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Classical solutions for a class of degenerate elliptic operators with a parameter

By

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Boundary value problems are generally investigated for elliptic differential operators, degenerating on the boundary of a domain ([1], [2], [3], etc.), which are well posed in appropriate Hilbert spaces. On the other hand, classical solutions seem little investigated except for equations of second order ([4], [5], etc.). In this paper, we consider a class of degenerate elliptic differential operators with a positive parameter, and we seek classical solutions, restricting the parameter small enough.

In §1, the regularity of solutions are considered for F-type operators, analogously in [6]. In §2, two types of half space problems are set for F-type operators corresponding to the location of the invertible zone. In §3, the existence of solutions for half space problems is considered for F-type elliptic operators with a parameter, using the energy estimates for adjoint operators ([7]). In §4, some examples of 4th order operators are given.

§1. Regularity for F-type operators.

1.1. F-type operators. Let $x=(x_1, x_2,\dots, x_n)=(x_1, x')\in R^n$, and let

$$A(x;\ D)=\sum_{\nu}a_\nu(x)D^\nu=\sum_{j=0}^{m}a_j(x;\ D')\ D_1^j,$$

where

$$D=(D_1,\ D_2,\dots,\ D_n)=(D_1,\ D'),\ D_j=D_{x_j}=-i\frac{\partial}{\partial x_j},$$

$$a_\nu(x)\in\mathscr{B}^\infty(R^n),\ a_m(x;\ \xi')\not\equiv 0\ \text{near}\ x_1=0,$$

and $a_{j_0}(0,\ x';\ \xi')\not\equiv 0$ for some $j_0(0\leqq j_0\leqq m)$. Let l_j be an integer such that

$$D_1^l a_j(0,\ x';\ \xi')\equiv 0\ \text{for}\ l=0,\ 1,\dots,\ l_j-1,$$

and

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$$D_1^{l_j}a_j(0, x'; \xi')\not\equiv 0,$$

where we denote

$$m'=\max_j (j-l_j) \quad (0\leqq m'\leqq m).$$

Then we say that A is of order (m, m') on $x_1=0$.

Let A be of order (m, m') on $x_1=0$, then we have

$$x_1^{m'}A(x; \xi)=\sum_{j=0}^{m} b_j(x; \xi')(x_1\xi_1)^j=\sum_\nu b_\nu(x)(x_1\xi_1)^{\nu_1}\xi'^{\nu'},$$

where $b_\nu(x)\in\mathscr{B}^\infty((-1, 1)\times R^{n-1})$, therefore we have

$$A(x, \xi)=\sum_{j=0}^{m-m'} b_{j+m'}(x; \xi')(x_1\xi_1)^j\xi_1^{m'}+\sum_{j=0}^{m'-1} a_j(x; \xi')\xi_1^j.$$

Hence we have

$$A(x, D)=\sum_{j=0}^{m-m'} \beta_{j+m'}(x, D')(x_1 D_1)^jD_1^{m'}+\sum_{j=0}^{m'-1} a_j(x, D')D_1^j$$
$$=\mathscr{B}(x; x_1D_1,D')D_1^{m'}+C(x; D_x),$$

where

$$\sum_j \beta_{j+m'}(x; \xi')\xi_1^j=\sum b_{j+m'}(x; \xi')\mathscr{M}^j(\xi_1),$$

$$\mathscr{M}^j(\xi_1)=(\xi_1+i(j-1))(\xi_1+i(j-2))\cdots(\xi_1+i)\xi_1.$$

Let us say that $P(x, \xi)\in\mathscr{F}_m$ if

$$P(x; \xi)=\sum_\nu p_\nu(x)\xi^\nu=\sum_{j=0}^{m} p_j(x; \xi')\xi_1^j,$$

where $p_\nu(x)\in\mathscr{B}^\infty((-1, 1)\times R^{n-1})$, then we have

Lemma 1.1. *Let A be of order (m,m') on $x_1=0$, then it is represented by*

$$A(x; D)=\mathscr{B}(x; x_1D_1, D')D_1^{m'}+C(x; D),$$

where $\mathscr{B}(x; \zeta)\in\mathscr{F}_{m-m'}$ and $C(x, \xi)\in\mathscr{F}_{m'-1}$.

Let A be of order (m, m') on $x_1=0$, that is,

$$A(x; D)=\mathscr{B}(x; x_1D_1,D')D_1^{m'}+C(x, D) \quad (\mathscr{B}\in\mathscr{F}_{m-m'}, C\in\mathscr{F}_{m'-1}),$$

then we denote

$$\Phi(x'; \zeta)=\mathscr{B}(0, x'; \zeta),$$

and we call it a *characteristic polynomial* of A on $x_1=0$. Moreover, we say that the interval (α_1, α_2) is an *invertible zone* of the characteristic operator $\Phi(z'; D_z)$ if, for any $\alpha\in(\alpha_1, \alpha_2)$, $\Phi_\alpha (z'; D_z)$ is invertible in $L^2(R^n)$ and there exists $c>0$ such that

$$\|\Phi_\alpha(z'; D_z)u\|_{L^2(R^n)} \geqq c\|u\|_{L^2(R^n)}$$

for $u \in H^\infty (R^n)$, where

$$\Phi_\alpha(z'; \zeta) = \Phi(z'; \zeta_1 + i\alpha, \zeta').$$

We say that A is a *F-type operator* on $x_1=0$ of order (m, m'), if there exists an invertible zone (α_1, α_2) of $\Phi(z'; D_z)$.

Lemma 1.2. *Let A be of order (m, m') on $x_1=0$ with a characteristic polynomial Φ, then $D_1^h A$ is of order $(m+h, m'+h)$ on $x_1=0$ with a characteristic polynomial Φ_{-h} ($h=0, 1, 2, \ldots$). Namely,*

$$D_1^h A = \Phi_{-h}(x; x_1 D_1, D') D_1^{h+m'} + x_1 \tilde{\mathscr{B}}(x; x_1 D_1, D') D_1^{h+m'} + \tilde{C}(x; D),$$

where $\tilde{\mathscr{B}}(x; \zeta) \in \mathscr{F}_{m-m'}$ and $\tilde{C}(x, \xi) \in \mathscr{F}_{h+m'-1}$.

