J. Math. Kyoto Univ. (JMKYAZ)
28-2 (1988) 343-372

On representations of finite groups in the space of Siegel modular forms and theta series

Dedicated to Professor Masayoshi Nagata
on his sixtieth birthday

By

Hiroyuki YOSHIDA

Let p be a prime. A representation $\pi_k^{(m)}$ of the symplectic group $Sp(m)$ over the finite field $\boldsymbol{Z}/p\boldsymbol{Z}$ is realized in the space of Siegel modular forms of genus m, of level p, and of weight k. When $m=1$, Hecke discovered that the difference of multiplicities of two specific irreducible representations in $\pi_k^{(m)}$ is equal to the class number of $\boldsymbol{Q}(\sqrt{-p})$, if $p\equiv 3 \bmod 4$, $p>3$, $k\geqq 2$; he also found a beautiful explanation of this fact by the special modular forms, called "eingliedrig forms", which correspond to L-functions with Grössencharacters of $\boldsymbol{Q}(\sqrt{-p})$ by the Mellin transformation. The basic philosophy suggested by this classical work is, in its raw form, that the existence of special (or "lifted") modular forms would produce a difference of multiplicities of certain representations in $\pi_k^{(m)}$.

This paper, in essence, is a document on experiments which are made to examine this picture in the case of Siegel modular forms of genus 2 and of level p. Main results obtained through the course of investigations are Theorems 2.6, 2.8 and 3.2, and examples in § 4.

We shall explain the contents of each section. In § 1, Hecke's work quoted above shall be briefly reviewed. In § 2, we shall first generalize Hecke's notion "eingliedrig" and "zweigliedrig" to Siegel modular forms of genus 2; for certain representations θ_9 and θ_{11} of $Sp(2)$ over $\boldsymbol{Z}/p\boldsymbol{Z}$, we shall introduce the notion of θ_9 and θ_{11}-eingliedrig forms. Then we shall prove the relation between the difference of multiplicities of θ_9 and θ_{11} in $\pi_k^{(2)}$ and the existence of eingliedrig forms (Theorem 2.8). We shall define the Hecke operator $T(p)$ for the level p of modular forms and determine the absolute value of its eigenvalues (Theorem 2.6); the eigenvalues of $T(p)$ are not real in general and the real eigenvalues correspond exactly to eingliedrig forms. This phenomenon is similar to Hecke's "Nebentypus" case, although the corresponding statement for the classical case does not seem to be rigorously proved.

In § 3, we shall prove that Siegel modular forms constructed from a pair of elliptic modular forms in our previous paper [15] are θ_{11}-eingliedrig in the case

Received February 5, 1987

of prime level. In §4, we shall decompose $S_k(\Gamma_0^{(2)}(p))$ (see §0, for the notation) into eigen spaces of Hecke operators for $p=3$, $k=2, 4, 6, 8$, $p=11$, $k=4$, and $p=7$, $k=6$. Though the theta series studied in §3 are "special", we can construct a major part of $S_k(\Gamma_0^{(2)}(p))$ by taking products of them of lower weights; and the theory developed in §2 can be applied efficiently to the explicit decomposition. We have calculated eigenvalues of $T(p)$ in these cases, which may have interesting arithmetical meanings as in the classical case.

In §5, we shall formulate Conjectures about eingliedrig forms suggested by these examples. The θ_{11}-eingliedrig forms would be precisely those constructed in §3 (Conjecture 5.1). For θ_9-eingliedrig forms, however, some complication shall arise. To clarify the points, we shall classify irreducible representations of a certain Hecke algebra in Appendix, and formulate a plausible Conjecture also for θ_9-eingliedrig forms (Conjecture 5.2). Roughly speaking, these Conjectures predict that the global nature of automorphic representations is strongly controlled if their local properties at a place, say p, are of special type.

Notation

Let R be a commutative ring. By $M(n, R)$, we denote the associative algebra of all $n\times n$ matrices with entries in R. For $A\in M(n, R)$, $\sigma(a)$ denotes the trace of A. Put $w=\begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}\in M(2m, R)$ and

$$GSp(m, R)=\{g\in GL(2m, R)\,|\,{}^t gwg=m(g)w \text{ with } m(g)\in R^\times\}.$$

If G is an algebraic group defined over a global field k, G_A denotes the adelization of G, and G_K denotes the group of all K-rational points of G for an extension K of k. For $z\in \boldsymbol{C}$, we set $e(x)=\exp(2\pi\sqrt{-1}\,z)$ and $\bar{z}$ denotes the complex conjugate of z.

Let G be a group (or an associative algebra) and π be a representation of G on a vector space V. Let $v\in V$ and V_1 be the smallest invariant subspace of V which contains v. The representation of G on V_1 is called the representation of G generated by v. If G is a locally compact group, $\hat{G}$ denotes the character group of G, and $\mathcal{S}(G)$ denotes the space of all Schwartz-Bruhat functions on G.

§0. Preliminaries

For a positive integer m, let $\mathfrak{H}_m$ denote the Siegel upper half space of genus m. Set

$$GSp^+(m, \boldsymbol{R})=\{g\in GSp(m, \boldsymbol{R})\,|\,m(g)>0\},$$

$$GSp^+(m, \boldsymbol{Q})=GSp(m, \boldsymbol{Q})\cap GSp^+(m, \boldsymbol{R});$$

$GSp^+(m, \boldsymbol{R})$ acts on $\mathfrak{H}_m$ in the usual manner. Let k be an integer. For a function f on $\mathfrak{H}_m$ and $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in GSp^+(m, \boldsymbol{R})$, we set

$$(f\mid[\gamma]_k)(z)=m(\gamma)^{mk/2}f(\gamma z)\det(cz+d)^{-k}, \qquad z\in\mathfrak{H}_m.$$

