde{M}_3\frac{t^j}{j!}. \tag{6.27}$$

Then $M_3(t)\equiv 0$. This means $w\equiv 0$. Thus we complete the proof of Theorem 2.

§7. Proof of Proposition 5.2.

$$Qu=v\in\mathscr{E}_t(\tilde{H}^\infty_{x,y}) \tag{5.8}$$

$$D_x^j u|_{x=0}=\psi_j(t,y)\in\mathscr{E}_t(H^\infty_y) \quad 0\leqq t\leqq T, \quad 0\leqq j\leqq l-1.$$

Q is hyperbolic with respect to the direction x. Here we consider that t is parameter. Because of the theory of hyperbolic equations we have the following lemma:

Lemma 7.1. *The Cauchy problem* (5.8) *has the unique solution* $u \in \mathscr{E}_x(H_y^\infty)$ *and it has the following estimate.*

$$\sum_{j=0}^{q'(i)+p} \|D_x^j\{\Gamma(q_{v-i})\Gamma(q_{v-i-1})\cdots\Gamma(q_1)u\}\|_{y,k+q'(i)+p-j} \leqq C(k, p)\{\sum_{j=0}^{l-1} \|\psi_j(t, y)\|_{y,k+p+l-1-j} + \int_{|x'|\leqq|x|} \sum_{j=0}^{p} \|D_{x'}^j v(x')\|_{y,k+p-j}dx'\}. \tag{7.1}$$

Proof of Prop. 5.2. *For fixed* t, *let*

$$X(t) = \max_{(t,x,y)\in\Omega(t)} |x|. \tag{7.2}$$

By (7.1), putting $k=0$ we have

$$\int_{|x|\leqq X(t)} \sum_{j=0}^{q'(i)+p} \|D_x^j\{\Gamma(q_{v-i})\Gamma(q_{v-i-1})\cdots\Gamma(q_1)u\}\|_{y,q'(i)+p-j}^2 dx \leqq C'(k, p)\Big\{\int_{|x|\leqq X(t)} \sum_{j=0}^{l-1} \|\psi_j(t, y)\|_{y,p+l-1-j}^2 dx + \int_{|x|\leqq X(t)} \sum_{j=0}^{p} \Big(\int_{|x'|\leqq|x|} \|D_{x'}^j v(x')\|_{y,p-j}dx'\Big)^2 dx\Big\}. \tag{7.3}$$

The left hand side of (7.3) equals $\|\Gamma(q_{v-i})\cdots\Gamma(q_1)u\|_{q'(i)+p,\Omega(t)}^2$. And

$$\begin{aligned} &\int_{|x|\leqq X(t)} \sum_{j=0}^{p} \Big(\int_{|x'|\leqq|x|} \|D_{x'}^j v(x')\|_{y,p-j}dx'\Big)^2 dx \\ &\leqq \int_{|x|\leqq X(t)} \sum_{j=0}^{p} \Big\{\int_{|x'|\leqq|x|} 1^2 dx' \int_{|x'|\leqq|x|} \|D_{x'}^j v(x')\|_{y,p-j}^2 dx'\Big\} dx \\ &\leqq \int_{|x|\leqq X(t)} \sum_{j=0}^{p} \Big\{2X(t)\int_{|x'|\leqq X(t)} \|D_{x'}^j v(x')\|_{y,p-j}^2 dx'\Big\} dx \\ &\leqq \{2X(t)\}^2 \|v\|_{p,\Omega(t)}^2. \end{aligned} \tag{7.4}$$

Then

$$\|\Gamma(q_{v-i})\cdots\Gamma(q_1)u\|_{q'(i)+p,\Omega(t)}^2 \leqq C'(0, p)\{2X(t)\sum_{j=0}^{l-1} \|\psi_j(t, y)\|_{y,p+l-1-j}^2 + (2X(t))^2\|v\|_{p,\Omega(t)}^2\}. \tag{7.5}$$

If $p'=0$ in (5.9), (5.9) is equivalent to (7.5).