Proof. Denoting $t=x_1$ and $\tau=\zeta_1$, we write

$$A = \mathscr{B}(t, tD_t) D_t^{m'} + \sum_{k=0}^{m'-1} c_k(t) D_t^k,$$

and

$$\mathscr{B}^{(j)}(t, \tau) = D_t^j \mathscr{B}(t, \tau), \quad c_k^{(j)}(t) = D_t^j c_k(t).$$

Remarking

$$D_t (tD_t)^k = t^{-1}(tD_t)^k t D_t = (tD_t - i)^k D_t,$$

we have

$$\begin{aligned} D_t^h A u &= \sum_{j=0}^{h} \binom{h}{j} \mathscr{B}_{-j}^{(h-j)}(t, tD_t) D_t^{m'+j} u + \sum_{k=0}^{m'-1} \sum_{j=0}^{h} \binom{h}{j} c_k^{(h-j)}(t) D_t^{m'+j} u \\ &= \mathscr{B}_{-h}^{(0)}(t, tD_t) D_t^{h+m'} u + \sum_{j=0}^{h-1} \binom{h}{j} \mathscr{B}_{-j}^{(h-j)}(t, tD_t) D_t^{m'+j} u + \sum_{j=0}^{h+m'-1} c_j'(t) D_t^j u \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \mathscr{B}_{-h}(0, tD_t) D_t^{h+m'} u + t\mathscr{B}'(t, tD_t) D_t^{h+m'} u, \\ I_2 &= \sum_{j=0}^{h-1} \binom{h}{j} \sum_{k=0}^{m-m'} \beta_k^{(h-j)}(t) \ (tD_t - ij)^k D_t^{m'+j} u \\ &= \sum_{j=0}^{h-1} \sum_{k=0}^{m-m'} \beta_{jk}(t) t^k D_t^{k+m'+j} u \\ &= \sum_{j=0}^{h-1} \sum_{k=h-j}^{m-m'} \beta_{jk}(t) t^k D_t^{k+m'+j} u + \sum_{j=0}^{h-1} \sum_{k=0}^{h-1-j} \beta_{jk}(t) t^k D_t^{k+m'+j} u \\ &= t\mathscr{B}''(t, tD_t) D_t^{h+m'} u + \sum_{j=m'}^{h+m'-1} c_j''(t) D_t^j u, \end{aligned}$$

where we have only to set

$$\tilde{\mathscr{B}}=\mathscr{B}'+\mathscr{B}'',\ \tilde{c}_j=c_j'+c_j''.$$

1.2. Regularity. Now, let us say that $u\in H_b^s(R_+^n)$ $(s=0, 1, 2,\cdots,\ b\in R)$ if

$$\hat{x}_1^{\nu_1+b-1/2}D^\nu u\in L^2(R_+^n)\quad(|\nu|\leqq s),$$

where $\hat{x}_1\in\mathscr{B}^\infty\ (R)$ is an increasing function satisfying

$$\hat{x}_1=\begin{cases}x_1 & \text{if } x_1<1,\\ 2 & \text{if } x_1>2.\end{cases}$$

Lemma 1.3. *Let s, k be positive integers $(s\geqq k)$, and let* supp $[u]\subset\{x_1<1\}$.
i) *Let $u\in H_{-k+1/2}^s(R_+^n)$, then we have $u\in H^k(R_+^n)$ and*

$$D_1^ju|_{x_1=0}=0\quad(0\leqq j\leqq k-1).$$

ii) *Conversely, let $u\in H^s(R_+^n)$ and let*

$$D_1^ju|_{x_1=0}=0\quad(0\leqq j\leqq k-1),$$

then $u\in H_{-k+1/2+\varepsilon}^{s-k}(R_+^n)$ for any $\varepsilon>0$.

Proof. i) First, since

$$\hat{x}^{\nu_1-k}D^\nu u\in L^2(R_+^n)\quad(|\nu|\leqq k),$$

we have $D^\nu u\in L^2(R_+^n)$ $(|\nu|\leqq k)$. Next, denoting

$$u_j=D_1^ju|_{x_1=0},$$

we have

$$\|D_1^ju(x_1,\ \cdot)-u_j\|_{L^2(R^{n-1})}\leqq C\hat{x}_1^{1/2}(0\leqq j\leqq k-1).$$

If $u_j\neq0$ for some $0\leqq j\leqq k-1$, then we have

$$\|D_1^ju(x_1,\ \cdot)\|_{L^2(R^{n-1})}\geqq c(>0)$$

near $x_1=0$. Therefore, we have

$$\|\hat{x}_1^{-1}D_1^ju\|_{L^2(R_+^n)}=+\infty,$$

which is a contradiction.
ii) There exists $C>0$ such that

$$\|D^\nu u(x_1,\ \cdot)\|_{L^2(R^{n-1})}\leqq C\hat{x}_1^{k-\nu_1-1/2}$$

if $\nu_1\leqq k-1$, $|\nu|\leqq s-k$, and we have

$$\|D^\nu u(x_1,\ \cdot)\|_{L^2(R^{n-1})}\leqq C$$

if $\nu_1\geqq k$, $|\nu|\leqq s-k$. Hence, we have

$$\|\hat{x}_1^{\varepsilon-k+\nu_1}D^\nu u\|_{L^2(R_+^n)}<+\infty\quad(\varepsilon>0)$$

if $|\nu|\leqq s-k$, that is,

$$u\in H^{s-k}_{-k+1/2+\varepsilon}(R^n_+).$$

Proposition 1.4. *Let A be a F-type operator of order (m, m') on $x_1=0$ with characteristic polynomial Φ. Let (α_1, α_2) be an invertible zone of Φ. Let* supp $[u]\subset\{x_1<1\}$, *$Au=f\in H^s_\alpha(R^n_+)$ in R^n_+ and $u\in H^s_{\beta-m'}(R^n_+)$ (s: large), where $\alpha_1<\alpha\leqq\beta<\alpha_2$ and $\beta-\alpha$ is an integer, then we have $D_1^{m'}u\in H_\alpha^{s-M(\beta-\alpha)}(R^n_+)$, where M is the differential order of A.*

Proof. From Lemma 1. 2, we have

$$\Phi_{-h}(tD_t)D_t^{h+m'}u=D_t^hf-t\tilde{\mathscr{B}}(t, tD)D_t^{h+m'}u-\sum_{k=0}^{h+m'-1}\tilde{c}_k(t)D_t^ku.$$

Hence, multiplying $t^{\sigma+h}$ on the both sides of the equality, we have

$$\Phi_\sigma(tD_t)t^{\sigma+h}D_t^{h+m'}u=t^{\sigma+h}\{D_t^hf-t\tilde{\mathscr{B}}(t, tD_t)D_t^{h+m'}u-\sum_{k=0}^{h+m'-1}\tilde{c}_k(t)D_t^ku\}.$$