Let Γ be a congruence subgroup of $Sp(m, \boldsymbol{Z})$. By $G_k(\Gamma)$ (resp. $S_k(\Gamma)$), we denote the space of all holomorphic modular (resp. cusp) forms of weight k with respect to Γ. For $f\in G_k(\Gamma)$ and $g\in S_k(\Gamma)$, we set

$$(0.1) \qquad (f, g)=\frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H}_m)}\int_{\Gamma\backslash\mathfrak{H}_m} f(z)\overline{g(z)}(\det y)^k dv(z),$$

where $z=x+\sqrt{-1}\,y$ with $x, y\in M(m, \boldsymbol{R})$ and $dv(z)$ denotes the invariant volume element on $\mathfrak{H}_m$ given by $dv(z)=(\det y)^{-m-1}dxdy$. Put $\bar{S}_k=\bigcup_{\Gamma} S_k(\Gamma)$ where Γ extends over all congruence subgroups. Then, for $f, g\in\bar{S}_k$, we can define (f, g) by (0.1) since it does not depend on the choice of Γ; $(,)$ is a positive hermitian inner product on $\bar{S}_k$.

Lemma 0.1. *If $f, g\in\bar{S}_k$ and $\gamma\in GSp^+(m, \boldsymbol{Q})$, then $(f\mid[\gamma]_k, g)=(f, g\mid[\gamma^{-1}]_k)$.*

This Lemma claims that the operator $f\to f\mid[\gamma]_k$ is unitary, which is trivial in adelized definition of cusp forms. The direct proof is also easy, so it is omitted.

Let N be a positive integer. We set

$$\Gamma^{(m)}(N)=\left\{\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in Sp(m, \boldsymbol{Z}) \mid a\equiv d\equiv 1_m,\ b\equiv c\equiv 0_m \bmod N\right\},$$

$$\Gamma_0^{(m)}(N)=\left\{\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in Sp(m, \boldsymbol{Z}) \mid c\equiv 0_m \bmod N\right\}.$$

As $\begin{pmatrix} 0 & 1\\ -N & 0\end{pmatrix}\in GSp^+(m, \boldsymbol{Q})$ normalizes $\Gamma_0^{(m)}(N)$, we can decompose $S_k(\Gamma_0^{(m)}(N))$ (and $G_k(\Gamma_0^{(m)}(N))$):

$$(0.2) \qquad S_k(\Gamma_0^{(m)}(N))=S_k^+(\Gamma_0^{(m)}(N))\oplus S_k^-(\Gamma_0^{(m)}(N)),$$

where $S_k^{\pm}(\Gamma_0^{(m)}(N))=\left\{f\in S_k(\Gamma_0^{(m)}(N)) \mid f\left|\begin{bmatrix} 0 & 1\\ -N & 0\end{bmatrix}_k\right.=\pm f\right\}$. If χ is a Dirichlet character modulo N, we set

$$S_k(\Gamma_0^{(m)}(N), \chi)$$
$$=\left\{f\in S_k(\Gamma^{(m)}(N)) \mid f\mid[\gamma]_k=\chi(\det a)f \quad \text{for any } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0^{(m)}(N)\right\}.$$

Since $\Gamma^{(m)}(N)$ is normal in $Sp(m, \boldsymbol{Z})$, we get a representation $\pi_k^{(m)}(N)$ of $Sp(m, \boldsymbol{Z}/N\boldsymbol{Z})\cong Sp(m, \boldsymbol{Z})/\Gamma^{(m)}(N)$ on $S_k(\Gamma^{(m)}(N))$ defined by

$$[(\pi_k^{(m)}(N))(\gamma \bmod N)]f=f\mid[\gamma]_k,\ f\in S_k(\Gamma^{(m)}(N)),\ \gamma\in Sp(m, \boldsymbol{Z}).$$

Let G be a finite group and B be a subgroup of G, and let $\mathcal{H}(G, B)$ denote the Hecke algebra of G with respect to B over $\boldsymbol{C}$. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) denote the category of the equivalence classes of all finite dimensional representations of $\mathcal{H}(G, B)$ over $\boldsymbol{C}$ (resp. G over $\boldsymbol{C}$ with non-trivial vectors fixed under B).

The following Lemma is well known (cf. N. Iwahori [7], Cor. 1.5, W. Casselman [3], Lemma 3.9).

Lemma 0.2. *The functor, $(\pi, V)\to$ the representation of $\mathcal{H}(G, B)$ on V^B, is the equivalence of categories C_2 and C_1, where $(\pi, V)\in C_2$ and V^B denotes the subspace of all B-fixed vectors of V.*

We shall also use the following Lemma.

Lemma 0.3. *Let G and B be as above. Let π be a representation of G on a finite dimensional vector space V over a field k. Let W be a subspace of V which is invariant under B, and let σ be the representation of B realized on W. If V is generated by W as a G-space, then π is a quotient representation of $\mathrm{Ind}_B^G\sigma$.*

Proof. The map $\varphi: k[G]\otimes_{k[H]}W\to V$ defined by

$$\varphi(\sum_i g_i\otimes w_i)=\sum_i \pi(g_i)w_i,\ g_i\in G,\ w_i\in W,$$

is a homomorphism of $k[G]$-modules. By the assumption, φ is surjective; hence the assertion follows.