Next, let us consider the estimate of the derivative of t direction. Notice that in (5.8) t is a parameter. We differentiate (5.8) by t. And in the nearly same way we have the estimate of the derivative of t direction.

§8. Appendix.

Lemma 8.1. *Let $P(z)$ be the polynomial of order n;*

(8.1) $$P(z)=z^n+a_1z^{n-1}+\cdots+a_n, \quad a_i\in C.$$

τ_i $(i=1, 2,\dots, n)$ are the roots of $P(z)=0$. Let Γ is a convex hull of $\{\tau_i;\ i=1, 2,\dots, n\}$. Then the root of $\frac{d}{dz}P(z)=0$ is contained in Γ.

Lemma 8.2. *Consider the following polynomial of τ:*

(8.2) $$B(\tau;\ \zeta,\ \eta)=a_0(\zeta,\ \eta)\tau^n+a_1(\zeta,\ \eta)\tau^{n-1}+\cdots+a_n(\zeta,\ \eta)$$

where $\zeta\in C^1$, $\eta\in C^l$ and $a_i(\zeta, \eta)$ $(i=1, 2,\dots, n)$ is holomorphic function in a domain $D\subset C^{l+1}$. Let $h(\zeta, \eta)$ is a holomorphic function in D. There exist holomorphic function (in D) $\tau_i(\zeta, \eta)$ $(i=1, 2,\dots, n+1)$ such that

(8.3) $$B(\tau_i(\zeta,\ \eta);\ \zeta,\ \eta)\equiv 0 \bmod h(\zeta,\ \eta), \qquad \textit{for} \quad i=1, 2,\dots, n+1$$

and

(8.4) $$\{(\zeta,\ \eta);\ (\zeta,\ \eta)\in D,\ \tau_i(\zeta,\ \eta)-\tau_j(\zeta,\ \eta)=0\}$$
$$\cap\{(\zeta,\ \eta);\ (\zeta,\ \eta)\in D,\ h(\zeta,\ \eta)=0\}=\phi, \qquad \textit{for} \quad i\neq j,\quad i, j=1, 2,\dots, n+1.$$

Then $B(\tau;\ \zeta,\ \eta)\equiv 0 \bmod h(\zeta, \eta)$. i.e. $a_j(\zeta, \eta)\equiv 0 \bmod h(\zeta, \eta)$, $j=0, 1,\dots, n$. Where $f(\zeta, \eta)\equiv 0 \bmod h(\zeta, \eta)$ means that there exists holomorphic function (in D) $g(\zeta, \eta)$ such that $f(\zeta, \eta)=h(\zeta, \eta)g(\zeta, \eta)$.

Proof of Lemma 8.2. We have

$$B(\tau_i;\ \zeta,\ \eta)-B(\tau_1;\ \zeta,\ \eta)$$
$$=a_0(\tau_i^n-\tau_1^n)+a_1(\tau_i^{n-1}-\tau_1^{n-1})+\cdots+a_{n-1}(\tau_i-\tau_1),$$

then $B(\tau_i;\ \zeta,\ \eta)-B(\tau_1;\ \zeta,\ \eta)$ is divisible by $\tau_i-\tau_1$. Let

(8.5) $$\{B(\tau_i;\ \zeta,\ \eta)-B(\tau_1;\ \zeta,\ \eta)\}/(\tau_i-\tau_1)=B^{(1)}(\tau_i;\ \zeta,\ \eta)$$
$$=a_0b_n^{(1)}(\tau_i)+a_1b_{n-1}^{(1)}(\tau_i)+\cdots+a_{n-2}b_2^{(1)}(\tau_i)+a_{n-1}, \quad i=2, 3,\dots, n+1$$

where $b_n^{(1)}(\tau_i)=(\tau_i^n-\tau_1^n)/(\tau_i-\tau_1)$, $b_{n-1}^{(1)}(\tau_i)=(\tau_i^{n-1}-\tau_1^{n-1})/(\tau_i-\tau_1),\dots$, $b_2^{(1)}(\tau_i)=(\tau_i^2-\tau_1^2)/(\tau_i-\tau_1)=\tau_i+\tau_1$, i.e. $b_k(\tau)$ is a polynomial of τ of degree $k-1$ and the coefficient of τ^{k-1} is 1. By (8.4), we have