First, let $\sigma=\beta-1$, then we have

$$\Phi_{\beta-1}(tD_t)t^{\beta-1+h}D_t^{h+m'}u=t^{\beta-1+h}\{D_t^hf-t\tilde{\mathscr{B}}(t, tD_t)D_t^{h+m'}u-\sum_{k=0}^{h+m'-1}\tilde{c}_k(t)D_t^ku\}\in H_0^{s-M-h}$$

if $\beta-1\geqq\alpha$, therefore we have

$$t^{\beta-1+h}D_t^{h+m'}u\in H_0^{s-M-h}.$$

Next, let $\sigma=\beta-2$, then we have

$$\Phi_{\beta-2}(tD_t)t^{\beta-2+h}D_t^{h+m'}u=t^{\beta-2+h}\{D_t^hf-t\tilde{\mathscr{B}}(t, tD_t)D_t^{h+m'}u-\sum_{k=0}^{h+m'-1}\tilde{c}_k(t)D_t^ku\}\in H_0^{s-M-h}$$

if $\beta-2\geqq\alpha$, therefore we have

$$t^{\beta-2+h}D_t^{h+m'}u\in H_0^{s-2M-h}.$$

In the same way, we have

$$t^{\alpha+h}D_t^{h+m'}u\in H_0^{s-(\beta-\alpha)M-h}.$$

Hence we have $D_t^{m'}u\in H_\alpha^{s-(\beta-\alpha)M}$.

Collorary. *Besides the assumptions in Prop. 1.4, we assume $\alpha<1$, then we have*

$$D_1^ju\in L^2(R^n_+)\ \ (0\leqq j\leqq m'-1).$$

§2. Boundary value problems for F-type operators.

2.1. Condition(Φ). Let A be a F-type operator of order (m, m') on $x_1=0$ with symbol

$$A(x;\ \xi)=\{\sum_\nu b_\nu(x)(x_1\xi_1)^{\nu_1}\xi'^{\nu'}\}\xi_1^{m'}+\sum_{\nu_1<m'}c_\nu(x)\xi^\nu,$$

where $b_\nu,c_\nu\in\mathscr{B}^\infty((0, 1)\times R^{n-1})$. Let Φ $(x';\zeta)$ be the characteristic polynomial of A and let (α_1, α_2) be an invertible zone of Φ. We assume

Condition (Φ). $0\in(\alpha_1, \alpha_2)$, i.e. $\alpha_1<0<\alpha_2$.

Now, denoting

$$\begin{aligned}A(x;\ D)&=\{\sum_\nu b_\nu(x)x_1^{\nu_1}D_1^{\nu_1}D'^{\nu'}\}D_1^{m'}+\sum_{\nu_1<m'}c_\nu(x)D^\nu\\ &=\{\sum_\nu \beta_\nu(x)(x_1D_1)^{\nu_1}D'^{\nu'}\}D_1^{m'}+\sum_{\nu_1<m'}c_\nu(x)D^\nu\\ &=\mathscr{B}(x;\ x_1D_1,\ D')D_1^{m'}+\sum_{j=0}^{m'-1}C_j(x;\ D')D_1^j,\end{aligned}$$

we have

$$\begin{aligned}x_1^{m'}A(x;\ D)&=\mathscr{B}_{m'}(x;\ x_1D_1,\ D')\mathscr{M}^{m'}(x_1D_1)\\ &\quad+\sum_{j=0}^{m'-1}C_j(x;\ D')x_1^{m'-j}\mathscr{M}^j(x_1D_1)=\mathscr{A}(x;\ x_1D_1,\ D'),\end{aligned}$$

where

$$\mathscr{B}_j(x;\ \zeta)=\mathscr{B}(x;\ \zeta_1+ij,\ \zeta').$$

Lemma 2.1. *We have the asymptotic expansion of* $\mathscr{A}$:

$$\mathscr{A}(x;\ x_1D_1,\ D')\sim\mathscr{A}_0(x';\ x_1D_1,D')+x_1\mathscr{A}_1(x';\ x_1D_1,D')+x_1^2\mathscr{A}_2(x';\ x_1D_1,\ D')+\cdots,$$

where $\mathscr{A}_j(x';\ \zeta)\in\mathscr{F}_m$,

$$\mathscr{A}_0(x';\ \zeta)=\Phi_{m'}(x';\ \zeta)\mathscr{M}^{m'}(\zeta_1),$$

and

$$\mathscr{A}_j(x';\ -ih,\ \zeta')=0\ \ (j\geqq 0,\ h\geqq 0,\ j+h\leqq m'-1).$$

Proof. In the expression

$$\mathscr{A}(x;\ x_1D_1,D')=\mathscr{B}_{m'}(x;\ x_1D_1,D')\mathscr{M}^{m'}(x_1D_1)+\sum_{j=0}^{m'-1}C_j(x;\ D')x_1^{m'-j}\mathscr{M}^j(x_1D_1),$$

we take the asymptotic expansions near $x_1=0$:

$$\mathscr{B}_{m'}(x;\ \zeta)\sim\sum_{k=0}^{\infty}\mathscr{B}_{m'}^{(k)}(x';\ \zeta)x_1^k,$$

$$C_j(x;\ \zeta')\sim\sum_{k=0}^{\infty}C_{m'}^{(k)}(x';\ \zeta')x_1^k,$$

then we have

$$\begin{aligned}\mathscr{A}(x;\ \zeta)\sim\mathscr{B}_{m'}^{(0)}(x';\ \zeta)\mathscr{M}^{m'}(\zeta_1)&+x_1\{\mathscr{B}_{m'}^{(1)}(x';\ \zeta)\mathscr{M}^{m'}(\zeta_1)+C_{m'-1}^{(0)}(x';\ \zeta')\mathscr{M}^{m'-1}(\zeta_1)\}\\ &+x_1^2\{\mathscr{B}_{m'}^{(2)}(x';\ \zeta)\mathscr{M}^{m'}(\zeta_1)+C_{m'-1}^{(1)}(x';\ \zeta')\mathscr{M}^{m'-1}(\zeta_1)\\ &+C_{m'-2}^{(0)}(x';\ \zeta')\mathscr{M}^{m'-2}(\zeta_1)\}+\cdots\cdots\end{aligned}$$

$$\sim \mathscr{A}_0(x'; \zeta)+x_1\mathscr{A}_1(x'; \zeta)+x_1^2\mathscr{A}_2(x'; \zeta)+\cdots,$$

where

$$\mathscr{A}_j(x'; \zeta)=\mathscr{B}_{m'}^{(j)}(x'; \zeta)\mathscr{M}^{m'}(\zeta_1)$$
$$+C_{m'-1}^{(j-1)}(x'; \zeta')\mathscr{M}^{m'-1}(\zeta_1)+\cdots+C_{m'-j}^{(0)}(x'; \zeta')\mathscr{M}^{m'-j}(\zeta_1)$$

