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On the Kazhdan-Lusztig conjecture for Kac-Moody algebras

By

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Introduction

Let $A=(a_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ integral matrix which satisfies the following conditions (i), (ii) and (iii):

(i) $a_{ii}=2$ for all $i=1,\ldots, n$,

(ii) $a_{ij} \leqq 0$ if $i \neq j$,

(iii) $a_{ij}=0$ if and only if $a_{ji}=0$.

We call such a matrix A a *generalized Cartan matrix* (GCM). For example, the Cartan matrix of a complex semisimple Lie algebra is a GCM.

One can associate a complex Lie algebra $\mathfrak{g}(A)$ to a GCM A (see §1 for detail). If A is the Cartan matrix of a complex semisimple Lie algebra, then $\mathfrak{g}(A)$ is the corresponding complex semisimple Lie algebra, and otherwise $\mathfrak{g}(A)$ is an infinite-dimensional Lie algebra. We call $\mathfrak{g}(A)$ a Kac-Moody algebra with the Cartan matrix A.

Kac-Moody algebras were introduced by [5] and [8] independently, as a generalization of complex semisimple Lie algebras. There are many theorems and conjectures on Kac-Moody algebras which are generalizations of corresponding results on complex semisimple Lie algebras. Among those, we study in this paper a generalization, given in [3], of the Kazhdan-Lusztig conjecture on the composition series of Verma modules. Let us explain this in more detail.

At first, we recall Kazhdan-Lusztig conjecture for complex semisimple Lie algebras. Let $\mathfrak{g}_0$ be a complex semisimple Lie algebra, $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$, and $\mathfrak{b}_0$ a Borel subalgebra containing $\mathfrak{h}_0$. Let Δ_0 be the root system of $(\mathfrak{g}_0, \mathfrak{h}_0)$, and $\rho_0 \in \mathfrak{h}_0^*$ half the sum of all the positive roots with respect to $\mathfrak{b}_0$. Let W_0 be the Weyl group of Δ_0, and S_0 the set of simple reflections. We denote by $M_0(\lambda)$ the Verma module with highest weight $\lambda \in \mathfrak{h}_0^*$ and $L_0(\lambda)$ the unique irreducible quotient of $M_0(\lambda)$.

Any irreducible subquotient of $M_0(\lambda)$ is of the form $L_0(\mu)$ for some $\mu \in \mathfrak{h}_0^*$ such that $\lambda-\mu \in Z\Delta_0$ and that $\mu=w(\lambda+\rho_0)-\rho_0$ for some $w \in W_0$. Every $M_0(\lambda)$ has a

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finite Jordan-Hölder series, and so the multiplicity $[M_0(\lambda): L_0(\mu)]$ is naturally defined. All the multiplicities $[M_0(\lambda): L_0(\mu)]$ for integral λ, $\mu \in \mathfrak{h}_0^*$ can be computed if $[M_0(y\rho_0-\rho_0): L_0(w\rho_0-\rho_0)]$ are known for all y, $w \in W_0$. Note that this index is zero unless $y \leqq w$, where $\leqq$ is the standard partial order on the Coxeter group (W_0, S_0) in which the unit is the smallest element. About these facts, one may refer to [4] for instance.

The following theorem on these multiplicities for complex semisimple Lie algebras is well-known as the Kazhdan-Lusztig conjecture, which was conjectured in [7] and proved in [2] and in [1] independently.

Theorem A [7, Conjecture 1.5]. *For all y, $w \in W_0$ such that $y \leqq w$, let $P_{y,w}$ be the Kazhdan-Lusztig polynomial for (W_0, S_0). Then, it holds that*

$$[M_0(y\rho_0-\rho_0): L_0(w\rho_0-\rho_0)] = P_{y,w}(1).$$

The Kazhdan-Lusztig polynomials were introduced in [7], related to the base change of Hecke algebras of Coxeter groups, and there were given the inductive formulas to compute these polynomials.

The Kazhdan-Lusztig conjecture was generalized, in [3], to Kac-Moody algebras with symmetrizable GCMs as follows.

Let $\mathfrak{h}(A)$ be the Cartan subalgebra of the Kac-Moody algebra $\mathfrak{g}(A)$, and ρ an element of $\mathfrak{h}(A)^*$ which takes the value 1 on each simple coroot. Let W be the Weyl group of $\mathfrak{g}(A)$, and $S=\{s_1,\dots, s_n\}$ the set of simple reflections in W. We denote by $M(\lambda)$ the Verma module over $\mathfrak{g}(A)$ with highest weight $\lambda \in \mathfrak{h}(A)^*$, and $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$. As in the case of complex semisimple Lie algebras, any irreducible subquotient of $M(\lambda)$ is of the form $L(\mu)$ $(\mu \in \mathfrak{h}(A)^*)$. Although $M(\lambda)$ does not have any finite Jordan-Hölder series in general, the multiplicity $[M(\lambda): L(\lambda)]$ is defined by means of local composition series as a generalization of that of complex semisimple Lie algebras (see §1).

Deodhar, Gabber and Kac proved the following result in [3] analogously to the complex semisimple case.

Theorem B [3, §5]. *Suppose that A is a symmetrizable GCM, that is, A is a GCM and there exists a non-degenerate diagonal $n \times n$ matrix D such that DA is a symmetric matrix. Let λ be a dominant integral emenent of $\mathfrak{h}(A)^*$ and $y \in W$. Then, all the irreducible subquotients of $M(y(\lambda+\rho)-\rho)$ are $L(w(\lambda+\rho)-\rho)$ $(w \in W, w \geqq y)$, and the multiplicity $[M(y(\lambda+\rho)-\rho): L(w(\lambda+\rho)-\rho)]$ is independent of λ. Here $\leqq$ is the standard partial order on the Coxeter group (W, S).*

Taking this result into account, they conjectured as follows.

Conjecture C [3, Conjecture 5.16]. *There holds that*

$$[M(y\rho-\rho): L(w\rho-\rho)] = P_{y,w}(1)$$

for all y, $w \in W$ such that $y \leqq w$, where $P_{y,w}$ are Kazhdan-Lusztig polynomials for (W, S).

We call this conjecture also the Kazhdan-Lusztig conjecture.

The aim of this paper is to prove that the conjecture above is affirmative for certain pairs (y, w), even if A is not necessarily symmetrizable. We explain this in more detail.

Let $A=(a_{ij})_{1\leqq i,j\leqq n}$ be a GCM. Take a subset I of $\{1,\dots, n\}$ such that $A_I=(a_{ij})_{i,j\in I}$ is the Cartan matrix of a complex semisimple Lie algebra $\mathfrak{g}_I$. Let $\mathfrak{h}_I$ be a Cartan subalgebra of $\mathfrak{g}_I$, $\mathfrak{b}_I$ a Borel subalgebra containing $\mathfrak{h}_I$, and so on. Then, $\mathfrak{g}_I$ and the Weyl group W_I are canonicaly embedded into $\mathfrak{g}(A)$ and W respectively in such a manner that $\mathfrak{h}_I\subset\mathfrak{h}(A)$, $\mathfrak{h}_I^*\subset\mathfrak{h}(A)^*$ and the set S_I of simple reflections in W_I is equal to $\{s_i \mid i\in I\}$. We define a category $\mathcal{O}$ of $\mathfrak{g}(A)$-modules and a category $\mathcal{O}_I$ of $\mathfrak{g}_I$-modules, containing all the highest weight modules, and define some exact functors from $\mathcal{O}$ to $\mathcal{O}_I$ corresponding to a decomposition of $\mathfrak{g}(A)$-modules as $\mathfrak{g}_I$-modules.