(8.6) $$B^{(1)}(\tau_i;\ \zeta,\ \eta)\equiv 0 \bmod h(\zeta,\ \eta), \quad i=2, 3,\dots, n+1.$$

Next, we consider $B^{(1)}(\tau_i;\ \zeta,\ \eta)-B^{(1)}(\tau_2;\ \zeta,\ \eta)$, $i=3, 4,\dots, n+1$. By (8.5) we have

$$B^{(1)}(\tau_i;\ \zeta,\ \eta)-B^{(1)}(\tau_2;\ \zeta,\ \eta)$$

$$=a_0\{b_n^{(1)}(\tau_i)-b_n^{(1)}(\tau_2)\}+a_1\{b_{n-1}^{(1)}(\tau_i)-b_{n-1}^{(1)}(\tau_2)\}+\cdots$$
$$+a_{n-2}\{b_2^{(1)}(\tau_i)-b_2^{(1)}(\tau_2)\}$$

$B^{(1)}(\tau_i;\zeta,\eta)-B^{(1)}(\tau_2;\zeta,\eta)$ is divisible by $\tau_i-\tau_2$, $i=3,4,\dots,n+1$. Let

(8.7) $$\{B^{(1)}(\tau_i;\zeta,\eta)-B^{(1)}(\tau_2;\zeta,\eta)\}/(\tau_i-\tau_2)$$
$$=B^{(2)}(\tau_i;\zeta,\eta)=a_0b_n^{(2)}(\tau_i)+a_1b_{n-1}^{(2)}(\tau_i)+\cdots+a_{n-2}.$$

$b_k^{(2)}(\tau)$ is a polynomial of degree $k-2$ and the coefficient τ^{k-2} is 1. By (8.4) and (8.6) we have

(8.8) $$B^{(2)}(\tau_i;\zeta,\eta)\equiv 0 \bmod h(\zeta,\eta).$$

In general we put

(8.9) $$\{B^{(s-1)}(\tau_i;\zeta,\eta)-B^{(s-1)}(\tau_s;\zeta,\eta)\}/(\tau_i-\tau_s)=B^{(s)}(\tau_i;\zeta,\eta)$$
$$=a_0b_n^{(s)}(\tau_i)+a_1b_{n-1}^{(s)}(\tau_i)+\cdots+a_{n-s},\quad i=s+1,s+2,\dots,n+1.$$

$b_k^{(s)}(\tau)$ is a polynomial of degree $k-s$ and the coefficient of τ^{k-s} is 1. And we have

(8.10) $$B^{(s)}(\tau_i;\zeta,\eta)\equiv 0 \bmod h(\zeta,\eta).$$

Last of all we have

$$B^{(n)}(\tau_i;\zeta,\eta)=a_0 \quad \text{for} \quad i=n+1,\quad B^{(n)}(\tau_{n+1};\zeta,\eta)\equiv 0 \bmod h(\zeta,\eta).$$

Then $a_0\equiv 0 \bmod h(\zeta,\eta)$. By (8.9) and (8.10) we have

$$B^{(n-1)}(\tau_i;\zeta,\eta)=a_0b_n^{(n-1)}(\tau_i)+a_1\equiv 0 \bmod h(\zeta,\eta),\quad i=n,n+1.$$

Then $a_1\equiv 0 \bmod h(\zeta,\eta)$.
In this way we have $a_2\equiv 0 \bmod h(\zeta,\eta)$, $a_3\equiv 0 \bmod h(\zeta,\eta),\dots, a_n\equiv 0 \bmod h(\zeta,\eta)$. After all we have $B(\tau;\zeta,\eta)\equiv 0 \bmod h(\zeta,\eta)$. q.e.d.

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