if $0\leqq j\leqq m'-1$, and

$$\mathscr{A}_j(x'; \zeta)=\mathscr{B}_{m'}^{(j)}(x'; \zeta)\mathscr{M}^{m'}(\zeta_1)$$
$$+C_{m'-1}^{(j-1)}(x'; \zeta')\mathscr{M}^{m'-1}(\zeta_1)+\cdots+C_0^{(j-m')}(x'; \zeta')$$

if $j\geqq m'$. Especially, we have

$$\mathscr{A}_0(x'; \zeta)=\Phi_{m'}(x'; \zeta)\mathscr{M}^{m'}(\zeta_1).$$

On the otherhand, since

$$\mathscr{M}^j(-ih)=0 \quad (h=0, 1,\dots, j-1),$$

we have

$$\mathscr{A}_j(x'; -ih, \zeta')=0 \quad (h=0, 1,\dots,m'-j-1).$$

2.2. Half space problem. Let us consider the following two types of half space problems. The first problem is a problem with no boundary data on the boundary, that is, the first problem (P-I) is to find a solution $u\in H^N(R_+^n)$ satisfying

$$Au=f \quad \text{in } R_+^n$$

for any $f\in H^{N'}(R_+^n)$. The second problem is a problem with full boundary data on the boundary, that is, the second problem (P-II) is to find a solution $u\in H^N(R_+^n)$ satisfying

$$\begin{cases} Au=f & \text{for } x\in R_+^n, \\ D_1^j u=\varphi_j \ (0\leqq j<m') & \text{for } x_1=0, \ x'\in R^{n-1} \end{cases}$$

for any $f\in H^{N'}(R_+^n)$ and $\phi_j\in H^{N'}(R^{n-1})$ $(0\leqq j<m')$.

Let u and f be smooth enough, satisfying $A\,u=f$, that is,

$$\mathscr{A}(x'; x_1D_1, D')u=x_1^{m'}f,$$

and let

$$u\sim\sum x_1^j u_j, \qquad f\sim\sum x_1^j f_j,$$

then we have

$$\mathscr{A}u\sim\sum x_1^{j+k}\mathscr{A}_j(x'; -ik, D')u_k,$$

because $x_1 D_1 x_1^k = -i\ k x_1^k$. Hence, remarking

$$\mathscr{A}_j(x';\ -ik,\ D')=0 \text{ if } j+k=l<m',$$

we have

$$\sum_{j+k=l} \mathscr{A}_j(x';\ -ik,\ D')u_k = f_{l-m'} \qquad \text{if } l \geqq m',$$

that is,

$$(*)\begin{cases} \mathscr{A}_0(x';\ -im',\ D')u_{m'} + \mathscr{A}_1(x';\ -i(m'-1),\ D')u_{m'-1} + \cdots\cdots \\ \qquad + \mathscr{A}_{m'}(x';\ 0,\ D')u_0 = f_0, \\ \mathscr{A}_0(x';\ -i(m'+1),\ D')u_{m'+1} + \mathscr{A}_1(x';\ -im',\ D')u_{m'} + \cdots\cdots \\ \qquad + \mathscr{A}_{m'}(x';\ -i,\ D')u_1 = f_1, \\ \qquad \cdots\cdots\cdots\cdots. \end{cases}$$

Now, we set $\phi_l=0$ $(0\leqq l\leqq m'-1)$ (apparent boundary values) in case when we consider (P-I). First, we set $u_l=\phi_l(0\leqq l\leqq m'-1)$, then, since $\mathscr{A}_0(x';\ -il,\ D')$ is invertible for $m'\leqq l<m'+|\alpha_1|$, $\{u_l\ (m'\leqq l<m'+|\alpha_1|)\}$ are defined by (*), using $\{\phi_l\ (0\leqq l<m')\}$ and $\{f\ (0\leqq l<|\alpha_1|)\}$.

Lemma 2.2. *For any $N>0$, there exists $N'>0$ as follows. For any $f\in H^{N'}(R^n)$ and $\phi_j\in H^{N'}(R^{n-1})$ $(0\leqq j<m')$, there exists $\bar{u}\in H^N(R^n)$ such that*

$$D_1^j(f-A\bar{u})=0 \ (0\leqq j<|\alpha_1|) \text{ for } x_1=0,\ x'\in R^{n-1},$$

$$D_1^j\bar{u}=\phi_j(0\leqq j<m') \text{ for } x_1=0,\ x'\in R^{n-1}.$$

Proof. We define

$$\bar{u}=\sum_{0\leqq j<m'+|\alpha_1|} x_1^j u_j \chi(x_1),$$

where $\chi\in C_0^\infty(R)$ and $\chi=1$ near $x_1=0$. Then, defining

$$g=f-A\bar{u}\sim\sum x_1^j g_j,$$

we remark

$$g_j=0\ (0\leqq j<|\alpha_1|).$$

From Lemma 2.2, the problems (P-I) and (P-II) can be reduced to the problems (P′-I) and (P′-II), where (P′-I) is to find a solution $u\in H^N({}^n_+R)$ satisfying

$$Au=f \qquad \text{in } R^n_+,$$

and (P′-II) is to the problem to find a solution $u\in H^N(R^n_+)$ satisfying

$$\begin{cases} Au=f & \text{for } x\in R^n_+, \\ D^j u=0\ (0\leqq j<m') & \text{for } x_1=0,\ x'\in R^{n-1} \end{cases}$$

for any $f \in H_\alpha^{N'}(R_+^n)$ satisfying

$$D_1^j f = 0 \qquad (0 \leqq j < |\alpha_1|) \qquad \text{for } x_1 = 0,\ x' \in R^{n-1}.$$

Moreover, remarking Lemma 1.3, the problem (P′-I) and (P′-II) are reduced to (P″-I) and (P″-II), where (P″-I) is to find a solution $u \in H^N(R_+^n)$ and (P″-II) is to find a solution $u \in H^N_{-m'+1/2}(R_+^n)$, satisfying

$$Au = f \quad \text{in } R_+^n,$$

for any $f \in H_\alpha^{N'}(R_+^n)$ $(\alpha > \alpha_1)$.

§3. Elliptic problems.

3.1. Assumptions. Let us consider F-typ e operators which are elliptic with a small positive parameter κ, degenerating on the boundary of a half space:

$$R_+^n = \{x = (x_1, x');\ x_1 > 0,\ x' = (x_2, \dots, x) \in R^{n-1}\}.$$

We assume the following conditions (A.1) and (A.2).