By applying these functors to a local composition series of a Verma module over $\mathfrak{g}(A)$, we can prove

Theorem2.3. *For any pair* (λ, μ) *in* $\mathfrak{h}(A)^*\times\mathfrak{h}(A)^*$ *such that* $\lambda-\mu\in \boldsymbol{Z}\Delta_I$, *we have the equality*

$$[M(\lambda)\colon L(\mu)]=[M_I(\lambda\mid\mathfrak{h}_I)\colon L_I(\mu\mid\mathfrak{h}_I)].$$

We see that the pairs $(\lambda, \mu)=(y\rho-\rho, w\rho-\rho)$ $(y, w\in W_I)$ satisfy the condition $\lambda-\mu\in \boldsymbol{Z}\Delta_I$ and that $(u\rho-\rho)|\mathfrak{h}_I=u\rho_I-\rho_I$ for any $u\in W_I$. So, if $y, w\in W_I$, Conjecture C is reduced to Theorem A by Theorem 2.3 above. Thus, we have one of our main results as follows.

Theorem 3.4. *Let* $A=(a_{ij})_{1\leqq i,j\leqq n}$ *be a GCM, and let I be a subset of* $\{1,\dots, n\}$ *such that* $A_I=(a_{ij})_{i,j\in I}$ *is the Cartan matrix of a complex semisimple Lie algebra. Denote by* W_I *the subgroup of W generated by* s_i $(i\in I)$. *Then, for all* $y, w\in W_I$ *such that* $y\leqq w$, *it holds that*

$$[M(y\rho-\rho)\colon L(w\rho-\rho)]=P_{y,w}(1).$$

Note that the symmetrizability of the GCM A is not assumed here.

Now we concentrate on the special case, that is, the case where A is a so-called extended Cartan matrix which is one of symmetrizable GCMs. In this case, the derived subalgebra $\mathfrak{g}'(A)$ of $\mathfrak{g}(A)$ is the universal central extension of a loop algebra $\boldsymbol{C}[t, t^{-1}]\otimes\mathfrak{g}_0$ with t an indeterminate, for a complex simple Lie algebra $\mathfrak{g}_0$, and $\mathfrak{g}(A)$ is the semidirect sum $\boldsymbol{C}d\ltimes\mathfrak{g}'(A)$ for a certain derivation d on $\mathfrak{g}'(A)$. We call $\mathfrak{g}(A)$ a non-twisted affine Lie algebra. There exists a subset I_0 of $\{1,\dots, n\}$ consisting of $n-1$ elements, such that $\mathfrak{g}_{I_0}=\mathfrak{g}_0$. We fix such a subset I_0.

We give a complete branching rule of Verma modules over $\mathfrak{g}(A)$ as $\mathfrak{g}_0$-modules (Proposition 4.2) by using the functors from $\mathcal{O}$ to $\mathcal{O}_{I_0}$, introduced before Theorem 2.3. By this branching rule, the problem of computing the multiplicities of irreducible subquotients of Verma modules over $\mathfrak{g}(A)$ is reduced to the problem of determining the branching rule of irreducible highest wiehgt modules over $\mathfrak{g}(A)$ as $\mathfrak{g}_0$-modules (see Remark 4.3). Solving this problem, we will get a useful tool to study Conjecture C in full generality for this type of $\mathfrak{g}(A)$.

This paper is organized as follows. In §1, we recall the definition of Kac-Moody algebras and some of their properties. In §2, we define certain subalgebras of Kac-Moody algebras, which we call canonical subalgebras, isomorphic to some complex semisimple Lie algebras. We prove some relations between highest weight modules over Kac-Moody algebras and those over their canonical subalgebras. Making use of these relations, we obtain a multiplicity equality (Theorem 2.3) which is our main tool in proving Theorem 3.4, one of our main results. In §3, we introduce a generalization of the Kazhdan-Lusztig conjecture after [3], and then prove that the generalized conjecture is true for certain good cases (Theorem 3.4), using Theorem 2.3. In §4, we consider the special case where the GCM A is an extended Cartan matrix. For such an A, we fix a canonical subalgebra $\mathfrak{g}_0$ of the Kac-Moody algebra $\mathfrak{g}(A)$ and give a complete branching rule of Verma modules over $\mathfrak{g}(A)$ as $\mathfrak{g}_0$-modules (Proposition 4.2).

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§1. Preliminaries for Kac-Moody algebras

We denote by $\boldsymbol{C}$, $\boldsymbol{Z}$ and $\boldsymbol{Z}_{\geqq 0}$ the set of all complex numbers, that of all integers and that of all non-negative integers, respectively. For any complex vector space V, the dual space of V is denoted by V^*. We denote by $\sharp(X)$ the cardinal number of a set X.

In this section, we recall the definition and some properties of Kac-Moody algebras and of their representations (cf., [6] for detailed discussion).

1.1. Definition of Kac-Moody algebras

Let $A=(a_{ij})_{i,j=1,\dots,n}$ be an $n\times n$ integral matrix. We call A a *generalized Cartan matrix* (GCM) if A satisfies the following conditions (i), (ii) and (iii):

(i) $a_{ii}=2$ for all $i=1,\dots, n$,

(ii) $a_{ij}\leqq 0$ if $i\neq j$,

(iii) $a_{ij}=0$ if and only if $a_{ji}=0$.

In this paper, we always assume A to be a GCM.

For a GCM $A=(a_{ij})$, there exists a (unique up to isomorphisms) complex Lie algebra $\mathfrak{g}(A)$ which has the following properties (i), (ii) and (iii):

(i) $\mathfrak{g}(A)$ has a commutative subalgebra $\mathfrak{h}(A)$ such that $\dim \mathfrak{h}(A)=2n-\operatorname{rank} A$ and

$$\mathfrak{g}(A)=\sum_{\alpha\in\mathfrak{h}(A)^*}\mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha=\{x\in\mathfrak{g}(A)\mid [h, x]=\alpha(h)x \text{ for all } h\in\mathfrak{h}(A)\},$$

$$g_0 = \mathfrak{h}(A).$$

(ii) There exist a linearly independent subset $\Pi = \{\alpha_1, \dots, \alpha_n\}$ of $\mathfrak{h}(A)^*$ and $2n$ elements $e_1, \dots, e_n, f_1, \dots, f_n$ in $\mathfrak{g}(A) \setminus \{0\}$ such that

a) $\mathfrak{g}_{\alpha_i} = \boldsymbol{C}e_i$ and $\mathfrak{g}_{-\alpha_i} = \boldsymbol{C}f_i$ for all $i = 1, \dots, n$,

b) $[e_i, f_j] = 0$ if $i \neq j$,

c) $\mathfrak{h}(A) \cup \{e_1, \dots, e_n, f_1, \dots, f_n\}$ generates $\mathfrak{g}(A)$ as a Lie algebra,

d) for each $i = 1, \dots, n$, we put $\alpha_i^\vee = [e_i, f_i]$, then $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is a linearly independent subset of $\mathfrak{h}(A)$ and it holds that $\alpha_j(\alpha_i^\vee) = a_{ij}$ for all $i, j = 1, \dots, n$.

(iii) Any ideal of $\mathfrak{g}(A)$ which intersects $\mathfrak{h}(A)$ trivially is zero.