§1. A review of a theory of Hecke

Let p be an odd prime. We set

$$G=SL(2, \boldsymbol{F}_p),\quad B=\left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\middle| a\in\boldsymbol{F}_p^\times, b\in\boldsymbol{F}_p\right\},\quad U=\left\{\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\middle| u\in\boldsymbol{F}_p\right\}.$$

Let χ_0 be the quadratic residue character of $\boldsymbol{F}_p^\times$. We define $\chi\in\hat{B}$ by $\chi\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right)=\chi_0(a)$. Let ψ be a non-trivial additive character of $\boldsymbol{F}_p$. For $a\in\boldsymbol{F}_p$, define $\psi_a\in\hat{U}$ by $\psi_a\left(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\right)=\psi(au)$. We see easily that

$$\mathrm{Ind}_B^G\chi\cong\mathfrak{C}_{p+1/2}\oplus\mathfrak{C}'_{p+1/2}, \tag{1.1}$$

where $\mathfrak{C}_{p+1/2}$ and $\mathfrak{C}'_{p+1/2}$ denote irreducible representations of G of degree $(p+1)/2$, which are not equivalent to each other and satisfy

$$\mathfrak{C}_{p+1/2}\mid U\cong\psi_0\oplus\Bigl(\bigoplus_{a\in(\boldsymbol{F}_p^\times)^2}\psi_a\Bigr) \tag{1.2}$$

$$\mathfrak{C}'_{p+1/2}\mid U\cong\psi_0\oplus\Bigl(\bigoplus_{a\in\boldsymbol{F}_p^\times-(\boldsymbol{F}_p^\times)^2}\psi_a\Bigr) \tag{1.3}$$

We normalize ψ by

$$\psi(x \bmod p)=e(x/p),\qquad x\in\boldsymbol{Z}. \tag{1.4}$$

Then the representation $\mathfrak{C}_{p+1/2}$ (resp. $\mathfrak{C}'_{p+1/2}$) given by (1.2) (resp. (1.3)) is called *Rest* (resp. *Nicht Rest*) in Hecke's terminology.

The representation $\pi_k^{(1)}(p)$ of G is realized on $S_k(\Gamma^{(1)}(p))$. For an irreducible representation ρ of G, let $m_k(\rho)$ denote the multiplicity of ρ in $\pi_k^{(1)}(p)$. Hecke obtained the formula

$$(1.5)\qquad m_k(\mathfrak{C}_{p+1/2})-m_k(\mathfrak{C}'_{p+1/2})=\begin{cases} h(\boldsymbol{Q}(\sqrt{-p})) & \text{if } k>2 \text{ is odd, and } p\equiv 3 \bmod 4, \\ 0 & \text{if } k \text{ is even, or } p\equiv 1 \bmod 4, \end{cases}$$

by the Riemann-Roch theorem, where $h(\boldsymbol{Q}(\sqrt{-p}))$ denotes the class number of $\boldsymbol{Q}(\sqrt{-p})$ and $p>3$ is assumed. Hereafter in this section, we shall abbreviate $\Gamma_0^{(1)}(p)$ to $\Gamma_0(p)$. Let $f(\neq 0)\in S_k(\Gamma_0(p), \left(\frac{-}{p}\right))$ and ρ_f be the representation of G generated by f. By Lemma 0.3, ρ_f is a subrepresentation of $\mathrm{Ind}_B^G\chi$. The key points of Hecke's theory are the following Proposition and Theorem.

Proposition 1.1. *Assume $f\in S_k(\Gamma_0(p), \left(\frac{-}{p}\right))$ is a non-zero common eigenfunction of all Hecke operators $T(n)$ for $p\nmid n$. Then $\rho_f\cong\mathfrak{C}_{p+1/2}$ or $\mathfrak{C}_{p+1/2}\oplus\mathfrak{C}'_{p+1/2}$.*

This Proposition states that $\rho_f\cong\mathfrak{C}'_{p+1/2}$ cannot occur. For the proof see Satz 26, [6], p. 842.

Hecke called a normalized eigen cusp form f *eingliedrig* (resp. *zweigliedrig*) if $\rho_f\cong\mathfrak{C}_{p+1/2}$ (resp. $\mathfrak{C}_{p+1/2}\oplus\mathfrak{C}'_{p+1/2}$) ([6], p. 841).

Theorem 1.2. *$m_k(\mathfrak{C}_{p+1/2})-m_k(\mathfrak{C}'_{p+1/2})$ is equal to the number of eingliedrig forms in $S_k(\Gamma_0(p), \left(\frac{-}{p}\right))$.*

For the proof, see [6], p. 841~843.

We can construct eingliedrig forms from a Grössencharacter χ of $\boldsymbol{Q}(\sqrt{-p})$. In fact, if χ is a Grössencharacter of $K=\boldsymbol{Q}(\sqrt{-p})$ of conductor 1 such that

$$\chi((\alpha))=\alpha^{k-1},\quad \alpha\in K^\times,\quad k>1,$$

and if $p\equiv 3 \bmod 4$, then

$$f(z)=\sum_{\mathfrak{a}}\chi(\mathfrak{a})\boldsymbol{e}(N(\mathfrak{a})z)\in S_k(\Gamma_0(p), \left(\frac{-}{p}\right))$$

is an eingliedrig form, where $\mathfrak{a}$ extends over all integral ideals of K and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$ (cf. [6], p. 893, G. Shimura [11], [12]); we obtain $h(\boldsymbol{Q}(\sqrt{-p}))$ eingliedrig forms in this way if $k>1$, $3<p\equiv 3 \bmod 4$. This "explains" the formula (1.5) (and characterizes eingliedrig forms). Hecke further showed that all the eigenvalues of $T(p)$ on $S_k(\Gamma_0(p), \left(\frac{-}{p}\right))$ are of absolute value $p^{(k-1)/2}$. We shall obtain a generalization of this theorem to Siegel modular forms of genus 2.