Condition (A.1). There exists $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ such that

i) $$\delta = \min_{1 \leqq i \leqq n} \delta_i > 0,$$

ii) $$A(x, \kappa;\ \kappa^\delta D) = \sum_{|\nu| \leqq m} a_\nu(x, \kappa)(\kappa^\delta D)^\nu,$$

where $a_\nu(x, \kappa) \in \mathscr{B}^\infty(R_+^n \times (0, 1))$ and $D = -i\partial_x$,

iii) for any $\varepsilon > 0$, there exists $c > 0$ such that

$$|A(x, 0;\ \tilde{\xi})| \geqq c(|\tilde{\xi}| + 1)^m$$

for $\varepsilon < x_1 < +\infty$, $x' \in R^{n-1}$, $\tilde{\xi} \in R^n$.

Condition (A.2). There exist m' and $\sigma = (\sigma_1, \dots, \sigma_n)$ satisfying the following i)~iii).

i) $$0 \leqq m' \leqq m,\ \sigma_1 = 1,\ \sigma_j \geqq 0\ (j = 2, \dots, n).$$

ii) $$x_1^{m'} A(x, \kappa;\ \tilde{\xi}) = \sum_{|\nu| \leqq m} x_1^{m'} a_\nu(x, \kappa)\tilde{\xi}^\nu = \sum_{|\nu| \leqq m} a'_\nu(x, \kappa)(x_1^\sigma \tilde{\xi})^\nu$$
$$= \sum_{|\nu| \leqq m} a'_\nu(x, \kappa)\tilde{\tilde{\xi}}^\nu = A'(x, \kappa;\ \tilde{\tilde{\xi}}),$$

where $a'_\nu(x, \kappa)$ is bounded in $R_+^n \times (0, 1)$ and

$$\tilde{\tilde{\xi}} = x_1^\sigma \tilde{\xi} = (x_1^{\sigma_1}\tilde{\xi}_1 \dots,\ x_1^{\sigma_n}\tilde{\xi}_n).$$

iii) There exists a non-zero zone (β_1, β_2) of A'. Namely, for any $\beta \in (\beta_1, \beta_2)$ there exists $c > 0$ such that

$$|A'(0, x', 0;\ \tilde{\tilde{\xi}}_1 + i\beta,\ \tilde{\tilde{\xi}}')| \geqq c(|\tilde{\tilde{\xi}}| + 1)^m$$

for $\tilde{\tilde{\xi}}\in R^n$, $x'\in R^{n-1}$.

Remark 1. Let $\beta\in(\beta_1, \beta_2)$, then $m/2$ of the roots of $A'(0, x', 0; \tilde{\tilde{\xi}})=0$ satisfy $\text{Im}\ \tilde{\tilde{\xi}}_1<\beta$ and the others satisfy $\text{Im}\ \tilde{\tilde{\xi}}_1>\beta$ if $\tilde{\tilde{\xi}}'\in R^{n-1}$, owing to the ellipticity of $A'(0, x', 0; \tilde{\tilde{\xi}})$.

Remark 2. It holds $m'\leqq m/2$, because the multiplicity of the root $\tilde{\tilde{\xi}}_1=0$ of

$$A'(0, x', 0; \tilde{\tilde{\xi}}_1, 0)=\sum_{j=m'}^{m} a'_{j0}(0, x', 0)\tilde{\tilde{\xi}}_1^j=0$$

is not smaller than m'.

Remark 3. 0 does not belong to (β_1, β_2) if $m'>0$, because $A'(0, x', 0; 0)=0$.

Remark 4. $m'\leqq\underline{\sigma}\ m$, owing to the ellipticity of $A'(0, x', 0; \tilde{\tilde{\xi}})$, where $\underline{\sigma}=\min_i \sigma_i$.

We say that A is of 0-type if $m'=0$, A is of I-type if $(\beta_1, \beta_2)\subset[0, \infty)$, and A is of II-type if $(\beta_1, \beta_2)\subset(-\infty, 0]$. We consider the problem (P-I) if A is of 0-type or of I-type, and consider the problem (P-II) if A is of II-type.

Example. Let us consider

$$A=\hat{x}_1(\kappa D_1)^2-ib(\kappa D_1)+\hat{x}_1^n(\kappa D_2)^2+1,$$

where b is a non-zero real constant and n is a non-negative integer. Let us see that it satisfies (A.2). Setting

$$\tilde{\xi}=\kappa\xi,\ \ \tilde{\tilde{\xi}}_1=x_1\tilde{\xi}_1,\ \ \tilde{\tilde{\xi}}_2=x_1^{(n+1)/2}\tilde{\xi}_2,$$

we have

$$A'=\tilde{\tilde{\xi}}_1^2-ib\tilde{\tilde{\xi}}_1+\tilde{\tilde{\xi}}_2^2+x_1$$

near $x_1=0$, where the zeros of $A'(0; \tilde{\tilde{\xi}})$ are

$$\tilde{\tilde{\xi}}_1=\frac{i}{2}\{b\pm(b^2+4\tilde{\tilde{\xi}}_2^2)^{1/2}\}.$$

Hence, $(0, b)$ is the non-zero zone of A' if $b>0$, and $(b, 0)$ is the non-zero zone of A' if $b<0$. Moreover, if n is odd, L is invariant under the change of the variable x_1 into $-x_1$.

Let us denote

$$e(z)=(e(z_1), z_2,\dots, z_n),$$

where $e(z_1)$ is a strictly increasing function satisfying

$$e(z_1)=\begin{cases} e^{z_1} & \text{if } z_1<-1,\\ z_1 & \text{if } z_1>1.\end{cases}$$

Moreover, we define

$$\tilde{\Lambda}^s u(x)=(|\tilde{D}_z|^2+1)^{s/2}u(e(z))|_{z=e^{-1}(x)}$$

for $u\in C_0^\infty(R_+^n)$. Then we have

Lemma 3.1. ([7]) *Assume the conditions* (A.1), (A.2). *Let* $\beta_1<\beta<\beta_2$, *and let* s *be real, then there exist* $\kappa_0>0$, $C>0$ *such that*

$$\sum_{|\nu|\leqq m}\|\hat{x}_1^{-1/2+\sigma\nu}D^\nu\tilde{\Lambda}^s\ \hat{x}_1^{\tilde{\beta}}u\|\leqq C\|\hat{x}_1^{-1/2}\tilde{\Lambda}^s\hat{x}^{\tilde{\beta}+m'}A(x,\ \kappa;\ \tilde{D})u\|$$

for $0<\kappa<\kappa_0$, $u\in H_{\tilde{\beta}}^\infty(R_+^n)$, *where* $\tilde{\beta}=\beta\kappa^{-\delta_1}$ *and* $\tilde{D}=\kappa^\delta D$.

