On the decomposition of non-negative finely harmonic functions

By

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Introduction.

It is well-known that every non-negative harmonic function can be uniquely expressed as the sum of a quasi-bounded harmonic function and a singular one (cf. Parreau [15, Theorem 12]). The main purpose of this paper is to show the same type of decomposition theorem for the finely harmonic functions introduced by Fuglede [10] in the potential theory on harmonic spaces. The quasi-bounded and singular functions for finely harmonic functions are defined in the usual way (see Definition 3 and 4 in §3). Then we can prove the following.

Main Theorem. Let $U \subseteq \mathbb{R}^d$ be a fine domain and $f$ a non-negative finely harmonic function in $U$. Then there exist uniquely a quasi-bounded function $f_1$ and a singular function $f_2$ such that

$$f = f_1 + f_2 \quad \text{in } U.$$

Now, Parreau's proof for his decomposition theorem is based on Perron's method and Harnack's inequality plays an important role. On the other hand, inequalities of this sort do not hold generally for finely harmonic functions. Thus the proof of Main Theorem is achieved by a probabilistic approach which is quite different from Perron's method.

In §1 we provide some definitions and results from potential and probability theories which are used in the sequel. In §2 we shall give the probabilistic characterization of finely harmonic functions. In §3 we shall first show that non-negative finely harmonic functions have the asymptotic values along almost all Brownian paths, and give some characterizations for quasi-bounded finely harmonic functions. By using these results, we shall give in §4 the proof of Main Theorem.

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§ 1. Preliminaries.

1. We introduce into the $d$-dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 2$) the weakest topology which makes the positive superharmonic functions in $\mathbb{R}^d$ continuous. This topology is called the fine topology (cf. [3, Ch. 1]). In this paper we use the usual topological term when using the Euclidean topology. As for the topological terms in fine topology, we add the word “fine” or “finely” before the corresponding Euclidean terms.

Let $V (\subset \mathbb{R}^d)$ be a compact set. We suppose that the fine interior $V'$ of $V$ is not empty. We consider a bounded domain $D$ containing $V$ and for $z \in V'$, we denote by $G^p_D$ Green's function for $D$ with pole at $z$ and by $\varepsilon$, the Dirac measure. Then, there exists a probability measure $\mu_z$ such that it is carried on the fine boundary $f - \partial V$ of $V$ and the equality

$$(G^p_D)_{D,V}(\zeta) = \int G^p_D d\mu_z \quad \text{(for every } \zeta \in D)$$

holds, where $(G^p_D)_{D,V}(\zeta) = \inf \{ s(\zeta) ; s \text{ is positive superharmonic in } D \text{ and majorizes } G_z \text{ quasi-everywhere on } D - V \}$. The $\mu_z$ dose not depend on any choice of $D$. We say $\mu_z$ the harmonic measure relative to $z$ and $V'$ and denote it by $\varepsilon_{z,V'}^D$, $CV$ being $\mathbb{R}^d - V$ (see [2, § 8] and [10, § 4.8]).

Definition 1 (Fuglede [10, Definition 8.3 and Theorem 14.1]). Let $U$ be a fine domain (finely connected and finely open set). A finely continuous mapping $f : U \to \mathbb{R}$ is called to be finely harmonic in $U$ if for every $x \in U$, there exists a compact fine neighbourhood $V$ of $x$ such that $f$ is bounded on $V$, and

$$f(z) = \int f d\varepsilon_{z,V'}^D \quad \text{for every } z \in V'.$$

Combining Fuglede’s theorem [11, Theorem 4.1] with Debiard and Gaveau’s theorem [5, Theorem 2], we obtain the following theorem.

Theorem A. Let $U$ be a fine domain and $f$ be a finely harmonic function in $U$. Then there exists an $\mathbb{R}^d$-valued function $h$ in $U$ satisfying the following condition: for every $x \in U$, there exist a compact fine neighbourhood $V$ of $x$ and a sequence $\{f_n\}_{n=1}^\infty$ of harmonic functions in neighbourhoods of $V$ such that $\{f_n\}_{n=1}^\infty$ converges uniformly to $f$ on $V$ and for every $z \in V'$, $\{\nabla f_n\}_{n=1}^\infty$ converges strongly to $h$ in $L^2(V, G^{p}_{D,V} dv)$, where $G^{p}_D$ is (the fine) Green's function for $V'$ with pole at $z$ (cf. [12]) and $dv$ is $d$-dimensional Lebesgue measure.

The above function $h$ is independent of any choice of $V$ and $\{f_n\}_{n=1}^\infty$. We call $h$ the gradient of $f$ and denote it by $\nabla f = (\partial f/\partial x_i)_{i=1}^d$.

2. For every $x \in \mathbb{R}^d$, a stochastic process $(B_t)_{t \geq 0}$, taking values in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{F}, P_2)$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$ (right continuous and increasing family of sub $\sigma$-fields of $\mathcal{F}$) is called a d-dimensional
Brownian motion defined on \((Q, \mathcal{F}, \mathcal{F}_t, P)\) starting at \(x\). If it has the following properties:

(i) \(B_t=x\) a.s. (=almost surely, that is almost everywhere on \(Q\) with respect to \(P\));
(ii) mapping \(t\to B_t\) is continuous a.s.;
(iii) for every \(t \geq 0\), a mapping \(\omega \to B_t(\omega)\) is measurable with respect to \(\mathcal{F}_t\), and for every \(s, t \geq 0\) and \(d\)-dimensional Borel set \(A\),

\[
P_x(B_{s+t} - B_t \in A) = \int_A (2\pi t)^{-d/2} \exp\left(-\frac{|z|^2}{2t}\right) dv(z),
\]

where \(dv\) is the \(d\)-dimensional Lebesgue measure;
(iv) if \(0 \leq t_0 < t_1 < \cdots < t_n\), then \(B_{t_0}, B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}\) are independent.