The Lie algebra $\mathfrak{g}(A)$ is called a Kac-Moody algebra, and the subalgebra $\mathfrak{h}(A)$ the Cartan subalgebra of $\mathfrak{g}(A)$.

For each $\alpha \in \mathfrak{h}(A)^* \setminus \{0\}$, we denote by $\mathrm{mult}(\alpha)$ the dimension of $\mathfrak{g}_\alpha$, and call it the multiplicity of α. If $\mathrm{mult}(\alpha) \neq 0$, α is called a root of $\mathfrak{g}(A)$, and the set of all roots of $\mathfrak{g}(A)$ is denoted by $\Delta(A)$ and is called the root system of $\mathfrak{g}(A)$. We call the decomposition $\mathfrak{g}(A) = \mathfrak{h}(A) + \sum_{\alpha \in \Delta(A)} \mathfrak{g}_\alpha$ the root space decomposition.

We put $Q = \sum_{1 \leqq i \leqq n} \boldsymbol{Z}\alpha_i$ and $Q_+ = \sum_{1 \leqq i \leqq n} \boldsymbol{Z}_{\geqq 0}\alpha_i$. Q is called the root lattice for $\mathfrak{g}(A)$. $\Delta(A)$ has the following properties.

(i) $\Delta(A) \subset Q_+ \cup (-Q_+)$.

(ii) For any $\alpha \in \mathfrak{h}(A)^*$, $\mathrm{mult}(\alpha) = \mathrm{mult}(-\alpha)$, in particular, $\alpha \in \Delta(A)$ if and only if $-\alpha \in \Delta(A)$.

(iii) Put $\Delta_+(A) = \Delta(A) \cap Q_+$, and it holds that $\Delta(A) = \Delta_+(A) \cup (-\Delta_+(A))$ (disjoint union).

We call each element of $\Delta_+(A)$ a positive root. Every element of $\Pi \subset \Delta_+(A)$ (resp. $\Pi^\vee$) is called a simple root (resp. a simple coroot). We define a partial order $\geqq$ on $\mathfrak{h}(A)^*$ by

$$\lambda \geqq \mu \quad \text{if and only if } \lambda - \mu \in Q_+ \quad (\lambda, \mu \in \mathfrak{h}(A)^*).$$

For all $\alpha, \beta \in \mathfrak{h}(A)^*$, we have

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta},$$

and so the subspaces $\mathfrak{n}_\pm = \sum_{\alpha \in \Delta_+(A)} \mathfrak{g}_{\pm\alpha}$ are both subalgebras of $\mathfrak{g}(A)$, and we have a triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- + \mathfrak{h}(A) + \mathfrak{n}_+ \quad \text{(direct sum)}.$$

1.2. The Weyl group

We define involutive linear operators s_i on $\mathfrak{h}(A)^*$ by

$$s_i\lambda = \lambda - \lambda(\alpha_i^\vee)\alpha_i \quad (1 \leqq i \leqq n)$$

for all $\lambda \in \mathfrak{h}(A)^*$. Let W be the subgroup of $GL(\mathfrak{h}(A)^*)$ generated by $S = \{s_1, \dots, s_n\}$.

We call W the Weyl group of $\mathfrak{g}(A)$. The pair (W, S) is a Coxeter system. For any $\alpha \in \mathfrak{h}(A)^*$ and any $w \in W$,

$$\operatorname{mult}(\alpha) = \operatorname{mult}(w\alpha),$$

and so $\Delta(A)$ is W-invariant.

We define the subset $\Delta^{re}(A)$ and $\Delta^{im}(A)$ of $\Delta(A)$ by

$$\Delta^{re}(A) = \{w\alpha_i \mid w \in W,\ i = 1, \ldots, n\},$$

$$\Delta^{im}(A) = \Delta(A) \setminus \Delta^{re}(A),$$

and call each element of $\Delta^{re}(A)$ or $\Delta^{im}(A)$ a real root or an imaginary root, respectively.

1.3. Category $\mathcal{O}$ of $\mathfrak{g}(A)$-modules

For any $\mathfrak{g}(A)$-module M and any $\mu \in \mathfrak{h}(A)^*$, we put

$$M_\mu = \{v \in M \mid hv = \mu(h)v \text{ for all } h \in \mathfrak{h}(A)\}.$$

M_μ is called the weight space of weight μ. If $M_\mu \neq 0$, μ is called a weight of M.

We define a category $\mathcal{O}$ of $\mathfrak{g}(A)$-modules as follows. The objects of $\mathcal{O}$ are $\mathfrak{g}(A)$-modules M which satisfy the following conditions (i) and (ii):

(i) M is $\mathfrak{h}(A)$-semisimple, i.e., $M = \sum_{\mu \in \mathfrak{h}(A)^*} M_\mu$, and $\dim M_\mu < +\infty$ for all $\mu \in \mathfrak{h}(A)^*$.

(ii) There exists a finite subset $\{\mu_1, \ldots, \mu_k\}$ of $\mathfrak{h}(A)^*$ such that $\mu \leqq \mu_i$ for some μ_i for any weight μ of M.

Let $\lambda \in \mathfrak{h}(A)^*$ and M a $\mathfrak{g}(A)$-module. We call M a highest weight module with highest weight λ if there exists $v \in M_\lambda \setminus \{0\}$ such that

$$\mathfrak{n}_+ v = 0 \quad \text{and} \quad M = U(\mathfrak{g}(A))v,$$

where for any Lie algebra $\mathfrak{a}$, we denote by $U(\mathfrak{a})$ the universal enveloping algebra of $\mathfrak{a}$. Then, we have

$$M = \sum_{\alpha \in Q_+} M_{\lambda - \alpha}, \quad M_\lambda = \mathbf{C}v,$$

and for $\alpha > 0$,

$$M_{\lambda - \alpha} = \sum \mathfrak{g}_{-\beta_1} \cdots \mathfrak{g}_{-\beta_j} v,$$

where the sum runs over $\beta_1, \ldots, \beta_j \in \Delta_+(A)$, $\beta_1 + \cdots + \beta_j = \alpha$. Hence, any highest weight module is an object of $\mathcal{O}$. The above v is called a highest weight vector.

For any $\lambda \in \mathfrak{h}(A)^*$, let I_λ be the left ideal of $U(\mathfrak{g}(A))$ generated by $\mathfrak{n}_+$ and $\{h - \lambda(h) \mid h \in \mathfrak{h}(A)\}$. Then, $M(\lambda) = U(\mathfrak{g}(A))/I_\lambda$ is a highest weight module with highest weight λ, and any highest weight module with highest weight λ is isomorphic to a quotient of the Verma module $M(\lambda)$.

$M(\lambda)$ has a unique irreducible quotient, denoted by $L(\lambda)$. Any irreducible object of $\mathcal{O}$ is isomorphic to $L(\lambda)$ for some $\lambda \in \mathfrak{h}(A)^*$. As a $U(\mathfrak{n}_-)$-module, $M(\lambda)$ is a free module of rank 1 and any highest weight vector is a free basis.

Each object in $\mathcal{O}$ does not necessarily have a finite Jordan-Hölder series, but has a local composition series as follows.

Proposition 1.1 [3, Proposition 3.2]. *Let $M \in \mathcal{O}$ and $\lambda \in \mathfrak{h}(A)^*$. Then, there exist a finite sequence $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ of $\mathfrak{g}(A)$-submodules of M and a subset J of $\{1, \ldots, t\}$ such that*

$$M_j/M_{j-1} \simeq L(\lambda_j) \quad \textit{for some} \quad \lambda_j \geqq \lambda \quad \textit{if} \quad j \in J,$$

$$(M_j/M_{j-1})_\mu = 0 \quad \textit{for any} \quad \mu \geqq \lambda \quad \textit{if} \quad j \notin J.$$

We call this sequence a *local composition series* of M at λ.