§ 2. Representations of finite groups in the space of Siegel modular forms of genus 2

We set $G=Sp(2, \boldsymbol{F}_p)$ and define subgroups B, P, P' of G as follows.

$$B=\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad P=\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad P'=\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

B is a Borel subgroup of G; P and P' are all proper parabolic subgroups which contain B. We have

Lemma 2.1. $\mathrm{Ind}_B^G(1_B)\cong 1_G\oplus 2\theta_9\oplus\theta_{11}\oplus\theta_{12}\oplus\theta_{13}$, $\mathrm{Ind}_P^G(1_P)\cong 1_G\oplus\theta_9\oplus\theta_{11}$, $\mathrm{Ind}_{P'}^G(1_{P'})\cong 1_G\oplus\theta_9\oplus\theta_{12}$. *Here 1_H denotes the trivial representation of H for a subgroup H of G; θ_9, θ_{11}, θ_{12}, and θ_{13} denote mutually non-equivalent irreducible representations of G which are labelled according to B. Srinivasan* [13] *when $p\neq 2$.*

The structure of the Hecke ring $\mathcal{H}(G, B)$ is given as follows (cf. [7]). Put

$$(2.1)\qquad w_1=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad w_2=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad w=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$S_1=Bw_1B$, $S_2=Bw_2B$. Then S_1 and S_2 satisfy the relations

$$(2.2)\qquad \begin{cases} S_i^2=(p-1)S_i+p, & i=1, 2, \\ (S_1S_2)^2=(S_2S_1)^2. \end{cases}$$

We have $\mathcal{H}(G, B)\cong \boldsymbol{C}[S_1, S_2]$, the associative algebra generated by S_1 and S_2 over $\boldsymbol{C}$ with relations (2.2). The (one dimensional) representations of $\mathcal{H}(G, B)$ which correspond to 1_G, θ_{11}, θ_{12} and θ_{13} by Lemma 0.2 are given as follows.

(2.3)

	1_G	θ_{11}	θ_{12}	θ_{13}
S_1	p	p	-1	-1
S_2	p	-1	p	-1

Corresponding to θ_9, we obtain the two dimensional representation $S_1\to\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}$, $S_2\to\begin{pmatrix}p & 0 \\ 0 & -1\end{pmatrix}$, where we may set

$$(2.4)\qquad \alpha=\frac{p-1}{p+1}, \quad \beta=\frac{1}{p+1}, \quad \gamma=\frac{2p(p^2+1)}{p+1}, \quad \delta=\frac{p(p-1)}{p+1}.$$

Similarly the representations which occur in $\mathrm{Ind}_P^G(1_P)$ are classified as follows: $\mathcal{H}(G, P)$ is generated by three Hecke operators P, PwP, Pw_2P. The eigenvalues of these Hecke operators which correspond to 1_G, θ_9 and θ_{11} are given in the following table.

(2.5)

	1_G	θ_9	θ_{11}
P	1	1	1
PwP	p^3	$-p$	p
Pw_2P	p^2+p	$p-1$	$-p-1$

Let $\phi: Sp(2, \boldsymbol{Z}) \to G$ be the canonical homomorphism. For a subgroup H of G, set $\Gamma_H = \phi^{-1}(H)$. Thus we have $\Gamma_P = \Gamma_0^{(2)}(p)$. For an irreducible representation ρ of G, let $m_k(\rho)$ denote the multiplicity with which ρ occurs in $\pi_k^{(2)}(p)$. T. Yamazaki, R. Tsushima and K. Hashimoto [5] have obtained the following formula:

(2.6)
$$m_k(\theta_9)+m_k(\theta_{10})-m_k(\theta_{11})-m_k(\theta_{12})$$
$$=(-1)^{k+1}h(\boldsymbol{Q}(\sqrt{-p}))^2 \times \begin{cases} 1/4 & \text{if } p\equiv 1, 5 \\ 4 & \text{if } p \equiv 3 \\ 1 & \text{if } p\equiv 7 \end{cases} \quad \bmod 8,$$

for $k \geqq 4$, $p \geqq 5$. In (2.6), a unipotent cuspidal representation θ_{10} of G appears which is of completely different nature from θ_9, θ_{11} and θ_{12} representation theoretically; thus (2.6) may not be "explained" as in Hecke's theory. However, if we consider $m_k(\theta_9)-m_k(\theta_{11})$, we can develop analogous theory to Hecke's.

For $F \in S_k(\Gamma_P)$, let ρ_F denote the representation of G generated by F. Three Hecke operators Γ_P, $W=\Gamma_P w \Gamma_P$, $W_2=\Gamma_P w_2 \Gamma_P$ act on $S_k(\Gamma_P)$, where w, $w_2 \in Sp(2, \boldsymbol{Z})$ are given by (2.1).

Lemma 2.2. *Let $F \in S_k(\Gamma_P)$, $F \neq 0$. Then*

(1) *ρ_F is a subrepresentation of* $\mathrm{Ind}_P^G(1_P) \cong 1_G \oplus \theta_9 \oplus \theta_{11}$.

(2) $\rho_F \cong 1_G \Leftrightarrow F|W=p^3F,\ F|W_2=(p^2+p)F \Leftrightarrow F|W=p^3F$.

$\rho_F \cong \theta_9 \Leftrightarrow F|W=-pF,\ F|W_2=(p-1)F \Leftrightarrow F|W=-pF$.

$\rho_F \cong \theta_{11} \Leftrightarrow F|W=pF,\ F|W_2=(-p-1)F \Leftrightarrow F|W=pF$.

Proof. (1) follows from Lemma 0.3, and (2) follows from the table (2.5).

Put $H=\begin{pmatrix} 0_2 & 1_2 \\ -p\cdot 1_2 & 0_2 \end{pmatrix}_k$.

Lemma 2.3. *With respect to the Petersson inner product $(\ ,\ )$, H, W and W_2 are self-adjoint operators acting on $S_k(\Gamma_P)$.*

Proof. Let F, $G \in S_k(\Gamma_P)$. By Lemma 0.1, we have

$$(F|H, G)=\left(F\left|\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_k\right., G\right)=\left(F, G\left|\begin{pmatrix} 0 & -p^{-1} \\ 1 & 0 \end{pmatrix}_k\right.\right)$$

$$=\left(F, G\left|\begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}_k\right.\right)=(F, G|H),$$

$$(F|W,\ G)=(\sum_u F\Big|[w]_k\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}_k,\ G)$$

$$=(F\Big|[w]_k,\ \sum_u G\Big|\begin{pmatrix}1 & -u\\ 0 & 1\end{pmatrix}_k)=p^3(F\Big|[w]_k,\ G),$$

where $\sum_u$ denotes the summation over the equivalence classes modulo p of $u\in M(2,\ \boldsymbol{Z})$, ${}^t u=u$. Similarly we get

$$(F,\ G|W)=p^3(F,\ G\Big|[w]_k)=p^3(F\Big|[w]_k,\ G).$$

Hence H and W are self-adjoint. We omit the proof for W_2 which is similar to the above.