The harmonic measure is characterized by a Brownian motion.

**Theorem B** (Debiard and Gaveau [4, Corollary of Lemma 1]). Let \(U (\subset \mathbb{R}^d)\) be a compact set such that \(U'\) is not empty. For \(x \in U'\), let \(\xi_x^U\) be the harmonic measure relative to \(U'\) and \(x\), \(\{B_t\}_{t \geq 0}\) a \(d\)-dimensional Brownian motion defined on \((Q, \mathcal{F}, \mathcal{F}_t, P)\) starting at \(x\), and \(\tau\) the first exit time of \(\{B_t\}_{t \geq 0}\) from \(U\), namely \(\tau = \inf\{t > 0; B_t \notin U\}\). Then,

\[
d\xi_x^U(\zeta) = P_x(B_\tau \in d\zeta).
\]

**§ 2. Finely harmonic functions in \(R^d\).**

To characterize the finely harmonic function, we state the definition of local martingales.

**Definition 2** (cf. [9, §2.3]) Let \((Q, \mathcal{F}, P)\) be a probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) (such a probability space is denoted by \((Q, \mathcal{F}, \mathcal{F}_t, P)\)), \(\tau\) a stopping time with respect to \(\mathcal{F}_t\), and \(\{X_t\}_{0 \leq t < \tau}\) a stochastic process on \((Q, \mathcal{F}, \mathcal{F}_t, P)\). Then \(\{X_t\}_{0 \leq t < \tau}\) is called to be a local martingale with respect to \(\mathcal{F}_t\) if there is a sequence \(\{\tau_n\}_{n=1}^\infty\) of stopping times satisfying the conditions:

(i) for each \(n\), \(\tau_n < \tau\) a.s. (=almost everywhere on \(Q\) with respect to \(P\));
(ii) \(\{\tau_n\}_{n=1}^\infty\) converges increasingly to \(\tau\) a.s.;
(iii) each \(\{X_t \wedge \tau_n\}_{t \geq 0}\) is a martingale with respect to \(\mathcal{F}_{t \wedge \tau_n}\), where \(t \wedge \tau_n = \min\{t, \tau_n\}\) and \(\mathcal{F}_{t \wedge \tau_n} = \{A \in \mathcal{F}; t \wedge \tau_n \leq s\} \cap \mathcal{F}_s\), for every \(s \geq 0\).

In the next lemma, we show the inverse of Øksendal’s result (cf. [14, Theorem 1]).

**Lemma 1.** Let \(U (\subset \mathbb{R}^d)\) be a fine domain and a mapping \(f: U \to \mathbb{R}\) finely continuous. Then the following two conditions are equivalent:

(i) \(f\) is finely harmonic in \(U\).
(ii) For all \(x \in U\), let \(\{B_t\}_{t \geq 0}\) be a \(d\)-dimensional Brownian motion starting at \(x\) and \(\tau\) the first exit time of \(\{B_t\}_{t \geq 0}\) from \(U\). Then \(\{f(B_t)\}_{0 \leq t < \tau}\) is a continuous local martingale with respect to \(\mathcal{F}_t\).
Proof. (i)⇒(ii): This is due to Øksendal. However, for the later purpose we give the full proof along with his idea.

Since $f$ is finely harmonic in $U$, we see from Theorem A that, for every $x \in U$, there exist a compact fine neighbourhood $V(x)$ of $x$ and a sequence $\{f_n\}_{n=1}^{\infty}$ of harmonic functions in neighbourhoods of $V(x)$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$ on $V(x)$, for every $z \in V(x)'$, $\{\nabla f_n\}_{n=1}^{\infty}$ converges strongly to $\nabla f$ in $L^2(V(x), G^r_{V(x)} du)$, and $f$ is bounded on $V(x)$. For every $z \in V(x)'$, let $\{B_t\}_{t \geq 0}$ be a $d$-dimensional Brownian motion starting at $z$, and $\tau(V(x))$ the first exit time of $\{B_t\}_{t \geq 0}$ from $V(x)$. By Itô's formula (cf. [9, §2.9 (5)]) and the optional sampling theorem (cf. [6, Ch. VI Theorem 15]), we have

$$f_n(B_{t \wedge \tau(V(x))}) = f_n(z) + \int_0^{t \wedge \tau(V(x))} \nabla f_n(B_s) dB_s.$$  

Hence, by the theory of stochastic integrals (cf. [9, §2.1]), letting $n$ be infinity, we have

$$(1) \quad f(B_{t \wedge \tau(V(x))}) = f(z) + \int_0^{t \wedge \tau(V(x))} \nabla f(B_s) dB_s,$$  

since $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$ on $V(x)$ and $\{\nabla f_n\}_{n=1}^{\infty}$ converges strongly to $\nabla f$ in $L^2(V(x), G^r_{V(x)} du)$. Thus we can cover $U$ with a family $\{V(\xi)' ; \xi \in U\}$ of finely open neighbourhoods such that the equation (1) holds for $V(\xi)$ when we replace $z$ by $\xi$. By the quasi-Lindelöf principle [8, Theorem 7.1], we can choose a sequence $\{\xi_n\}_{n=1}^{\infty}$ ($\subseteq U$) such that $U - \bigcup_{n=1}^{\infty} V(\xi_n)'$ is a polar set and $\xi_1 = x$.