Let $\mu \in \mathfrak{h}(A)^*$. Take a $\lambda \in \mathfrak{h}(A)^*$ such that $\mu \geqq \lambda$. For any object M of $\mathcal{O}$, let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a local composition series of M at λ, and λ_j $(j \in J)$ be as in Proposition 1.1. We put

$$[M : L(\mu)] = \sharp\{j \in J \mid \mu = \lambda_j\}.$$

Then, $[M : L(\mu)]$ is independent of λ and the local composition series. We call $[M : L(\mu)]$ the multiplicity of $L(\mu)$ in M. If $[M : L(\mu)] \neq 0$, we call $L(\mu)$ an irreducible component of M. For any $\mu \in \mathfrak{h}(A)^*$, $L(\mu)$ is an irreducible component of M if and only if $L(\mu)$ is isomorphic to a subquotient of M.

1.4. Contravariant bilinear form

$\mathfrak{g}(A)$ has a unique involutive automorphism ω such that

$$\omega(h) = -h \quad (h \in \mathfrak{h}(A)),$$

$$\omega(e_i) = -f_i \quad (1 \leqq i \leqq n).$$

Let M be a $\mathfrak{g}(A)$-module and B a bilinear form on M. B is called contravariant if

$$B(xv_1, v_2) = -B(v_1, \omega(x)v_2) \quad (x \in \mathfrak{g}(A),\ v_1, v_2 \in M).$$

For any $\lambda \in \mathfrak{h}(A)^*$, $L(\lambda)$ has a (unique up to scalar multiples) contravariant symmetric non-degenerate bilinear form, and the decomposition of $L(\lambda)$ into weight spaces is an orthogonal decomposition with respect to this bilinear form. Conversely, any highest weight module which has a contravariant symmetric non-degenerate bilinear form is irreducible.

§2. Relations between highest weight modules over $\mathfrak{g}(A)$ and those over their canonical subalgebras

In this section, we define some finite-dimensional subalgebras of $\mathfrak{g}(A)$, and prove some relations between highest weight modules over $\mathfrak{g}(A)$ and those over these subalgebras. Making use of these relations, we have a multiplicity equality which plays the principal role in proving the Kazhdan-Lusztig conjecture.

2.1. Canonical subalgebras and branchig rules for them

Let I be a subset of the index set $\{1, \ldots, n\}$ of A. We put

$$\mathfrak{h}_I=\sum_{i\in I} C\alpha_i^\vee,\quad Q_I=\sum_{i\in I} Z\alpha_i,$$

$$(2.1)\qquad \Delta_I=\Delta(A)\cap Q_I,\quad \Delta_{I,+}=\Delta_+(A)\cap Q_I,$$

$$\mathfrak{n}_{I,\pm}=\sum_{\alpha\in\Delta_{I,+}}\mathfrak{g}_{\pm\alpha},\quad \mathfrak{g}_I=\mathfrak{n}_{I,-}+\mathfrak{h}_I+\mathfrak{n}_{I,+}.$$

We call I *of finite type* if $A_I=(a_{ij})_{i,j\in I}$ is one of classical Cartan matrices, that is, the Cartan matrix of a complex semisimple Lie algebra.

In this section, we assume I to be of finite type. For such an I, it is proved by using Serre's structure theory of split semisimple Lie algebras, that $\mathfrak{g}_I$ is a finite-dimensional complex semisimple Lie algebra which has the Cartan matrix A_I, a Cartan subalgebra $\mathfrak{h}_I$, and a Borel subalgebra $\mathfrak{h}_I+\mathfrak{n}_{I,+}$ (cf., [9] and [6, Exercise 1.2]). We call such a $\mathfrak{g}_I$ a canonical subalgebra.

We define a category $\mathcal{O}_I$ of $\mathfrak{g}_I$-modules similarly to $\mathcal{O}$ as follows. The objects of $\mathcal{O}_I$ are $\mathfrak{g}_I$-modules M satisfying the following conditions.

(i) M is $\mathfrak{h}_I$-semisimple with finite-dimensional weight spaces.

(ii) There exists a finite subset $\{\mu_1,\dots,\mu_k\}$ of $\mathfrak{h}_I^*$ such that $\mu\leq\mu_i$ for some i for any weight μ of M.

The morphisms in $\mathcal{O}_I$ are $\mathfrak{g}_I$-homomorphisms.

Let $M_I(\lambda)$ be the Verma module over $\mathfrak{g}_I$ with highest weight $\lambda\in\mathfrak{h}_I^*$, and $L_I(\lambda)$ the unique irreducible quotient of $M_I(\lambda)$. As in the case of $\mathfrak{g}(A)$, every highest weight module over $\mathfrak{g}_I$ is an object of $\mathcal{O}_I$, and so $M_I(\lambda)$ and $L_I(\lambda)$ belong to $\mathcal{O}_I$ for any $\lambda\in\mathfrak{h}_I^*$.

We consider the quotient $\mathfrak{h}(A)^*/Q_I$ as additive groups. Let $\Lambda\in\mathfrak{h}(A)^*/Q_I$. We define M^Λ for any object M of $\mathcal{O}$ by

$$M^\Lambda=\sum_{\lambda\in\Lambda} M_\lambda.$$

It is clear that M^Λ is an object of $\mathcal{O}_I$ and that M decomposes into the direct sum of M^Λ's as a g_I-module. We have

Lemma 2.1. *For any $\Lambda\in\mathfrak{h}(A)^*/Q_I$, the functor*

$$\mathcal{O}\ni M\longmapsto M^\Lambda\in\mathcal{O}_I$$

is exact.

Proof. Let $M\in\mathcal{O}$ and N a $\mathfrak{g}(A)$-submodule of M. Since $N_\lambda=N\cap M_\lambda$, we have

$$N^\Lambda=\sum_{\lambda\in\Lambda}(N\cap M_\lambda)\subset N\cap(\sum_{\lambda\in\Lambda} M_\lambda),$$

and so $N^\Lambda\subset N\cap M^\Lambda$. Conversely, take $v\in N\cap M^\Lambda$. It is written as

$$v=\sum_{\lambda\in\Lambda} v_\lambda\quad(v_\lambda\in M_\lambda).$$

Because N is $\mathfrak{h}(A)$-invariant, $v_\lambda\in N$ for all $\lambda\in\Lambda$ and so $v\in\sum_{\lambda\in\Lambda}(N\cap M_\lambda)=N^\Lambda$. Thus, $N\cap M^\Lambda\subset N^\Lambda$, whence $N^\Lambda=N\cap M^\Lambda$. This implies that

$$\mathrm{Ker}\,(p\,|\,M^{\Lambda})=N^{\Lambda},$$

where p is the canonical projection from M onto M/N.

On the other hand, p is a surjective $\mathfrak{g}(A)$-homomorphism so that

$$p(M_{\lambda})=(M/N)_{\lambda}$$

for every $\lambda\in\mathfrak{h}(A)^*$, and so

$$p(M^{\Lambda})=(M/N)^{\Lambda}.$$

Therefore

$$M^{\Lambda}/N^{\Lambda}\simeq(M/N)^{\Lambda}$$

through p, which implies the lemma. Q.E.D.