Lemma 2.4. *Let V be a finite dimensional vector space over $\boldsymbol{C}$ and $(\ ,\ )$ be a positive hermitian inner product on V. Let A and B be endomorphisms of V which satisfy*

$$A^2=\varepsilon_A\cdot 1_V,\quad B^2=\varepsilon_B t\cdot 1_V,\quad A^*=\varepsilon_A A,\quad B^*=\varepsilon_B B.$$

Here $\varepsilon_A=\pm 1$, $\varepsilon_B=\pm 1$, 1_V is the identical automorphism of V, t is a positive real number and A^* (resp. B^*) denotes the adjoint of A (resp. B) with respect to $(\ ,\ )$. Then

(1) *AB is semi-simple and all eigenvalues of AB have absolute value $t^{1/2}$.*

(2) *V is a direct sum of irreducible invariant subspaces under the actions of A and B.*

(3) *Assume that V is irreducible. Then $\dim V\leqq 2$. Let λ be an eigenvalue of AB. Then, if $\varepsilon_A=\varepsilon_B$,*

$$\dim V=1\Leftrightarrow\lambda\in\boldsymbol{R},\quad \dim V=2\Leftrightarrow\lambda\notin\boldsymbol{R}.$$

Proof. By the assumptions, we see immediately that A and $B/\sqrt{t}$ are unitary; hence $AB/\sqrt{t}$ is unitary and (1) follows. If V_1 is an invariant subspace of V, then the orthogonal complement V_2 of V_1 is an invariant subspace and $V=V_1+V_2$ holds. Hence we get (2).

Now we assume that V is irreducible. Let v_1 $(\neq 0)$ be an eigenvector of AB. Put $ABv_1=\lambda v_1$, $\lambda\in\boldsymbol{C}$, $v_2=Av_1$, $W=\boldsymbol{C}v_1+\boldsymbol{C}v_2$. Then we find $\lambda\neq 0$. Obviously W is invariant under A. Since

$$Bv_1=\varepsilon_A\lambda v_2\quad\text{and}\quad Bv_2=(\varepsilon_A\varepsilon_B t/\lambda)v_1,$$

W is invariant under B. Hence $V=W$ and $\dim V\leqq 2$. To prove the latter half of (3), it suffices to show that $\dim V=1$ if $\lambda\in\boldsymbol{R}$. Put $\varepsilon=\varepsilon_A=\varepsilon_B$ and take $\mu\in\boldsymbol{C}$ so that $\mu^2=\varepsilon$. We find

$$A(v_1+\mu v_2)=\varepsilon\mu(v_1+\mu v_2),\quad B(v_1+\mu v_2)=\mu\lambda(v_1+\mu v_2),$$

if $\lambda=\bar{\lambda}$. This proves (3).

Let $\mathcal{L}$ denote the commutative algebra generated over $\boldsymbol{C}$ by the Hecke operators $T(1, 1, l, l)$ and $T(1, l, l, l^2)$ for all primes $l \neq p$ (cf. Andrianov [1]). The action of $\mathcal{L}$ on $S_k(\Gamma_P)$ is semi-simple; we can take common eigenfunctions of $\mathcal{L}$ as a basis of $S_k(\Gamma_P)$.

Lemma 2.5. *The operators W, W_2 and H commute with operators in $\mathcal{L}$.*

Since the proof is easy (trivial in adelized definition of automorphic forms), we omit it.

Let $S_k^1(\Gamma_P)$ denote the smallest invariant subspace under the actions of H, W and W_2 which contains $S_k(Sp(2, \boldsymbol{Z}))$ (the space of "old forms"). Let $S_k^0(\Gamma_P)$ denote the orthogonal complement of $S_k^1(\Gamma_P)$ in $S_k(\Gamma_P)$ (the space of "new forms"). Of course, $S_k^0(\Gamma_P)=S_k(\Gamma_P)$ if $S_k(Sp(2, \boldsymbol{Z}))=\{0\}$.

Theorem 2.6. *$S_k^0(\Gamma_P)$ is an invariant subspace under the action of $\mathcal{L}$, H and W. Put $T(p)=HW$. Then we have*
(1) *As a basis of $S_k^0(\Gamma_P)$, we can take common eigenfunctions of operators in $\mathcal{L}$ and $T(p)$.*
(2) *All eigenvalues of $T(p)$ have absolute value p.*

Proof. The invariance under $\mathcal{L}$ follows from Lemma 2.5 and the fact that the operators in $\mathcal{L}$ are hermitian. The invariance for H and W follows from Lemma 2.3. Take $f\in S_k^0(\Gamma_P)$. By Lemma 2.2, we have $f\big|W^2-p^2f\in S_k(Sp(2, \boldsymbol{Z}))$. Hence $W^2=p^2$ on $S_k^0(\Gamma_P)$. Obviously we have $H^2=1$. Now, by Lemma 2.1, (1) and Lemma 2.5, the operators in $\mathcal{L}$ and $T(p)$ are mutually commutative and semi-simple. Hence we obtain (1); (2) follows from Lemma 2.3 and Lemma 2.4, (1).

Remark. (1) The eigenvalues of $T(p)$ can be both real and non-real. The examples shall be given in §4.
(2) We can prove Hecke's original theorem by the same method.