Now we consider a $d$-dimensional Brownian motion $\{B_t\}_{t \geq 0}$ starting at $y \in \mathbb{R}^d$. Let $\tau_n$ be the first exit time of $\{B_t\}_{t \geq 0}$ from $\bigcup_{k=1}^{n} V(\xi_k)$ for $y \in \bigcup_{k=1}^{n} V(\xi_k)'$. If we set $y=x$, we see from the probabilistic characterization of the potential-theoretic notion “thinness” (cf. [1, Theorem 7.5]) that for every $n$, $\tau_n \leq \tau_{n+1} < \tau$ a.s., since $U - \bigcup_{k=1}^{n} V(\xi_k)$. To prove the assertion of (ii), we have to prove that $f(B_{t \wedge \tau_n})$ is a continuous martingale with respect to $\mathcal{F}_{t \wedge \tau_n}$ for every $y \in \bigcup_{k=1}^{n} V(\xi_k)'$. The proof of this assertion is carried out by induction on $n$.

Consider first the case $n=1$. By equation (1) replacing $z$ by $y$ and the well-known result on stochastic integral (cf. [3, Ch. II Proposition 1.1 (1.17)]), we have

$$f(B_{t \wedge \tau_n}) = f(y) + \int_0^t I(s \leq \tau_n) \nabla f(B_s) dB_s,$$

where $I(A)$ stands for the characteristic function of a set $A$. Hence we see that the assertion holds since a stochastic integral with respect to the Brownian motion is a continuous martingale (cf. [9, §2.1 and §2.14]).

Next we suppose that the assertion holds for $n \leq m$ ($m \geq 1$) and consider the case $n=m+1$. Clearly, for every $t \geq 0$, $f(B_{t \wedge \tau_{m+1}})$ is measurable with respect to $\mathcal{F}_{t \wedge \tau_{m+1}}$, and $E_y |f(B_{t \wedge \tau_{m+1}})| < +\infty$, since $f$ is bounded on $\bigcup_{k=1}^{m+1} V(\xi_k)$. Thus

$$f(B_{t \wedge \tau_{m+1}}) = f(y) + \int_0^t I(s \leq \tau_{m+1}) \nabla f(B_s) dB_s,$$

where $I(A)$ stands for the characteristic function of a set $A$. Hence we see that the assertion holds since a stochastic integral with respect to the Brownian motion is a continuous martingale (cf. [9, §2.1 and §2.14]).
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we have only to show that for \( s < t, \)

\[
E_y (f(B(t \wedge \tau_{m+1}) | \mathcal{F}_{t \wedge \tau_{m+1}}) = f(B_s \tau_{m+1}) \quad \text{a.s.,}
\]

since a local martingale with respect to the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) generated by a \( d \)-dimensional Brownian motion is continuous (cf. [9, p. 86 (1) and p. 52 Exercise 1]). It is sufficient to prove this statement for \( y \in \bigcup_{k=1}^{m} V(\zeta_k)' \), since we can prove it for \( y \in V(\zeta_{m+1})' \) by using the same argument as in the following proof. To see this, we have only to prove that for \( s < t, \)

\[
E_y (f(B(t \wedge \tau_{m+1})) I(\tau_m > s) | \mathcal{F}_s) = f(B_s) I(\tau_m > s) \quad \text{a.s.,}
\]

and

\[
E_y (f(B(t \wedge \tau_{m+1})) I(\tau_{m+1} > s) | \mathcal{F}_s) = f(B_s) I(\tau_{m+1} > s) \geq \tau_m) \quad \text{a.s.}
\]

In fact, we have

\[
E_y (f(B(t \wedge \tau_{m+1})) | \mathcal{F}_{t \wedge \tau_{m+1}})
\]

\[
= E_y (f(B(t \wedge \tau_{m+1})) I(t \wedge \tau_{m+1} = s \wedge \tau_{m+1}) | \mathcal{F}_{t \wedge \tau_{m+1}})
+ E_y (f(B(t \wedge \tau_{m+1})) I(t \wedge \tau_{m+1} > s \wedge \tau_{m+1}) | \mathcal{F}_{t \wedge \tau_{m+1}})
\]

\[
= f(B_{t \wedge \tau_{m+1}}) I(t \wedge \tau_{m+1} = s \wedge \tau_{m+1}) + E_y (f(B(t \wedge \tau_{m+1})) I(\tau_{m+1} > s) | \mathcal{F}_{t \wedge \tau_{m+1}})
\]

\[
= f(B_{\tau_{m+1}}) I(s \geq \tau_{m+1}) + E_y (f(B(t \wedge \tau_{m+1})) I(\tau_{m+1} > s) | \mathcal{F}_s) | \mathcal{F}_{t \wedge \tau_{m+1}})
\]

\[
= E_y (f(B(t \wedge \tau_{m+1})) I(\tau_{m+1} > s) | \mathcal{F}_s) | \mathcal{F}_{t \wedge \tau_{m+1}})
\]

\[
= f(B_{t \wedge \tau_{m+1}}).
\]