For an element $\lambda\in\mathfrak{h}(A)^*$, we denote by $[\lambda]$ the residue class containing λ in $\mathfrak{h}(A)^*/Q_I$. We have the following lemma which explains what $M^{[\lambda]}$ is for a highest weight module M over $\mathfrak{g}(A)$ with the highest weight λ.

Lemma 2.2. *Let M be a highest weight module over $\mathfrak{g}(A)$ with highest weight $\lambda\in\mathfrak{h}(A)^*$. Then, $M^{[\lambda]}$ is a highest weight module over $\mathfrak{g}_I$ with highest weight $\lambda|\mathfrak{h}_I$. Moreover, we have the following isomorphisms*

$$M(\lambda)^{[\lambda]}\simeq M_I(\lambda\,|\,\mathfrak{h}_I), \tag{2.2}$$

$$L(\lambda)^{[\lambda]}\simeq L_I(\lambda\,|\,\mathfrak{h}_I). \tag{2.3}$$

Proof. Let v_0 be a non-zero highest weight vector of M. By definition

$$U(\mathfrak{g}_I)v_0\subset M^{[\lambda]}.$$

On the other hand, if μ is a weight of M, then

$$M_{\mu}=\sum\mathfrak{g}_{-\beta_1}\cdots\mathfrak{g}_{-\beta_j}v_0,$$

where the sum runs over $\beta_1,\dots,\beta_j\in\Delta_+(A)$, $\beta_1+\cdots+\beta_j=\lambda-\mu$. This implies that $M_{\mu}\subset U(\mathfrak{g}_I)v_0$ for $\mu\in[\lambda]$, because if $\beta_1,\dots,\beta_j\in\Delta_+(A)$ and $\beta_1+\cdots+\beta_j\in Q_I$, then $\beta_1,\dots,\beta_j\in\Delta_I$. Hence we have

$$U(\mathfrak{g}_I)v_0=M^{[\lambda]}.$$

This proves the first assertion of the lemma.

It is clear that $U(\mathfrak{g}_I)v_0$ is $U(\mathfrak{n}_{I,-})$-free if M is $U(\mathfrak{n}_-)$-free, which implies (2.2).

Let B be a (unique up to scalar multiples) non-degenerate contravariant symmetric bilinear form on $L(\lambda)$. Since the weight space decomposition of $L(\lambda)$ is an orthogonal decomposition with respect to B, we have $B(L(\lambda)^{\Lambda}, L(\lambda)^{\Theta})=0$ for Λ and Θ two different residue classes in $\mathfrak{h}(A)^*/Q_I$. Hence, the restriction $B\,|\,L(\lambda)^{[\lambda]}$ is non-degenerate. Thus, we have a non-degenerate contravariant symmetric bilinear form on $L(\lambda)^{[\lambda]}$. On the other hand, $L(\lambda)^{[\lambda]}$ is a highest weight module with highest weight $\lambda\,|\,\mathfrak{h}_I$ as we have proved above, so that $L(\lambda)^{[\lambda]}$ is isomorphic to $L_I(\lambda\,|\,\mathfrak{h}_I)$ as is stated in (2.3). This completes the proof of the Lemma. Q.E.D.

2.2. Multiplicity equalities

Now we can prove the following theorem, using the above two lemmas. This is one of our main results, and plays the principal role in proving the Kazhdan-Lusztig conjecture.

Theorem 2.3. *Let $I \subset \{1,\ldots, n\}$ be of finite type. Let $\lambda, \lambda' \in \mathfrak{h}(A)^*$ and assume $\lambda - \lambda' \in Q_I$. Then, there holds the following equality for the multiplicities*

$$[M(\lambda): L(\lambda')] = [M_I(\lambda \mid \mathfrak{h}_I): L_I(\lambda' \mid \mathfrak{h}_I)]. \tag{2.4}$$

Proof. If $\lambda \not\geqq \lambda'$, the left hand side of (2.4) is zero. On the other hand, if $\lambda \not\geqq \lambda'$ and $\lambda - \lambda' \in Q_I$, then $\lambda \mid \mathfrak{h}_I \not\geqq \lambda' \mid \mathfrak{h}_I$ and so the right hand side of (2.4) is also equal to zero.

Assume now $\lambda \geqq \lambda'$. Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M(\lambda)$ be a local composition series of $M(\lambda)$ at λ', that is, there exists a subset J of $\{1,\ldots, t\}$ such that if $j \in J$, $M_j/M_{j-1} \simeq L(\lambda_j)$ for some $\lambda_j \geqq \lambda'$, and if $j \notin J$, $(M_j/M_{j-1})_\mu = 0$ for any $\mu \geqq \lambda'$.

Let $j \in J$. Then, $\lambda \geqq \lambda_j$ because λ_j is a weight of $M(\lambda)$. So, $\lambda_j - \lambda' \in Q_I$ because $0 \leqq \lambda_j - \lambda' \leqq \lambda - \lambda' \in Q_I$. This implies $[\lambda_j] = [\lambda'] = [\lambda]$ for any $j \in J$. So, by Lemma 2.1, (2.2) and (2.3), we have a filtration of $M_I(\lambda \mid \mathfrak{h}_I)$

$$0 = M_0^{[\lambda]} \subset M_1^{[\lambda]} \subset \cdots \subset M_t^{[\lambda]} = M_I(\lambda \mid \mathfrak{h}_I)$$

such that for every $j \in J$

$$M_j^{[\lambda]}/M_{j-1}^{[\lambda]} \simeq (M_j/M_{j-1})^{[\lambda]} \simeq L_I(\lambda_j \mid \mathfrak{h}_I).$$

Let $j \notin J$. Then, by definition, any weight of $M_j^{[\lambda]}/M_{j-1}^{[\lambda]} \simeq (M_j/M_{j-1})^{[\lambda]}$ is of the form $\mu \mid \mathfrak{h}_I$, where μ is a weight of M_j/M_{j-1} such that $\mu - \lambda' \in Q_I$ and $\mu \not\geqq \lambda'$. Hence, $(M_j^{[\lambda]}/M_{j-1}^{[\lambda]})_{\mu'} = 0$ for any $\mu' \geqq \lambda' \mid \mathfrak{h}_I$.

Putting these together, we see that the series

$$0 = M_0^{[\lambda]} \subset M_1^{[\lambda]} \subset \cdots \subset M_t^{[\lambda]} = M_I(\lambda \mid \mathfrak{h}_I)$$

is a local composition series at $\lambda' \mid \mathfrak{h}_I$, and so

$$[M_I(\lambda \mid \mathfrak{h}_I): L_I(\lambda' \mid \mathfrak{h}_I)] = \sharp\{j \in J \mid \lambda_j \mid \mathfrak{h}_I = \lambda' \mid \mathfrak{h}_I\}.$$

On the other hand, for any $j \in J$, $[\lambda_j] = [\lambda']$ and so $\lambda_j \mid \mathfrak{h}_I = \lambda' \mid \mathfrak{h}_I$ implies $\lambda_j = \lambda'$. Therefore

$$\sharp\{j \in J \mid \lambda_j \mid \mathfrak{h}_I = \lambda' \mid \mathfrak{h}_I\} = \sharp\{j \in J \mid \lambda_j = \lambda'\}$$
$$= [M(\lambda): L(\lambda')].$$

This proves the theorem. Q.E.D.