Let $F\in S_k^0(\Gamma_P)$ and let $F(z)=\sum_N A(N)e(\sigma(Nz))$ be its Fourier expansion. Then, by a direct computation, we have

$$(F|T(p))(z)=p^{3-k}\sum_{N\equiv 0 \bmod p} A(p^{-1}N)e(\sigma(Nz)). \tag{2.7}$$

Now we are going to look the space $S_k^0(\Gamma_P)$ more closely. The representation of $\mathcal{L}$ on $S_k^0(\Gamma_P)$ decomposes into a direct sum of one dimensional representations. For a one dimensional representation λ of $\mathcal{L}$, let $S_k^0(\Gamma_P)_\lambda$ denote the λ-isotypic component of $S_k^0(\Gamma_P)$. By Lemma 2.5, $S_k^0(\Gamma_P)_\lambda$ is a $\boldsymbol{C}[H, W]$-module. Since H and W are hermitian with respect to the Petersson inner product and $H^2=1$, $W^2=p^2$ on $S_k^0(\Gamma_P)$, we have, by Lemma 2.4,

$$S_k^0(\Gamma_P)_\lambda=\oplus V_i, \tag{2.8}$$

where V_i is an irreducible $\boldsymbol{C}[H, W]$-module such that $\dim_C V_i \leqq 2$. If $\dim_C V_i = 2$, the eigenvalues of $T(p)$ on V_i are not real and mutually complex conjugate by Lemma 2.4, (3). Thus the action of $T(p)$ on V_i is semi-simple. This consideration proves (1) of Theorem 2.6 again and also justifies the following definition.

Definition 2.7. Let $F(\neq 0) \in S_k^0(\Gamma_P)$ be a common eigenfunction of operators in $\mathcal{L}$ and of $T(p)$. We call F *zweigliedlig* (resp. *eingliedrig*) if F generates a two (resp. one) dimensional irreducible $\boldsymbol{C}[H, W]$-module. Assume that F is eingliedrig. If $\rho_F \cong \theta_9$ (resp. θ_{11}), F is called θ_9 (resp. θ_{11})-*eingliedrig*.

Remark. Let F be as above. Then $\rho_F \cong \theta_9 \oplus \theta_{11}$ or θ_9 or θ_{11}; F is eingliedrig if and only $\rho_F \cong \theta_9$ or θ_{11}; F is zweigliedrig if and only if $\rho_F \cong \theta_9 \oplus \theta_{11}$. Put $F|T(p) = \mu F$, $\mu \in \boldsymbol{C}$, $|\mu| = p$. Then F is eingliedrig (resp. zweigliedrig) if and only if $\mu = \pm p$ (resp. $\mu \notin \boldsymbol{R}$).

Theorem 2.8. *Assume* $S_k(Sp(2, \boldsymbol{Z})) = \{0\}$. *Then* $m_k(\theta_9) - m_k(\theta_{11}) = \dim_C \langle \theta_9$-*eingliedrig forms*$\rangle - \dim_C \langle \theta_{11}$-*eingliedrig forms*$\rangle$.

Proof. By the assumption and Lemma 2.2, we have $W^2 = p^2$ on $S_k(\Gamma_P)$ and $m_k(\theta_9)$ (resp. $m_k(\theta_{11})$) is the multiplicity of the eigenvalue $-p$ (resp. p) of W. We consider the decomposition (2.8). If $\dim V_i = 2$, the set of eigenvalues of W on V_i is $\{p, -p\}$, since otherwise W acts as a scalar on V_i and V_i cannot be irreducible. Thus a two dimensional component V_i gives no contribution to $m_k(\theta_9) - m_k(\theta_{11})$. Assume $\dim V_i = 1$ and V_i is spanned by $F_i \in S_k(\Gamma_P)$. Then we have $F_i|W = -pF_i$ (resp. pF_i) if and only if F_i is a θ_9 (resp. θ_{11})-eingliedrig form. We see, by the Jordan-Hörder theorem, that the number of such V_i's does not depend on the particular choice of the decomposition (2.8). This completes the proof.

Remark. We can obtain analogous results for $S_k(\Gamma_{P'})$ and $m_k(\theta_9) - m_k(\theta_{12})$.

We are going to study arithmetic properties of Fourier coefficients of eingliedrig forms. Let U be the subgroup of G defined by

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \, b \in M(2, \boldsymbol{F}_p),\ {}^t b = b \right\}.$$

Hereafter in this section, we assume $p \neq 2$. For a symmetric matrix $S \in M(2, \boldsymbol{F}_p)$, define $\eta_S \in \hat{U}$ by $\eta_S\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \psi(\sigma(Sb))$, where ψ is given by (1.4); we see easily that all characters of U are of this form. For symmetric matrices $S_1, S_2 \in M(2, \boldsymbol{F}_p)$, let us write $S_1 \sim S_2$ if there exists $T \in GL(2, \boldsymbol{F}_p)$ such that ${}^t T S_1 T = S_2$. We have

$$\theta_9 | U \cong \Big(\bigoplus_{S \sim S_1} \eta_S\Big) \oplus \Big(\bigoplus_{S \sim S_3} \eta_S\Big) \oplus \Big(\bigoplus_{S \sim S_4} \eta_S\Big) \oplus (p+1) 1_G, \tag{2.9}$$

(2.10) $$\theta_{11}|U\cong\Big(\bigoplus_{S\sim S_2}\eta_S\Big)\oplus\Big(\bigoplus_{S\sim S_3}\eta_S\Big)\oplus\Big(\bigoplus_{S\sim S_4}\eta_S\Big)\oplus 1_G,$$

where $S_1=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $S_2=\begin{pmatrix}1&0\\0&-\omega\end{pmatrix}$, $S_3=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $S_4=\begin{pmatrix}\omega&0\\0&0\end{pmatrix}$ with $\omega\in\boldsymbol{F}_p^\times-(\boldsymbol{F}_p^\times)^2$. These formulas are similar to (1.2) and (1.3). We need the following

Lemma 2.9. *Let v_1 (resp. v_2) be a non-zero P-fixed vector in a representation space of θ_{11} (resp. θ_9). Then*

$$\sum_{u\in M(2,\boldsymbol{F}_p),\,{}^tu=u}\theta_{11}\Big(\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\Big)\eta(u)^{-1}v_1\neq 0$$

if $\eta=\eta_S$ with $-\det S\in\boldsymbol{F}_p^\times-(\boldsymbol{F}_p^\times)^2$,

$$\sum_{u\in M(2,\boldsymbol{F}_p),\,{}^tu=u}\theta_9\Big(\begin{pmatrix}1&u\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\Big)\eta(u)^{-1}v_2\neq 0$$

if $\eta=\eta_S$ with $-\det S\in(\boldsymbol{F}_p^\times)^2$.