The proof of equation (2). By the strong Markov property (cf. [9, p. 21 (2)]), we have

the left hand side of (2)

\[
= (E_y (f(B_s) I(\tau_m \geq t) | \mathcal{F}_s) + E_y (f(B_{t \wedge \tau_{m+1}}) I(\tau_m < t) | \mathcal{F}_{t \wedge \tau_{m+1}}) (\mathcal{F}_s)) I(\tau_m > s)
\]

\[
= E_y (f(B_s) I(\tau_m \geq t) | \mathcal{F}_s) + E_y (f(B_{t \wedge \tau_{m+1}}) I(\tau_m < t, \tau_m = \tau_{m+1}) | \mathcal{F}_{t \wedge \tau_{m+1}}) (\mathcal{F}_s) I(\tau_m > s)
\]

\[
= E_y (f(B_s) I(\tau_m \geq t) | \mathcal{F}_s) + E_y (f(B_{t \wedge \tau_{m+1}}) I(\tau_m < t, \tau_m = \tau_{m+1}) | \mathcal{F}_{t \wedge \tau_{m+1}}) (\mathcal{F}_s) I(\tau_m > s)
\]

\[
= E_y (f(B_s) I(\tau_m \geq t) | \mathcal{F}_s) + E_y (f(B_{t \wedge \tau_{m+1}}) I(\tau_m < t, \tau_m = \tau_{m+1}) | \mathcal{F}_{t \wedge \tau_{m+1}}) (\mathcal{F}_s) I(\tau_m > s).
\]

Suppose that for every \( y \in V(\zeta_{m+1})' \cap \left( -\partial \left( \bigcup_{k=1}^{m} V(\zeta_k) \right) \right), \)

\[
f(y) = E_y (f(B_{t \wedge \tau_{m+1}})).
\]

By the induction hypothesis and the equation (4), we have

the left hand side of (2)
To complete the proof of (2), it remains to prove that (4) holds. To see this, we set

\[ d_1 = (f - \partial V(\zeta_{m+1})) \cap \left( \bigcup_{k=1}^{m} V(\zeta_k) \right), \]

and

\[ d_2 = V(\zeta_{m+1})' \cap \left( f - \partial V(\zeta_{m+1}) \right). \]

We define inductively sequences \( \{\sigma_n\}_{n=1}^{\infty} \) and \( \{\delta_n\}_{n=1}^{\infty} \) of stopping times as follows:

\[
\sigma_1 = \tau(V(\zeta_{m+1})), \\
\delta_1 = \begin{cases} 
\sigma_1 + \tau_m \theta_{\sigma_1} & \text{on } B_{\sigma_1} \subseteq d_1 \\
\infty & \text{on } \Omega - B_{\sigma_1} \subseteq d_1 
\end{cases}, \\
\sigma_n = \begin{cases} 
\sigma_{n-1} + \tau(V(\zeta_{m+1})) \theta_{\delta_{n-1}} & \text{on } B_{\delta_{n-1}} \subseteq d_1 \\
\infty & \text{on } \Omega - B_{\delta_{n-1}} \subseteq d_1 
\end{cases}, \\
\delta_n = \begin{cases} 
\sigma_n + \tau_m \theta_{\sigma_n} & \text{on } B_{\sigma_n} \subseteq d_1 \\
\infty & \text{on } \Omega - B_{\sigma_n} \subseteq d_1 
\end{cases}, \quad \text{for } n \geq 2,
\]

where for a stopping time \( \sigma \), we denote the random shift by \( \theta_{\sigma} \) (cf. [9, p. 20 (6)]). Then, by the induction hypothesis, we have

\[
E_y(f(B_{t \wedge \tau_{m+1}})) = \sum_{k=1}^{\infty} (E_y(f(B_{t \wedge \delta_k}) ; \tau_{m+1} = \sigma_k) + E_y(f(B_{t \wedge \delta_k}) ; \tau_{m+1} = \delta_k)) \\
= E_y(f(B_{t \wedge V(\zeta_{m+1})})).
\]

The proof of the equation (3): By the strong Markov property, we have the left hand side of (3)

\[
E_y(f(B_{t \wedge \tau_{m+1}}) \mid \mathcal{F}_s) I(\tau_{m+1} > s) \geq \tau_m) \\
= E_{B_y}(f(B_{t \wedge \zeta_{m+1}})) I(\tau_{m+1} > s) \geq \tau_m).
\]

Using the same argument as in the proof of (2), we have

\[
f(B_y) = E_{B_y}(f(B_{t \wedge \zeta_{m+1}})) \quad \text{a.s. on } \{\tau_{m+1} > s \geq \tau_m\}.
\]

Thus we obtain the equation (3). Therefore the assertion of (2) was proved.

(ii) \Rightarrow (i): Since \( f \) is finely continuous in \( U \), for every \( x \in U \), there is a compact fine neighbourhood \( V(x) \) (\( \subset U \)) of \( x \) such that \( f \) is bounded on \( V(x) \).
For every \( z \in V(X)' \), let \( \{ B_t \}_{t \geq 0} \) be a \( d \)-dimensional Brownian motion starting at \( z \) and \( \tau(V(x)) \) the first exit time of \( \{ B_t \}_{t \geq 0} \) from \( V(x) \). We see from the assertion of (ii) that there is a sequence \( \{ \tau_n \}_{n=1}^\infty \) of stopping times satisfying the conditions:

(a) for each \( n, \tau_n < \tau \) a.s.;
(b) \( \{ \tau_n \}_{n=1}^\infty \) converges increasingly to \( \tau \) almost surely;
(c) each \( \{ f(B_{t \wedge \tau_n}) \}_{t \geq 0} \) is a martingale with respect to \( \F_{t \wedge \tau_n} \).