3.2. Existence theorem. Let A^* be the formal adjoint operator of A in L^2 (R_+^n). Namely, let

$$A=\sum a_\nu(x,\ \kappa)\tilde{D}^\nu,$$

then

$$A^*=\sum\tilde{D}^\nu\overline{a_\nu(x,\ \kappa)}.$$

Assume that the conditions (A.1) and (A.2) are satisfied for A, then they are also satisfied for A^* with the non-zero zone $(-\beta_2,\ -\beta_1)$. Hence we have from [7]

Lemma 3.2. *Let us assume the conditions* (A.1) *and* (A.2). *Let* β *satisfy* $\beta_1<\beta$ $<\beta_2$ *and let* s *be a real number, then there exist* κ_0 *and* C *such that*

$$\sum_{|\nu|\leqq m}\|\hat{x}_1^{-1/2+\sigma\nu}D^\nu\tilde{\Lambda}^s\hat{x}_1^{-\tilde{\beta}}u\|\leqq C\|\hat{x}_1^{-1/2}\tilde{\Lambda}^s\hat{x}_1^{-\tilde{\beta}}A^*u\|$$

if $u\in H_{-\tilde{\beta}}^\infty(R_+^n)$ *and* $0<\kappa<\kappa_0$.

Let us denote

$$\Lambda^s u(x)=(|D_z|^2+1)^{s/2}u(e(z))|_{z=e^{-1}(x)}$$

and

$$\Lambda'^s u(x)=\{D_{z_1}^2+(e'(z_1)^{\sigma_2}D_{z_2})^2+\cdots+(e'(z_1)^{\sigma_n}D_{z_n})^2+1\}^{s/2}u((z))|_{z=e^{-1}(x)}$$

for $u\in C_0^\infty(R_+^n)$.

Setting s and κ^{-1} large enough, we define a Hilbert space H with an inner product:

$$(w,\ \phi)_H=(\hat{x}_1^{-1/2}\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}+m'}A^*w,\ \hat{x}_1^{-1/2}\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}+m'}A^*\phi)_{L^2(R_+^n)},$$

then $w\in H$ is equivalent to

$$\|\hat{x}_1^{-1/2}\Lambda'^m\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}}w\|<+\infty.$$

Proposition 3.3. *Assume* (A.1) *and* (A.2). *Let* $\beta_1<\beta<\beta_2$, $s>s_0$, $0<\kappa<\kappa_0$, *and*

$$\|\hat{x}_1^{-1/2}\Lambda'^{-m}\Lambda^s\hat{x}_1^{\tilde{\beta}}f\|<+\infty,$$

then there exists a solution satisfying

$$Au=f \ in R_{+}^{n}$$

and

$$\|\hat{x}_1^{1/2}\Lambda^s\hat{x}_1^{\tilde{\beta}-m'}u\|<+\infty.$$

Proof. Let

$$\|\hat{x}_1^{1/2}\Lambda'^{-m}\Lambda^s\hat{x}_1^{\tilde{\beta}}f\|<+\infty,$$

then we have for $\phi\in H$

$$|(f,\ \phi)|=|(\hat{x}_1^{1/}\Lambda'^{-m}\Lambda^s\hat{x}_1^{\tilde{\beta}}f,\ \hat{x}_1^{-1/2}\Lambda'^m\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}}\phi)|\leqq\|\hat{x}_1^{1/2}\Lambda'^{-m}\Lambda^s\hat{x}_1^{\tilde{\beta}}f\| \quad \|\phi\|_H.$$

Owing to Riesz' theorem, there exists $w\in H$ such that

$$(f,\ \phi)=(w,\ \phi)_H=(\hat{x}_1^{-1/2}\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}+m'}A^*w,\ \hat{x}_1^{-1/2}\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}+m'}A^*\phi).$$

Let

$$u=\hat{x}_1^{-\tilde{\beta}+m'}\Lambda^{-s}\hat{x}_1^{-1}\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}+m'}A^*w,$$

then we have

$$Au=f \text{ in } R_{+}^{n}$$

and

$$\hat{x}_1^{1/2}\Lambda^s\hat{x}_1^{\beta-m'}u=\hat{x}_1^{-1/2}\Lambda^{-s}\hat{x}_1^{-\tilde{\beta}+m'}A^*w\in L^2(R_{+}^{n}).$$

3.3. Regularity. From the condition (A.2), A is of order $(m,\ m')$ on $x_1=0$ and

$$A(x,\ \kappa;\ \tilde{\xi})=\sum_{|\nu|\leqq m-m'}b_\nu(x,\ \kappa)(x_1\tilde{\xi}_1)^{\nu_1}\tilde{\xi}'^{\nu'}\tilde{\xi}_1^{m'}+\sum_{\nu_1<m'}a_\nu(x,\ \kappa)\tilde{\xi}^\nu$$

$$=B(x,\ \kappa;\ x_1\tilde{\xi}_1,\ \tilde{\xi}')\tilde{\xi}_1^{m'}+C(x,\ \kappa;\ \tilde{\xi}),$$

where $b_\nu(x,\ \kappa)\in\mathscr{B}^\infty(R_+\times(0,\ 1))$. Denoting

$$A(x,\ \kappa;\ \tilde{D})=\mathscr{B}(x,\ \kappa;\ x_1\tilde{D}_1,\ \tilde{D}')\tilde{D}_1^{m'}+C(x,\ \kappa;\ \tilde{D}),$$

we have

$$\mathscr{B}(0,\ x',\ 0;\ \tilde{\zeta})=B(0,\ x',\ 0;\ \tilde{\zeta}),$$

which we denote $\Phi(x';\ \tilde{\zeta})$. Let $\sigma_2=\cdots=\sigma_k=0<\sigma_{k+1}\leqq\cdots\leqq\sigma_n$, then

$$\Phi(x';\ \tilde{\zeta})\tilde{\zeta}_1^{m'}=A'(0,\ x',\ 0;\ \tilde{\zeta}_1,\ \tilde{\zeta}_2,\dots,\ \tilde{\zeta}_k,\ 0,\dots,\ 0),$$

therefore

$$\Phi(x';\ \tilde{\zeta})\neq0 \text{ if } \beta_1<\text{Im } \tilde{\zeta}_1<\beta_2,\ \tilde{\zeta}'\in R^{n-1}.$$

Moreover, if $m'>0$, then $\underline{\sigma}>0$, therefore $\Phi(x',\ \tilde{\zeta})$ is independent of $\tilde{\zeta}'$, which we denote $\Phi\ (x';\ \tilde{\zeta}_1)$. Now, we assume

Condition (**A.3**).

i) In case when $m'>0$, there exists (α_1, α_2) such that

$$0\in(\alpha_1, \alpha_2),\ (\beta_1, \beta_2)\subset(\alpha_1, \alpha_2),$$

and $\Phi(x'; \tilde{\zeta}_1)\neq 0$ if $\alpha_1<\mathrm{Im}\ \tilde{\zeta}_1<\alpha_2$.

ii) In case when $m'=0$, $0\in(\beta_1, \beta_2)$.