§ 3. A partial solution of the Kazhdan-Lusztig conjecture for Kac-Moody algebras

In this section, at first, we recall the Kazhdan-Lusztig conjecture for complex semisimple Lie algebras, and then introduce a generalization of it to Kac-Moody algebras after [3]. Making use of Theorem 2.3, we prove that the generalized conjecture is true for certain good cases (see §3.3).

3.1. Kazhdan-Luszitg conjecture

Let $\mathfrak{g}_0$ be a complex semisimple Lie algebra, $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$, and $\mathfrak{b}_0$ a Borel subalgebra containing $\mathfrak{h}_0$. Let ρ_0 be the element of $\mathfrak{h}_0^*$ which takes the value 1 on each simple coroot corresponding to $\mathfrak{b}_0$. Let W_0 be the Weyl group of $(\mathfrak{g}_0, \mathfrak{h}_0)$ and S_0 the set of simple reflections. For each $\lambda \in \mathfrak{h}_0^*$, let $M_0(\lambda)$ be the Verma module with highest weight λ and $L_0(\lambda)$ the unique irreducible quotient of $M_0(\lambda)$.

The following theorem on multiplicities of irreducible subquotients of Verma modules is well-known as the Kazhdan-Lusztig conjecture, which was conjectured in [7] and proved in [2] and in [1] independently.

Theorem 3.1 [7, Conjecture 1.5]. *For all y, w in W_0 such that $y \leqq w$, where $\leqq$ is the standard partial order on the Coxeter group (W_0, S_0), let $P_{y,w}$ be the Kazhdan-Lusztig polynomial for (W_0, S_0). Then, we have*

$$[M_0(y\rho_0-\rho_0)\colon L_0(w\rho_0-\rho_0)]=P_{y,w}(1).$$

Note that the computation of multiplicities of irreducible subquotients of Verma modules with integral highest weights is reduced to the case in the above theorem (cf., [4], for example).

3.2. Generalization of the conjecture

In the case where A is a symmetrizable GCM, Deodhar, Gabber and Kac proved the following result in [3] analogously to the complex semisimple case.

Theorem 3.2 [3, §5]. *Let λ be a dominant integral element of $\mathfrak{h}(A)^*$ and $y \in W$. Then, all the irreducible subquotients of $M(y(\lambda+\rho)-\rho)$ are $L(w(\lambda+\rho)-\rho)$ $(w \in W,\ w \geqq y)$, and the multiplicity $[M(y(\lambda+\rho)-\rho)\colon L(w(\lambda+\rho)-\rho)]$ is independent of λ. Here ρ is an element of $\mathfrak{h}(A)^*$ which takes the value 1 on each simple coroot and $\geqq$ is the standard partial order on the Coxeter group (W, S).*

Taking this result into account, they conjectured as follows.

Conjecture 3.3 [3, Conjecture 5.16]. *For all y, w in W such that $y \leqq w$,*

$$[M(y\rho-\rho)\colon L(w\rho-\rho)]=P_{y,w}(1),$$

where $P_{y,w}$ is the Kazhdan-Lusztig polynomial for (W, S).

We call this conjecture also the Kazhdan-Lusztig conjecture.

Note that the polynomial $P_{y,w}$ and the multiplicity $[M(y\rho-\rho)\colon L(w\rho-\rho)]$ are both equal to zero if $y \nleqq w$.

3.3. Main theorem

Making use of Theorem 2.3 in §2, we can prove the following theorem which implies that Conjecture 3.3 holds for some pairs (y, w).

Theorem 3.4. *Let I be any subset of finite type of $\{1,\dots, n\}$ and W_I the subgroup of W generated by s_i $(i\in I)$. Then, for all y, w in W_I such that $y\leqq w$, we have the following expression for the multiplicity*:

$$[M(y\rho-\rho)\colon L(w\rho-\rho)]=P_{y,w}(1)\,.$$

Proof. Notations are as in §2. It is clear that W_I is isomorphic to the Weyl group of $(\mathfrak{g}_I, \mathfrak{h}_I)$, whose standard order as a Coxeter group is the restriction of $\leqq$ in (W, S). And so, by Theorem 3.1, we have

$$(*)\qquad P_{y,w}(1)=[M_I(y\rho_I-\rho_I)\colon L_I(w\rho_I-\rho_I)]$$

for all y, w in W_I such that $y\leqq w$, where ρ_I is half the sum of all elements in $\Delta_{I,+}$.

On the other hand, for any $i\in I$

$$\rho_I(\alpha_i^\vee)-\rho(\alpha_i^\vee)=1-1=0.$$

Hence, we have $(\rho_I-\rho)\,|\,\mathfrak{h}_I=0$, which implies that

$$y(\rho_I-\rho)=\rho_I-\rho$$

for all $y\in W_I$. Therefore, it holds that for all $y\in W_I$

$$\begin{aligned}y\rho_I-\rho_I&=y(\rho+\rho_I-\rho)-\rho_I\\&=y\rho+\rho_I-\rho-\rho_I=y\rho-\rho.\end{aligned}$$

From this and the fact that $y\rho_I-w\rho_I$ belongs to Q_I for any y, w in W_I, we see that the right hand side of $(*)$ is equal to

$$[M(y\rho-\rho)\colon L(w\rho-\rho)]$$

by Theorem 2.3. Q.E.D.

§4. Branching rules for Verma modules over non-twisted affine Lie algebras

In this section, we consider the case where the GCM A is a so-called extended Cartan matrix. In this case, $\mathfrak{g}(A)$ is called a non-twisted affine Lie algebra. As the subset I in §2, we take the most natural and maximal one in the sense that the derived subalgebra of $\mathfrak{g}(A)$ is the universal central extension of the loop algebra $\boldsymbol{C}[t, t^{-1}]\otimes\mathfrak{g}_I$.

For this subset I, we give a branching rule for Verma modules over $\mathfrak{g}(A)$ considered as $\mathfrak{g}_I$-modules.

4.1. Non-twisted affine Lie algebras

Let $\mathfrak{g}_0$ be a complex simple Lie algebra and let $\mathfrak{h}_0$, $\mathfrak{b}_0$, W_0 and S_0 be as in 3.1. We denote by Δ_0 and $\Delta_{0,+}$ the root system of $(\mathfrak{g}_0, \mathfrak{h}_0)$ and the set of positive roots in Δ_0 corresponding to $\mathfrak{b}_0$, respectively. Let $\{\alpha_1,\dots, \alpha_l\}$ be the set of simple roots in $\Delta_{0,+}$ and θ the highest root. We define an $(l+1)\times(l+1)$ matrix $A=(a_{ij})_{0\leqq i,j\leqq l}$ by

$$a_{ij}=\begin{cases} 2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i) & \text{if } 1\leqq i,j\leqq l,\\ -2(\theta,\alpha_j)/(\theta,\theta) & \text{if } i=0 \text{ and } 1\leqq j\leqq l,\\ -2(\alpha_i,\theta)/(\alpha_i,\alpha_i) & \text{if } 1\leqq i\leqq l \text{ and } j=0,\\ 2 \text{ if } i=j=0, \end{cases}$$

where $(\cdot,\cdot)$ is the bilinear form on $\mathfrak{h}_0^*$ induced by the Killing form $K(\cdot,\cdot)$ on $\mathfrak{g}_0$. Then, A is called an extended Cartan matrix and is a symmetrizable GCM, and the Kac-Moody algebra $\mathfrak{g}=\mathfrak{g}(A)$ is called a non-twisted affine Lie algebra. In this section, A will always denote the extended Cartan matrix given above.