By the optional sampling theorem (cf. [6, Ch. VI Theorem 15]), we have

\[
f(z) = E_z(f(B_{\tau})|V(x')).
\]

By the bounded convergence theorem and Theorem B, we have

\[
f(z) = \lim_{n \to \infty} \lim_{t \to \infty} E_z(f(B_{t \wedge \tau_n})|V(x')) = E_z(f(B_{\tau})|V(x')) = \int f d\mu^{V(x')},
\]

for every \( z \in V(x)' \).

Thus we have the assertion of (i). q.e.d.

§ 3. Quasi-bounded functions.

1. To start with, we study the probabilistic asymptotic values of non-negative finely harmonic functions.

**Theorem 1.** Let \( U \subset R^d \) be a fine domain and \( f \) a non-negative finely harmonic function in \( U \). For every \( x \in U \), let \( \{ B_t \}_{t \geq 0} \) be a \( d \)-dimensional Brownian motion starting at \( x \) and \( \tau \) the first time of \( \{ B_t \}_{t \geq 0} \) from \( U \). Then \( \lim_{t \to \tau=0} f(B_t) \) exists almost surely.

**Proof.** For every \( x \in U \), let \( \{ B_t \}_{t \geq 0} \) be a \( d \)-dimensional Brownian motion starting at \( x \) and \( \{ \tau_n \}_{n=1}^\infty \) a sequence of stopping times defined in proving the implication: (i) \( \Rightarrow \) (ii) in Lemma 1. For each \( n \), \( \{ f(B_{t \wedge \tau_n}) \}_{t \geq 0} \) is a continuous martingale with respect to \( \F_{t \wedge \tau_n} \) and

\[
\sup_{t \geq 0} |f(B_{t \wedge \tau_n})| < +\infty.
\]

From the optional sampling theorem and the bounded convergence theorem, we see that \( \{ f(B_{\tau_n}) \}_{n=1}^\infty \) is a martingale with respect to \( \F_{\tau_n} \). By the general theory of martingales (cf. [6, Ch. V Theorem 28]), \( \lim_{n \to \infty} f(B_{\tau_n}) \) exists almost surely and \( E_x(\lim f(B_{\tau})) = +\infty \), since \( f \) is non-negative in \( U \). We denote \( \lim f(B_{\tau}) \) by \( X \) and set

\[
f(B_{t \wedge \tau}) = \begin{cases} f(B_t) & \text{for } t < \tau, \\ X & \text{for } t \geq \tau. \end{cases}
\]

Clearly, \( f(B_{t \wedge \tau}) \) is measurable with respect to \( \F_{t \wedge \tau} \) for every \( t \geq 0 \). Further, by
the monotone convergence theorem, we have

\[ E_x(f(B_t \wedge \tau)) = E_x(f(B_t)I(\tau < t)) + E_x(XI(\tau \geq t)) \leq \lim_{n \to \infty} E_x(f(B_t)I(\tau < n_0)) + E_x(X) \leq \lim_{n \to \infty} E_x(f(B_t \wedge n_0)) + E_x(X) = E_x(f(B_t)) + E_x(X) < +\infty, \]

since each \( \{f(B_t \wedge \tau)\}_{t \geq 0} \) is a martingale with respect to \( \{\tau \wedge n_0\}_{t \geq 0} \). Thus, by the general theory of supermartingales (cf. [6, Ch. VI Theorem 6]), to prove the assertion of this theorem, it is enough to prove that \( \{f(B_t \wedge \tau)\}_{t \geq 0} \) is a supermartingale with respect to \( \{\tau \wedge n_0\}_{t \geq 0} \). For this purpose we have only to show that for \( t > s \geq 0 \),

\[ E_x(f(B_f \wedge \tau)|\tau \wedge n_0) \leq f(B_s \wedge \tau) \text{ a.s..} \]

Since \( \{f(B_t \wedge \tau)\}_{t \geq 0} \) is a continuous martingale with respect to \( \{\tau \wedge n_0\}_{t \geq 0} \), we have, by the strong Markov property,

\[ f(B_s \wedge \tau) = E_x(f(B_f \wedge \tau)|\tau \wedge n_0) \]

\[ = f(B_s \wedge \tau)|I(s \geq \tau_n) = E_x(f(B_f \wedge \tau)|\tau \wedge n_0)I(t \wedge \tau_n > s \wedge \tau_n) \]

\[ = f(B_s)I(s \geq \tau_n) + E_x(f(B_f \wedge \tau)|\tau \wedge n_0)I(s < \tau_n) \]

\[ = f(B_s)I(s \geq \tau_n) + E_x(f(B_f \wedge \tau)|\tau \wedge n_0)I(s < \tau_n) \]

Letting \( n \) be infinity, by Fatou's lemma, we have

\[ f(B_\tau) \geq XI(s \geq \tau) + E_x(f(B_t \wedge \tau)|\tau)I(s < \tau) \]

\[ = f(B_\tau)I(s \geq \tau) + E_x(f(B_t \wedge \tau)|\tau)I(s < \tau) \]

\[ = E_x(f(B_t \wedge \tau)|\tau) \text{ a.s..} \]

q.e.d.