Setting $(\alpha_1, \alpha_2)=(\beta_1, \beta_2)$ when $m'=0$, we have

Lemma 3.4. *Assume* (A.2) *and* (A.3), *then* $(\tilde{\alpha}_1, \alpha_2)$ *is an invertible zone of* $\Phi(z'; \tilde{D}_2)$ *for* $0<\kappa<\kappa_0$, *where*

$$\tilde{\alpha}_j=\kappa^{-\delta_1}\alpha_j\ (j=1, 2).$$

Theorem 3.5. *Assume that the conditions* (A.1)~(A.3) *are satisfied. If A is of 0-type or of I-type, then, for $N>0$, there exist $\kappa_0>0$ and $N'>0$ such that there exists a unique solution $u\in H^N(R_+^n)$ of the half space problem* (P-I), *satisfying*

$$Au=f\in H^{N'}(R_+^n)\ in\ R_+^n,$$

if $0<\kappa<\kappa_0$. If A is of II-type, then, for $N>0$, there exist $\kappa_0>0$ and $N'>0$ such that there exists a unique solution $u\in H^N(R_+^n)$ of the half space problem (P-II), *satisfying*

$$\begin{cases} Au=f\in H^{N'}(R_+^n)\ in\ R_+^n, \\ D_1^j u=\phi_j\in H^{N'}(R^{n-1})\ on\ \{x_1=0\}\times R^{n-1}\ (j=0, 1,\ldots, m'-1), \end{cases}$$

if $0<\kappa<\kappa_0$.

3.4. Whole space problem. Let us consider A in the whole space R^n:

$$A=\sum_\nu a_\nu(x, \kappa)\tilde{D}^\nu, \qquad a_\nu(x, \kappa)\in\mathscr{B}^\infty(R^n\times(0, 1)).$$

Let us say that the conditions (A.1)~(A.3) are satisfied in $R_+^n\cup R_-^n$, if (A.1)~(A.3) are satisfied not only for A but also for $\tilde{A}$, where

$$\tilde{A}(x, \kappa; \tilde{D})=A(-x_1, x', \kappa; -\tilde{D}_1, \tilde{D}').$$

We remark that A and $\tilde{A}$ are of the same type, because

$$\tilde{A}'(0, x', 0; \tilde{\tilde{\xi}}_1, 0)=A'(0, x', 0; \tilde{\tilde{\xi}}_1, 0).$$

Remarking Lemma 2.2, we have

Proposition 3.6. *Assume that* (A.1)~(A.3) *are satisfied in $R_+^n\cup R_-^n$, and assume that A is of II-type. Then, for $N>0$, there exist $\kappa_0>0$ and $N'>0$ such that there exists a unique solution $u\in H^N(R_+^n)$ of the whole space problem with datas on a intermediate hypersurface*:

$$\begin{cases} Au=f\in H^{N'}(R^n) \qquad in\ R^n, \\ D_1^j u=\phi_j\in H^{N'}(R^{n-1}) \qquad on\ \{x_1=0\}\times R^{n-1}\ (j=0, 1,\ldots, m'-1), \end{cases}$$

if $0<\kappa<\kappa_0$.

Remarking the freedom of the choice of $\{\phi_j\}$, obviously we have

Corollary. *Let A satisfy the assumptions of Prop.* 3.6, *then there exists $\kappa_0>0$ such that A is not hypoelliptic on $x_1=0$ if $0<\kappa<\kappa_0$.*

§4. Examples.

Let

$$P(\beta)=a\beta^4+b\beta^3+c\beta^2+d\beta+e$$

be a polynomial of β, where $a(>0)$, b, c, d, e are real. Let us assume that zeros of $P(\beta)$ are real, where we denote

$$P(\beta)=a\prod_{j=1}^{4}(\beta+b_j),\ b_1\leqq b_2\leqq b_3\leqq b_4.$$

Lemma 4.1. *Assume*

$$b_1\leqq b_2<b_3\leqq b_4,$$

then we have $b_0\in I$, where

$$b_0=(b_2+b_3)/2,\ I=(b_2,\ b_3)\cap[(b_1+b_3)/2,\ (b_2+b_4)/2].$$

Moreover, let $\beta\in I$, then

$$\mathrm{Re}\ P(i\zeta)\geqq a(|\mathrm{Re}\ \zeta|^2+\delta^2)^2 \quad \textit{for}\ \mathrm{Im}\ \zeta=\beta,$$

where $\delta=\min(\beta-b_2,\ b_3-\beta)$ (>0).

Proof. Let $\zeta=\xi+i\beta$, then

$$P(i\zeta)=a\Pi(i\zeta+b_j)=a\Pi(\zeta-ib_j)=a\Pi(\xi+i\beta-ib_j).$$

First, let us prove the inequality for $\xi\leqq 0$. Let us denote

$$ib_j-(\xi+i\beta)=r_j(\xi)e^{i\theta_j(\xi)}\ (r_j>0,\ |\theta_j|\leqq\pi/2),$$

then $r_j(\xi)$, $\theta_j(\xi)$ are continuous functions of $\xi(\leqq 0)$, and satisfy

$$r_j(\xi)\geqq(|\xi|^2+\delta^2)^{1/2},$$

$$-\pi/2\leqq\theta_j(\xi)<0\ \ (j=1,\ 2),\ \ 0<\theta_j(\xi)\leqq\pi/2(j=3,\ 4),$$

and $\theta_j(\xi)\longrightarrow 0$ as $\xi\longrightarrow-\infty$. Since

$$(b_1+b_3)/2\leqq\beta\leqq(b_2+b_4)/2,$$

we have

$$(b_1-\beta)+(b_3-\beta)\leqq 0,\ (b_2-\beta)+(b_4-\beta)\geqq 0,$$

that is,

$$\theta_1+\theta_3\leqq 0,\ \theta_2+\theta_4\geqq 0.$$

Therefore we have

$$-\pi/2\leqq\theta_1<\theta_1+\theta_3\leqq\theta_1+\theta_2+\theta_3+\theta_4\leqq\theta_2+\theta_4<\theta_4\leqq\pi/2.$$

Remarking that $\theta_j\longrightarrow 0$ as $\xi\longrightarrow-\infty$, we have

$$|\theta_1+\theta_2+\theta_3+\theta_4|\leqq\delta' \ \ (<\pi/2).$$

Hence we have

$$\mathrm{Re}\ \Pi(r_je^{i\theta_j})=(\Pi r_j)\cos(\theta_1+\theta_2+\theta_3+\theta_4)\geqq(\cos\ \delta')(|\xi|^2+\delta^2)^2.$$

The rest of the proof for $\xi\geqq 0$ is shown just in the same way as for $\xi\leqq 0$.