The non-twisted affine Lie algebra $\mathfrak{g}$ has a simple realization. We describe this realization without proof (cf., [6, Chapter 7], for example). Let $\boldsymbol{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t with coefficients in $\boldsymbol{C}$. As a vector space, $\mathfrak{g}$ is isomorphic to

$$\hat{\mathfrak{g}}_0=\boldsymbol{C}d\oplus\boldsymbol{C}c\oplus(\boldsymbol{C}[t,t^{-1}]\otimes_{\boldsymbol{C}}\mathfrak{g}_0)$$

and the bracket in $\hat{\mathfrak{g}}_0$ induced by this isomorphism is given by

$$\begin{aligned}&[a_1d+b_1c+P_1\otimes x_1,\ a_2d+b_2c+P_2\otimes x_2]\\ &\quad=\mathrm{Res}\,((dP_1/dt)P_2)K(x_1,x_2)c+a_1t(dP_2/dt)\otimes x_2+\\ &\quad\quad+a_2t(dP_1/dt)\otimes x_1+P_1P_2\otimes[x_1,x_2]\end{aligned}$$

for $a_i, b_i\in\boldsymbol{C}$, $P_i\in\boldsymbol{C}[t,t^{-1}]$ and $x_i\in\mathfrak{g}_0$ $(i=1,2)$, where for $P\in\boldsymbol{C}[t,t^{-1}]$, $\mathrm{Res}\,(P)$ is the coeffitient of t^{-1} in P.

We identify $\mathfrak{g}$ with $\hat{\mathfrak{g}}_0$. Then, we have an injective Lie algebra homomorphism from $\mathfrak{g}_0$ into $\mathfrak{g}$:

$$\mathfrak{g}_0\ni x\longmapsto 1\otimes x\in\mathfrak{g},$$

and so we may (and do) consider $\mathfrak{g}_0$ to be a subalgebra of $\mathfrak{g}$. In these identifications, the Cartan subalgebra $\mathfrak{h}=\mathfrak{h}(A)$ of $\mathfrak{g}$ is $\boldsymbol{C}c\oplus\boldsymbol{C}d\oplus\mathfrak{h}_0$. We regard $\mathfrak{h}_0^*$ as a subspace of $\mathfrak{h}^*$ by

$$\lambda(c)=\lambda(d)=0 \qquad \text{for} \quad \lambda\in\mathfrak{h}_0^*.$$

We define $\delta\in\mathfrak{h}^*$ by

$$\delta(c)=0,\quad \delta(d)=1 \quad \text{and} \quad \delta\,|\,\mathfrak{h}_0=0.$$

Then, we have

$$\alpha_0\ (=\text{the 0-th simple root})=\delta-\sum_{1\leqq i\leqq l}c_i\alpha_i,$$

where c_i are positive integers, and α_i is the i-th simple root for each $i=1,\dots,l$. Moreover, we have

$$\Delta=\Delta(A)=\{\alpha+j\delta\,|\,\alpha\in\Delta_0,\ j\in\boldsymbol{Z}\}\cup\{j\delta\,|\,j\in\boldsymbol{Z}\smallsetminus\{0\}\},$$

$$\Delta_+ = \Delta_+(A) = \{\alpha + j\delta \mid \alpha \in \Delta_0 \cup \{0\}, j \in \boldsymbol{Z}, j > 0\} \cup \Delta_{0,+} ;$$

$$\mathfrak{g}_{\alpha+j\delta} = t^j \otimes \mathfrak{g}_{0,\alpha} \quad \text{for} \quad \alpha \in \Delta_0, \quad j \in \boldsymbol{Z},$$

$$\mathfrak{g}_{j\delta} = t^j \otimes \mathfrak{h}_0 \quad \text{for} \quad j \in \boldsymbol{Z} \setminus \{0\},$$

where $\mathfrak{g}_{0,\alpha}$ is the root space in $\mathfrak{g}_0$ corresponding to $\alpha \in \Delta_0$.

4.2. Branching rules relative to $\mathfrak{g}_0$

Notaions are as above. We take $\{1, \dots, l\}$ as a subset I of the index set $\{0, 1, \dots, l\}$ in 2.1, then it holds that $\mathfrak{g}_I = \mathfrak{g}_0$, $\mathfrak{h}_I = \mathfrak{h}_0$, $\Delta_I = \Delta_0$ and so on. In this section, we write Q_0 for Q_I, $M_0(\lambda)$ for $M_I(\lambda)$ etc. We put

$$\mathfrak{u}_{0,\pm} = \sum \mathfrak{g}_{\pm(\alpha+j\delta)},$$

where the sum runs over $\alpha \in \Delta_0 \cup \{0\}$ and $j \in \boldsymbol{Z}, j > 0$. Then, $\mathfrak{u}_{0,\pm}$ is subalgebras of $\mathfrak{n}_\pm$, and we have

$$\mathfrak{n}_\pm = \mathfrak{n}_{0,\pm} + \mathfrak{u}_{0,\pm} \qquad \text{(direct sum)}.$$

For any $\lambda \in \mathfrak{h}^*$, the set of all weights $P(M(\lambda))$ is contained in $\lambda - Q_+$, and so if $\Lambda \in \mathfrak{h}^*/Q_0$ and $M(\lambda)^\Lambda \neq 0$, then we have

$$\Lambda = [\lambda - \alpha] \qquad \text{for some} \quad \alpha \in Q_+$$

Since $\alpha_0 \equiv \delta$ modulo Q_0, we can take $j\delta$ as α above for some non-negative integer j. Thus, $M(\lambda)^\Lambda = M(\lambda)^{[\lambda - j\delta]}$ ($j \in \boldsymbol{Z}_{\geqq 0}$) if $M(\lambda)^\Lambda \neq 0$, and so, in the rest of this section, we determine what $M(\lambda)^{[\lambda - j\delta]}$ is as a $\mathfrak{g}_0$-module, that is, we give a branching rule for $M(\lambda)$ as a $\mathfrak{g}_0$-module.

For $j = 0$, $M(\lambda)^{[\lambda]}$ is isomorphic to $M_0(\lambda \mid h_0)$ as proved in Lemma 2.2 already, and so we assume that $j > 0$. We write $\precsim$ the lexicographical order on Q with respect to the coefficients of $\alpha_0, \alpha_1, \dots, \alpha_l$. It is clear that if $\alpha, \beta \in Q$ and $\alpha \leqq \beta$, then $\alpha \precsim \beta$. Let v_λ be a non-zero highest weight vector of $M(\lambda)$. Then, we have

Lemma 4.1. *For any positive integer j,*

$$M(\lambda)^{[\lambda - j\delta]} = \sum_{\gamma \in Q_0}^{\oplus} \sum_{(\gamma, j)}^{\oplus} U(\mathfrak{n}_{0,-}) \mathfrak{g}_{\beta_k - n_k\delta} \cdots \mathfrak{g}_{\beta_1 - n_1\delta} v_\lambda,$$

where $\sum_{(\gamma,j)}$ denotes the summation taken over $\beta_1, \dots, \beta_k \in \Delta_0 \cup \{0\}$, $n_1, \dots, n_k \in \boldsymbol{Z}_{\geqq 0} \setminus \{0\}$ such that

$$\beta_1 - n_1\delta \precsim \cdots \precsim \beta_k - n_k\delta,$$

$$(\beta_1 - n_1\delta) + \cdots + (\beta_k - n_k\delta) = \gamma - j\delta.$$

Proof. By Poincaré-Birkhoff-Witt theorem, we have

$$\begin{aligned} M(\lambda) &= U(\mathfrak{n}_-) v_\lambda \\ &= U(\mathfrak{n}_{0,-}) U(\mathfrak{u}_{0,-}) v_\lambda \\ &= U(\mathfrak{n}_{0,-}) v_\lambda + \sum_{\substack{j \in \boldsymbol{Z} \\ j > 0}} \sum_{\gamma \in Q_0} \sum_{(\gamma, j)} U(\mathfrak{n}_{0,-}) \mathfrak{g}_{\beta_k - n_k\delta} \cdots \mathfrak{g}_{\beta_1 - n_1\delta} v_\lambda \end{aligned}$$

and this sum is direct since $M(\lambda)$ is $U(\mathfrak{n}_-)$-free. Q. E. D.