For a non-negative finely harmonic function \( f \), we denote the above random variable \( \lim_{t \to \tau} f(B_t) \) by \( f(B_\tau) \).

2. Definition 3. Let \( U \) be a fine domain and \( f \) a non-negative finely harmonic function in \( U \). Then \( f \) is called to be quasi-bounded in \( U \), if there exists a sequence \( \{f_n\}_{n=1}^\infty \) of bounded and non-negative finely harmonic functions in \( U \) such that \( \{f_n\}_{n=1}^\infty \) converges increasingly to \( f \) on \( U \).

We give the characterization of quasi-bounded functions by using the same argument as that of Doob [7, Theorem 4.1].

Theorem 2. Let \( U (\subset \mathbb{R}^d) \) be a fine domain and a mapping \( f : U \to \mathbb{R} \) a non-negative finely continuous function. Then the following conditions are equivalent:

(a) \( f \) is quasi-bounded in \( U \).

(b) For every \( x \in U \), let \( \{B_t\}_{t \geq 0} \) be \( d \)-dimensional Brownian motion starting...
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at x and \( \tau \) the first exit time of \( \{B_t\}_{t \geq 0} \) from U. Then \( \{f(B_t)\}_{t \geq 0} \) is a continuous local martingale with respect to \( \{\mathcal{F}_t\}_{t \geq 0} \) and has the following property:

\[
\lim_{N \to \infty} \sup_{t \leq \tau_N} E_x(f(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N) = 0,
\]

where \( \{\tau_n\}_{n=1}^\infty \) is a sequence of stopping times defined as in Definition 2.

(c) For every \( x \in U \), let \( \{B_t\}_{t \geq 0} \) be a d-dimensional Brownian motion and \( \tau \) the first exit time of \( \{B_t\}_{t \geq 0} \) from U. Then \( f(B_t) \) exists almost surely and \( f(x) = E_x(f(B_t)) \).

(d) For every \( x \in U \), there exists a sequence \( \{K_n\}_{n=1}^\infty \) of compact sets in U satisfying the following properties:

(i) for every \( n \), \( x \in K_n' \) and \( K_n \subseteq K_{n+1} (\subseteq U) \);

(ii) \( U - \bigcup_{n=1}^\infty K_n' \) is a polar set;

(iii) for each \( n \), \( f \) is quasi-bounded in \( K_n' \) and

\[
\lim_{N \to \infty} \sup_{x \in U} \left( \int_{\{\xi \in \partial K_n'; f(\xi) > N\}} f(\xi) d\mu(\xi) \right) = 0.
\]

Proof. (a)\( \Rightarrow \) (b): It follows from Lemma 1 that \( \{f(B_t)\}_{t \geq 0} \) is a continuous local martingale with respect to \( \{\mathcal{F}_t\}_{t \geq 0} \).

Next we show that the equality (5) holds. Since \( f \) is quasi-bounded in U, there is a sequence \( \{f_m\}_{m=1}^\infty \) of bounded and non-negative finely harmonic functions in U such that \( \{f_m\}_{m=1}^\infty \) converges increasingely to \( f \) on U. Therefore, for any \( \epsilon > 0 \), there is a natural number \( m_\epsilon \) such that for \( m \geq m_\epsilon \), \( 0 \leq f(x) - f_m(x) < \epsilon \). Using the same argument as in the proof of implication: (i)\( \Rightarrow \) (ii) in Lemma 1, we may suppose that \( \{f(B_t \wedge \tau_N)\}_{t \geq 0} \) and \( \{f_m(B_t \wedge \tau_N)\}_{t \geq 0} \) are continuous martingales with respect to \( \{\mathcal{G}_t\}_{t \geq 0} \). Thus we have

\[
0 \leq E_x(f(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N)
\]

\[
= E_x(f(B_{t \wedge \tau_N}) - f_m(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N) + E_x(f_m(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N)
\]

\[
\leq E_x(f(B_{t \wedge \tau_N}) - f_m(B_{t \wedge \tau_N}) + E_x(f_m(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N)
\]

\[
= f(x) - f_m(x) + E_x(f_m(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N)
\]

\[
\leq \epsilon + \sup_{x \in U} |f_m(\xi)| P(f(B_{t \wedge \tau_N}) > N).
\]

and

\[
f(x) = E_x(f(B_{t \wedge \tau_N})
\]

\[
\geq E_x(f(B_{t \wedge \tau_N}); f(B_{t \wedge \tau_N}) > N)
\]

\[
\geq NP_x(f(B_{t \wedge \tau_N}) > N).
\]

We see from (7) that

\[
P_x(f(B_{t \wedge \tau_N}) > N) \leq f(x)N^{-1}.
\]

Combining (6) with (8), we have
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\[ 0 \leq \sup_{t \geq 0} \mathbb{E}_x (f(B_{t \wedge \tau_n}) \mid f(B_{t \wedge \tau_n}) \geq N) \leq \epsilon + \sup_{\xi \in \partial \Omega} |f_{\mu}(\xi)| f(x) N^{-1}. \]

Hence we have

\[ 0 \leq \lim_{N \to \infty} \sup_{t \geq 0} \mathbb{E}_x (f(B_{t \wedge \tau_n}) \mid f(B_{t \wedge \tau_n}) \geq N) \leq \epsilon. \]

Since \( \epsilon \) is arbitrary, we obtain the assertion of (b).