Example 1: $(m, m')=(4, 0)$. Let us consider

$$A=ax_1^4\tilde{\partial}_1^4+bx_1^3\tilde{\partial}_1^3+cx_1^2\tilde{\partial}_1^2+dx_1\tilde{\partial}_1+e+\tilde{\partial}_2^4 \ \ (0<x_1<1),$$

where $a(>0)$, b, c, d, e are rea constants and $\tilde{\partial}_j=\kappa\partial_j$. Then

$$\begin{aligned}A(x, \xi)&=ax_1^4\tilde{\xi}_1^4-ibx_1^3\tilde{\xi}_1^3-cx_1^2\tilde{\xi}_1^2+idx_1\tilde{\xi}_1+e+\tilde{\xi}_2^4\\&=a\tilde{\tilde{\xi}}_1^4-ib\tilde{\tilde{\xi}}_1^3-c\tilde{\tilde{\xi}}_1^2+id\tilde{\tilde{\xi}}_1+e+\tilde{\tilde{\xi}}_2^4=A'(\tilde{\tilde{\xi}}).\end{aligned}$$

Let us denote

$$a\beta^4+b\beta^3+c\beta^2+d\beta+e=a\prod_{j=1}^{4}(\beta+b_j),$$

where we assume

$$(*)\begin{cases}b_1\leqq b_2<b_3\leqq b_4\\ 0\in(b_2,\ b_3)\cap[(b_1+b_3)/2,\ (b_2+b_4)/2].\end{cases}$$

Then (A.2) and (A.3) are satisfied. In fact, let

$$\beta\in(b_2,\ b_3)\cap[(b_1+b_3)/2,\ (b_2+b_4)/2],$$

then we have from Lemma 4.1

$$\mathrm{Re}\ A'(\xi_1+i\beta,\ \xi_2)=\mathrm{Re}\ \{a\prod_j(\xi_1+i\beta-ib_j)+\xi_2^4\}\geqq\delta(\xi_1^2+\delta^2)^2+\xi_2^4\ (\delta>0)$$

for $(\xi_1,\ \xi_2)\in R^2$.

Example 2: $(m, m')=(4,1)$. Let us consider

$$A=ax_1^3\tilde{\partial}_1^4+bx_1^2\tilde{\partial}_1^3+cx_1\tilde{\partial}_1^2+d\tilde{\partial}_1+\tilde{\partial}_2^4\ (0<x_1<1),$$

where $a(>0)$, b, c, d are real constants. Then we have

$$\begin{aligned}x_1A(x, \tilde{\xi})&=ax_1^4\tilde{\xi}_1^4-ibx_1^3\tilde{\xi}_1^3-cx_1^2\tilde{\xi}_1^2+idx_1\tilde{\xi}_1+x_1\tilde{\xi}_2^4\\&=a\tilde{\tilde{\xi}}_1^4-ib\tilde{\tilde{\xi}}_1^3-c\tilde{\tilde{\xi}}_1^2+id\tilde{\tilde{\xi}}_1+\tilde{\tilde{\xi}}_2^4=A'(\tilde{\tilde{\xi}}).\end{aligned}$$

Let us denote

$$a\beta^3+b\beta^2+c\beta+d=a\prod_{j=1}^{3}(\beta+b_j'),$$

where we assume

$$(*)\begin{cases} \text{i)} & b_1'<0<b_2'\leqq b_3' \\ & \text{or} \\ \text{ii)} & b'\leqq b_2'<0<b_3'. \end{cases}$$

Then (A.2) and (A.3) are satisfied, setting

$$(\alpha_1,\ \alpha_2)=\begin{cases} (b_1',\ b_2') & \text{in case of i)}, \\ (b_2',\ b_3') & \text{in case of ii)}. \end{cases}$$

In fact, we remark that $b_0=b_2'/2\in(\alpha_1,\ \alpha_2)$ and we have from lemma 4.1

$$\operatorname{Re} A'(\xi_1+ib_0,\ \xi_2)\geqq\delta(\xi_1^2+\delta^2)^2+\xi_2^4\ (\delta>0)$$

for $(\xi_1,\ \xi_2)\in R^2$.

Example 3: $(m,\ m')=(4,2)$. Let us consider

$$A=ax_1^2\tilde{\partial}_1^4+bx_1\tilde{\partial}_1^3+c\tilde{\partial}_1^2+\tilde{\partial}_2^4\ (0<x_1<1),$$

where $a(>0)$, b, c are real constants. Then we have

$$x_1^2A(x,\ \tilde{\xi})=ax_1^4\tilde{\xi}_1^4-ibx_1^3\tilde{\xi}_1^3-cx_1^2\tilde{\xi}_1^2+x_1^2\tilde{\xi}_1^4=a\tilde{\tilde{\xi}}_1^4-ib\tilde{\tilde{\xi}}_1^3-c\tilde{\tilde{\xi}}_1^2+\tilde{\tilde{\xi}}_2^4=A'(\tilde{\tilde{\xi}}).$$

Let us denote

$$a\beta^2+b\beta+c=a\prod_{j=1}^{2}(\beta+b_j''),$$

where we assume

$$(*)\begin{cases} \text{i)} & 0<b_1''\leqq b_2'' \\ & \text{or} \\ \text{ii)} & b_1''\leqq b_2''<0. \end{cases}$$

Then (A.2) and (A.3) are satisfied, setting

$$(\alpha_1,\ \alpha_2)=\begin{cases} (-\infty,\ b_1'') & \text{in case of i)}, \\ (b_2'',\ +\infty) & \text{in case of ii)}. \end{cases}$$

In fact, setting

$$b_0=\begin{cases} b_1''/2 & \text{in case of i)}, \\ b_2''/2 & \text{in case of ii)}, \end{cases}$$

we have $b_0\in(\alpha_1,\ \alpha_2)$ and

$$\operatorname{Re} A'(\xi_1+ib_0,\ \xi_2)\geqq\delta(\xi_1^2+\delta^2)^2+\xi_2^4\ (\delta>0)$$

for $(\xi_1,\ \xi_2)\in R$.

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