Since the proof is easy, it is omitted.

Proposition 2.10. *Let $F\in S_k^0(\Gamma_P)$, $F\neq 0$ and $F(z)=\sum_N A(N)e(\sigma(Nz))$ be the Fourier expansion of F, where N extends over positive definite half integral symmetric matrices. We assume that F is an eigenfunction of H and of operators in $\mathcal{L}$ and $p\neq 2$. Then*

(1) *F is θ_9-eingliedrig if and only if $A(N)=0$ whenever $\Big(\frac{-\det 2N}{p}\Big)=-1$.*

(2) *F is θ_{11}-eingliedrig if and only if $A(N)=0$ whenever $\Big(\frac{-\det 2N}{p}\Big)=1$.*

Proof. Since $F|H=cF$ with $c^2=1$, we get

(2.11) $$(F\,|\,[w]_k)(z)=\Big(F\,\Big|\begin{pmatrix}0&1\\-p&0\end{pmatrix}_k\begin{pmatrix}p^{-1}&0\\0&1\end{pmatrix}_k\Big)(z)=cp^{-k}\sum_N A(N)e(\sigma(Nz/p)).$$

For $\eta=\eta_S$, put

$$F_\eta=\sum_{u\in M(2,\boldsymbol{F}_p),\,{}^tu=u}F\,\Big|\,[w]_k\begin{pmatrix}1&\tilde{u}\\0&1\end{pmatrix}_k\eta(u)^{-1},$$

where $\tilde{u}$ denotes a symmetric matrix in $M(2,\boldsymbol{Z})$ such that $\tilde{u}\bmod p=u$. Then we have

(2.12) $$F_\eta\,\Big|\begin{pmatrix}1&u\\0&1\end{pmatrix}_k=\eta(u\bmod p)F_\eta,\quad u\in M(2,\boldsymbol{Z}),\ {}^tu=u.$$

By a direct computation using (2.11), we obtain

(2.13) $$F_\eta(z)=cp^{3-k}\sum_{N\bmod p=S}A(N)e(\sigma(Nz/p)).$$

Now assume F is θ_{11}-eingliedrig, i.e. $\rho_F\cong\theta_{11}$. By (2.10) and (2.12), we see that $F_\eta=0$ if $-\det S\in(\boldsymbol{F}_p^\times)^2$. Therefore, by (2.13), we obtain $A(N)=0$ if

$\left(\frac{-\det 2N}{p}\right)=1$. Conversely assume $A(N)=0$ if $\left(\frac{-\det 2N}{p}\right)=1$. Assume $\rho_F\cong\theta_9\oplus\theta_{11}$ or θ_9. By Lemma 2.9, we get $F_\eta\neq0$ for some (actually any) $\eta=\eta_S$ with $-\det S\in(F_p^\times)^2$. Hence we get $A(N)\neq0$ for some N such that $\left(\frac{-\det 2N}{p}\right)$ $=1$ by (2.13). This is a contradiction. Therefore we must have $\rho_F\cong\theta_{11}$. This proves (2). The assertion (1) can be proved in a similar way.

§ 3. A construction of θ_{11}-eingliedrig forms

In this section, we shall show that some of Siegel's modular forms constructed in our previous paper [15] are θ_{11}-eingliedrig. First we shall recall this construction briefly. Let p be a prime and D be the quaternion algebra over $\boldsymbol{Q}$ which ramify only at p and at the archimedean prime ∞ (called (p, ∞)-quaternion algebra); $*$ denotes the main involution of D. Let $D_A^\times$ denote the adelization of $D^\times$. We take a maximal order R of D and set $D_l=D\otimes_{\boldsymbol{Q}}\boldsymbol{Q}_l$, $R_l=R\otimes_{\boldsymbol{Z}}\boldsymbol{Z}_l$ for a prime l, $K=\prod_l R_l^\times\times\boldsymbol{H}^\times$. Here $\boldsymbol{H}=D\otimes_{\boldsymbol{Q}}\boldsymbol{R}$ is (isomorphic to) the Hamilton quaternion algebra. For $0\leqq n\in\boldsymbol{Z}$, let σ'_{2n} denote the symmetric tensor representation of degree n of $GL(2, \boldsymbol{C})$ on $V\cong\boldsymbol{C}^{2n+1}$. Fixing a splitting $\boldsymbol{H}\otimes_{\boldsymbol{R}}\boldsymbol{C}\cong M(2, \boldsymbol{C})$, we consider σ'_{2n} as a representation of $\boldsymbol{H}^\times$ and put $\sigma_{2n}(g)=\sigma'_{2n}(g)N(g)^{-n}$. We set $S(R, 2n)=\{\varphi\mid \varphi$ is a V-valued function of $D_A^\times$ which satisfies $\varphi(\gamma hk)=\varphi(h)\sigma_{2n}(k_\infty)$ for any $\gamma\in D_{\boldsymbol{Q}}^\times$, $h\in D_A^\times$, $k\in K\}$.