(b) \( \Rightarrow \) (c): We see from the assertion of (b) and Lemma 1 that there exists a sequence \( \{\tau_n\}_{n=1}^\infty \) such that each \( \{f(B_{t \wedge \tau_n})\}_{t \geq 0} \) is a continuous martingale with respect to \( \{\mathcal{F}_{t \wedge \tau_n}\}_{t \geq 0} \) and

\[ \sup_{n \geq 0} f(B_{t \wedge \tau_n}) < +\infty. \]

Hence we have

\[ (9) \quad f(x) = \mathbb{E}_x (f(B_{t \wedge \tau})). \]

Moreover, we see from the assertion of (b) and Lemma 1 that \( f \) is non-negative finely harmonic in \( U \), and so by Theorem 1, \( f(B_z) \) exists almost surely. In (9), letting \( t \) and \( n \) be infinity, by Fatou’s lemma, we have

\[ (10) \quad f(x) \geq \mathbb{E}_x (f(B_z)). \]

The condition (5) implies that for any \( \epsilon > 0 \), there exists a natural number \( N_0 \) such that for \( N \geq N_0 \),

\[ (11) \quad \sup_{n \geq 0} \mathbb{E}_x (f(B_{t \wedge \tau_n}) \mid f(B_{t \wedge \tau_n}) \geq N) \leq \epsilon. \]

By (9), (10) and (11), we have

\[ (12) \quad 0 \leq f(x) - \mathbb{E}_x (f(B_z)) = \mathbb{E}_x (f(B_{t \wedge \tau_n})) - \mathbb{E}_x (f(B_z)) \]

\[ = \mathbb{E}_x (f(B_{t \wedge \tau_n}) \mid f(B_{t \wedge \tau_n}) \leq N) + \mathbb{E}_x (f(B_{t \wedge \tau_n}) \mid f(B_{t \wedge \tau_n}) > N) \]

\[ - \mathbb{E}_x (f(B_z) \mid f(B_z) \leq N) - \mathbb{E}_x (f(B_z) \mid f(B_z) > N) \]

\[ \leq \mathbb{E}_x (f(B_{t \wedge \tau_n}) \mid f(B_{t \wedge \tau_n}) \leq N) - \mathbb{E}_x (f(B_z) \mid f(B_z) \leq N) + \epsilon. \]

In (12), letting \( t \) and \( n \) be infinity, by the bounded convergence theorem we have

\[ 0 \leq f(x) - \mathbb{E}_x (f(B_z)) \leq \epsilon. \]

Since \( \epsilon \) is arbitrary, \( f(x) = \mathbb{E}_x (f(B_z)) \). Thus we have the assertion of (c).

(c) \( \Rightarrow \) (a): For every \( x \in U \), we take a compact fine neighbourhood \( V \subset U \) of \( x \). For every \( z \in \mathbb{V} \), let \( \{B_t\}_{t \geq 0} \) be a \( d \)-dimensional Brownian motion and \( \tau(V) \) the first exit time of \( \{B_t\}_{t \geq 0} \) from \( V \). Applying the strong Markov property and Theorem B to the assertion of (c), we have

\[ f(z) = \mathbb{E}_z (f(B_z)) \]

\[ = \mathbb{E}_z (\mathbb{E}_z (f(B_z) \mid \mathcal{F}_{\tau(V)})) \]

\[ = \mathbb{E}_z (\mathbb{E}_{B_{\tau(V)}} (f(B_z))). \]
Thus we see from Fuglede’s theorem [10, Theorem 14.6] that $u_m$ is a bounded and non-negative finely harmonic function in $U$, since $u_m$ is a bounded measurable with respect to $d\varepsilon^U$. Therefore we obtain the assertion of (a), since \( \{f_m\}_{m=1}^\infty \) converges increasingly to $f$ in $U$.

(a)\(\Rightarrow\)(d): Using the same argument as in the proof of implication: (a)\(\Rightarrow\)(b) in this theorem, the assertion of (d) follows from Theorem B and Lemma 1.

(d)\(\Rightarrow\)(c): For every $x \in U$, let \( \{K_n\}_{n=1}^\infty \) be a sequence of compact sets in $U$ satisfying the properties stated in the assertion of (d), \( \{B_t\}_{t\geq 0} \) a $d$-dimensional Brownian motion starting at $x$, and $\tau_n$ the first exit time of $\{B_t\}_{t\geq 0}$ from $K_n$. Since $f$ is quasi-bounded in $K_n$, and (a) and (c) are equivalent, we have

\[
    f(x) = E_x(f(B_{\tau_n})).
\]

Hence, using the argument similar to that in the proof of implication: (b)\(\Rightarrow\)(c) in this theorem, we see from Theorem 1 that the assertion of (c) holds.

q.e.d.

Finally, we introduce the definition of singular functions.

**Definition 4.** Let $U$ be a fine domain and $f$ a non-negative finely harmonic function in $U$. Then $f$ is called to be singular in $U$, if, for a bounded and non-negative finely harmonic function $u$ in $U$ such that $u \leq f$ in $U$, $u = 0$ in $U$.