For a positive integer j, let $\Delta_0(j)$ be the set of elements γ in Q_0 which can be written as $\gamma=\beta_1+\cdots+\beta_j$ for some $\beta_1,\ldots,\beta_j\in\Delta_0\cup\{0\}$. We define a subset D of $(\Delta_+\setminus\Delta_0)\times\boldsymbol{Z}$ by

$$D=\{(\beta,k)\in(\Delta_+\setminus\Delta_0)\times\boldsymbol{Z}\mid 1\leqq k\leqq\mathrm{mult}(\beta)\},$$

and for each element $\alpha\in Q_+$, we put

$$\mathscr{P}_0(\alpha)=\sharp\{\tau\colon D\longmapsto\boldsymbol{Z}_{\geqq 0}\mid\alpha=\sum_{(\beta,k)\in D}\tau(\beta,k)\beta\}.$$

Then, $\mathscr{P}_0(\alpha)$ is the number of partitions of α into a sum of elements of $\Delta_+\setminus\Delta_0$, where each element β of $\Delta_+\setminus\Delta_0$ is counted with its multiplicity $\mathrm{mult}(\beta)$.

Recall

$$M(\lambda)=\sum_{\Lambda\in h^*/Q_0}^{\oplus}M(\lambda)^{\Lambda}=\sum_{j\in\boldsymbol{Z}_{\geqq 0}}^{\oplus}M(\lambda)^{[\lambda-j\delta]}$$

as noted in §2. Now, we have the following proposition which, together with the above decomposition, gives a branching rule of $M(\lambda)$ as a $\mathfrak{g}_0$-module.

Proposition 4.2. *For each positive integer j, we put*

$$\Delta_0(j)=\{\gamma_1,\ldots,\gamma_s\},\quad k<m\quad\text{if}\quad\gamma_k>\gamma_m,$$

where $s=\sharp\Delta_0(j)$. *Then, there exists an increasing sequence* $0=M^{(0)}\subset\cdots\subset M^{(s)}=M(\lambda)^{[\lambda-j\delta]}$ *of* $\mathfrak{g}_0$-*submodules of* $M(\lambda)^{[\lambda-j\delta]}$ *such that* $M^{(k)}/M^{(k-1)}$ *is isomorphic to the direct sum of* $\mathscr{P}_0(-\gamma_k+j\delta)$-copies of $M_0(\lambda\mid\mathfrak{h}_0+\gamma_k)$ *for every* $k=1,\ldots,s$.

Proof. For each $k=1,\ldots,s$, we put

$$M^{(k)}=\sum_{1\leqq p\leqq k}\sum_{(\gamma_p,j)}U(\mathfrak{n}_{0,-})\mathfrak{g}_{\beta_m-n_m\delta}\cdots\mathfrak{g}_{\beta_1-n_1\delta}v_\lambda,$$

where v_λ is a non-zero highest weight vector of $M(\lambda)$. By definition, $M^{(k)}$ is $(\mathfrak{h}_0+\mathfrak{n}_{0,-})$-invariant. For any $\alpha\in\Delta_{0,+}$, we have in $U(\mathfrak{g})$

$$\begin{aligned}\mathfrak{g}_\alpha\mathfrak{g}_{\beta_p-n_p\delta}&\subset[\mathfrak{g}_\alpha,\mathfrak{g}_{\beta_p-n_p\delta}]+\mathfrak{g}_{\beta_p-n_p\delta}\mathfrak{g}_\alpha\\&\subset\mathfrak{g}_{\alpha+\beta_p-n_p\delta}+\mathfrak{g}_{\beta_p-n_p\delta}\mathfrak{g}_\alpha,\end{aligned}$$

and so, by the ordering of $\gamma_1,\ldots,\gamma_s$, $M^{(k)}$ is also $\mathfrak{n}_{0,+}$-invariant. Hence $M^{(k)}$ is a $\mathfrak{g}_0$-submodule of $M(\lambda)^{[\lambda-j\delta]}$. By Lemma 4.1, we have a $U(\mathfrak{n}_{0,-})$-isomorphism

$$M^{(k)}/M^{(k-1)}\simeq\sum_{(\gamma_k,j)}U(\mathfrak{n}_{0,-})\mathfrak{g}_{\beta_m-n_m\delta}\cdots\mathfrak{g}_{\beta_1-n_1\delta}v_\lambda.$$

Hence, $M^{(k)}/M^{(k-1)}$ is a free module as a $U(\mathfrak{n}_{0,-})$-module, and any basis of $V=\sum_{(\gamma_k,j)}\mathfrak{g}_{\beta_m-n_m\delta}\cdots\mathfrak{g}_{\beta_1-n_1\delta}v_\lambda$ as a complex vector space is a basis of $M^{(k)}/M^{(k-1)}$ as a $U(\mathfrak{n}_{0,-})$-module. Therefore, it suffices for the proof of the proposition to show that the dimension of V over $\boldsymbol{C}$ is equal to $\mathscr{P}_0(-\gamma_k+j\delta)$. In turn, this holds because $Q=Q_0\oplus\boldsymbol{Z}\delta$ and so $(\beta_1-n_1\delta)+\cdots+(\beta_m-n_m\delta)=\gamma_k-j\delta$ if and only if $\beta_1+\cdots+\beta_m=\gamma_k$ and $n_1+\cdots+n_m=j$. Q. E. D.

Remark 4.3. If λ, $\lambda' \in \mathfrak{h}^*$ are integral and $\lambda \geqq \lambda'$, we can write $\lambda - \lambda' = \alpha + j\delta$ with some $\alpha \in Q_0$ and some non-negative integer j. Then, by the results of §2, we have

$$(*) \qquad [M(\lambda)^{[\lambda']}: L_0(\lambda' \mid \mathfrak{h}_0)] = \sum_{\lambda \geqq \lambda'' \geqq \lambda'} [M(\lambda): L(\lambda'')][L(\lambda'')^{[\lambda']}: L_0(\lambda' \mid \mathfrak{h}_0)].$$

The left hand side of $(*)$ can be computed by Proposition 4.2, Theorem 3.1 and the translation principle in the complex semisimple case. On the other hand, the coeffitient of $[M(\lambda): L(\lambda')]$ in the right hand side of $(*)$ is equal to

$$[L(\lambda')^{[\lambda']}: L_0(\lambda' \mid \mathfrak{h}_0)] = [L_0(\lambda' \mid \mathfrak{h}_0): L_0(\lambda' \mid \mathfrak{h}_0)] = 1$$

by (2.3). Hence, if we determine the branching rule of $L(\lambda'')$ as a $\mathfrak{g}_0$-module for every integral $\lambda'' \in \mathfrak{h}^*$, we can compute $[M(\lambda): L(\lambda')]$ inductively.

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