§ 4. **Proof of Main Theorem.**

**Proof of Main Theorem.** Suppose that $f$ is a non-negative finely harmonic function in $U$. For every $x \in U$, let \( \{B_t\}_{t\geq 0} \) be a $d$-dimensional Brownian motion starting at $x$ and $\tau$ the first exit time of $\{B_t\}_{t\geq 0}$ from $U$. We see from the proof of Theorem 1 that \( \{f(B_t\wedge \tau)\}_{t\geq 0} \) is a supermartingale with respect to \( \{\mathcal{F}_t\}_{t\geq 0} \). Therefore, $f(x) \geq E_x(f(B_{\tau}))$. By Fatou’s lemma,

\[
    f(x) \geq \lim_{t \to \tau} E_x(f(B_{\tau})) \geq E_x(f(B_{\tau})).
\]

(13)

We set

\[
    f_1(x) = E_x(f(B_t)) \quad \text{and} \quad f_2(x) = f(x) - E_x(f(B_t)).
\]

First, the same argument as in the proof of implication: (c)\(\Rightarrow\)(a) in Theorem 2 gives that $f_1$ is quasi-bounded in $U$. 
Next we show that $f_z$ is singular in $U$. For this purpose we have only to show that

$$f_1(B_r)=f(B_r) \quad \text{a.s.} \quad (14)$$

In fact, we suppose that, for a bounded and non-negative finely harmonic function $u$ in $U$, $u \leq f_z$ in $U$. By Theorem 2, $u(B_r)$ exists almost surely and $u(x) = E_x(u(B_r))$. Moreover, we see from Theorem 1 and (14) that $f_z(B_r)=0$ a.s.. Therefore, $u(B_r)=0$ a.s., and so $u=0$ in $U$. Now we prove the fact (14). From Theorem 1 and (13), we see that $f_1(B_r)$ exists and $f_1(B_r) \geq f_z(B_r)$ almost surely. Since $f_1$ is quasi-bounded in $U$, we see from Theorem 2 that $f_1(x) = E_x(f_1(B_r))$. Therefore,

$$f_1(x) = E_x(f_1(B_r)) \leq E_x(f(B_r)) = f_1(x),$$

and so, $f_1(B_r) = f(B_r)$ a.s..

Finally we show the uniqueness of the decomposition of $f$. Suppose that for a quasi-bounded function $u_1$ in $U$ and a singular function $u_2$ in $U$, $f = u_1 + u_2$ in $U$. We have only to prove that $u_1(B_r)=0$ a.s.. To see this, we set

$$g(x) = E_x(u_2(B_r)) \quad \text{and} \quad g_m(x) = E_x(\min\{u_2(B_r), m\}).$$

In the same way as in the proof of implication: $(c) \Rightarrow (a)$ in Theorem 2, we see that $g$ and $g_m$ are quasi-bounded in $U$. Moreover, each $g_m$ is bounded in $U$ and $(g_m)_m$ converges increasingly to $g$ in $U$. Putting $f = u_2$ into (13), $g \leq u_2$ in $U$. Therefore, $g_m \leq u_2$ in $U$, for every $m$. Since $u_2$ is singular in $U$, $g_m=0$ in $U$, for every $m$. Therefore, $g=0$ in $U$ and so, $u_1(B_r)=0$ a.s.. q.e.d.

We call the functions $f_1$ and $f_z$ defined in Main Theorem the \textit{quasi-bounded} part of $f$ and the \textit{singular} part of $f$ respectively.

\textbf{Corollary.} Let $U (\subset \mathbb{R}^d)$ be a fine domain, $f$ and $g$ non-negative finely harmonic functions in $U$, $f_1$ and $g_1$ the quasi-bounded parts of $f$ and $g$ respectively, and $f_2$ and $g_2$ the singular parts of $f$ and $g$ respectively. Suppose that $f \geq g$ in $U$, then $f_i \geq g_i$ in $U$, for $i=1, 2$.

\textit{Proof.} Applying Main Theorem to $f-g$, there exist the quasi-bounded part $h_1$ of $f-g$ and the singular part $h_2$ of $f-g$. Therefore,

$$f=f_1+f_2=g_1+h_1+g_2+h_2.$$ 

It is obvious that $g_1+h_1$ is quasi-bounded in $U$. We show that $g_2+h_2$ is singular in $U$. Suppose that for a bounded and non-negative finely harmonic function $u$ in $U$, $u \leq g_2+h_2$ in $U$. For every $x (\in U)$, let $\{B_t\}_{t \geq 0}$ be a $d$-dimensional Brownian motion starting at $x$ and $\tau$ the first exit time of $\{B_t\}_{t \geq 0}$ from $U$. In the same way as in the proof of Main Theorem, we see that $g_2(B_r)+h_2(B_r)$ a.s.. Since $u(B_r) \leq g_2(B_r)+h_2(B_r)$ a.s., $u(B_r)=0$ a.s.. By Theorem 2, $u(x) = E_x(u(B_r))$. Therefore, $u=0$ in $U$, and so, $g_2+h_2$ is singular in $U$. By the uniqueness of the decomposition of $f$, $f_1=g_1+h_1$ in $U$ and $f_2=g_2+h_2$ in $U$. 

Therefore, $f_1 \geq g_1$ in $U$ and $f_2 \geq g_2$ in $U$. q.e.d.

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References


Added in proof: Recently, Professor Fuglede kindly informed me that he could obtain a non-probabilistic proof of our Main theorem in a personal communication.