For any prime $l\neq p$, we can define Hecke operator $T'(l)$ which acts on $S(R, 2n)$ (cf. [15], p. 210). If φ is a common eigenfunction of $T'(l)$ and $T'(l)\varphi=\lambda_l\varphi$, we put

$$L(s, \varphi)=\prod_{l\neq p}(1-\lambda_l l^{-s}+l^{1-2s})^{-1}.$$

Let ϖ_p be a prime element of D_p. As ϖ_p normalizes $R_p^\times$, we have

$$S(R, 2n)=S^+(R, 2n)\oplus S^-(R, 2n),$$

where $S^\pm(R, 2n)=\{\varphi\in S(R, 2n)\mid\varphi(h\varpi_p)=\pm\varphi(h)$ for any $h\in D_A^\times\}$. We consider D as a quadratic space over $\boldsymbol{Q}$ by the reduced norm N. Set

$$X=D\oplus D,\ G=Sp(2),\ H=\{(a, b)\in D^\times\times D^\times \mid N(a)=N(b)=1\};$$

G and H are considered as algebraic groups over $\boldsymbol{Q}$. We let $D^\times\times D^\times$ act on X on the right by

$$\rho(a, b)(x_1, x_2)=(a^{-1}x_1b,\ a^{-1}x_2b),\ x_1, x_2\in D,\ a, b\in D^\times.$$

Take an additive character ψ of $\boldsymbol{Q}_A$ so that $\psi=\prod_v\psi_v$,

$$\psi_\infty(x)=\boldsymbol{e}(x),\ x\in\boldsymbol{R}=\boldsymbol{Q}_\infty,\ \psi_l(x)=\boldsymbol{e}(-Fr(x)),\ x\in\boldsymbol{Q}_l,$$

where $Fr(x)$ denotes the fractional part of x. Then ψ is trivial on $\boldsymbol{Q}$. Let π denote the Weil representation of G_A realized on $\mathcal{S}(X_A)$ associated with D and ψ. Take $\varphi_1\in S(R, 0)$, $\varphi_2\in S(R, 2n)$ and let V_1 be the representation space of

$\sigma_0 \otimes \sigma_{2n}$. Let $\langle , \rangle$ be a hermitian inner product on V_1 such that $\sigma_0 \otimes \sigma_{2n}$ is unitary with respect to $\langle , \rangle$. For $f \in \mathcal{S}(X_A) \otimes V_1$, set

$$\Phi_f(g)=\int_{H_Q \backslash H_A} \langle \sum_{x \in X_Q} (\pi(g)f)(\rho(h)x),\ \varphi(h) \rangle dh\,, \qquad g \in G_A\,, \tag{3.1}$$

where $\varphi=\varphi_1 \otimes \varphi_2$ is the V_1-valued function on $D_A^\times \times D_A^\times$.

We choose f more explicitly in the following form: $f=\prod_l f_l \times f_\infty$; $f_l \in \mathcal{S}(X_l)$ is the characteristic function of $R_l \oplus R_l$; $f_\infty \in \mathcal{S}(X_\infty) \otimes V_1$ is of the form $f_\infty(x_1, x_2) = \eta(x_1, x_2)\exp(-2\pi(N(x_1)+N(x_2)))$, $(x_1, x_2) \in X_\infty = \boldsymbol{H} \oplus \boldsymbol{H}$. Here η is a V_1-valued polynomial function on X_∞ such that $\eta(x_1, x_2)=P(x_1^* x_2)$ for $x_1, x_2 \in \boldsymbol{H}$ with a V_1-valued homogeneous harmonic polynomial P of degree n of three variables depending on the pure quaternion part of $x_1^* x_2$. We assume

$$\eta(\rho(a, b)x)=(\sigma_0(a) \otimes \sigma_{2n}(b))\eta(x) \qquad \text{for } a, b \in \boldsymbol{H}^1, \tag{3.2}$$

where $\boldsymbol{H}^1=\{a \in \boldsymbol{H} \mid N(a)=1\}$.

For $g \in G_\infty$, define $\tilde{g} \in G_A$ by $\tilde{g}_\infty = g$, $\tilde{g}_l=1$ (l is a finite prime). Let K_∞ be the stabilizer of $\sqrt{-1}\,1_2 \in \mathfrak{H}_2$. In general, if Φ_0 is a function on G_A which satisfies

$$\Phi_0(gk_\infty)=\Phi_0(g)\det(a+b\sqrt{-1})^k, \quad k_\infty=\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K_\infty\,, \tag{3.3}$$

then a function F_0 on $\mathfrak{H}_2$ can be defined by

$$F_0(g(\sqrt{-1}\ 1_2))=\Phi_0(\tilde{g})\det(c\sqrt{-1}+d)^k \tag{3.4}$$

for $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty$. We set $F_0=\Psi(\Phi_0)$.

Put $F=\Psi(\Phi_f)$. By the choice of f as above, we have $F \in G_{n+2}(\Gamma^{(2)}(p))$; and $F \in S_{n+2}(\Gamma_0^{(2)}(p))$ if $n>0$. We can easily prove the following

Lemma 3.1. *Let $\gamma \in Sp(2, \boldsymbol{Z})$. Then we have $F|[\gamma]_k=\Psi(\Phi_{f'})$, where $f' \in \mathcal{S}(V_A) \otimes V_1$ is given by $f'=\prod_{v \neq p} f_v \times \pi_p(\gamma^{-1})f_p$. Here π_p denotes the local Weil representation of G_{Q_p} on $\mathcal{S}(X_{Q_p})$.*

The explicit form of F is given as follows. Let $D_A^\times = \bigcup_{i=1}^h D_Q^\times y_i K$ be a double coset decomposition with

$$N(y_i)=1 \in \boldsymbol{Q}_A^\times\,, \quad (y_i)_\infty=1\,. \tag{3.5}$$

Define a lattice on D by $L_{ij}=D_Q \cap y_i \prod R_l y_j^{-1}$. Then L_{ii} is a maximal order of D; put $e_i=|L_{ii}^\times|$. Up to a constant multiple, we have

$$F(z)=\sum_{i=1}^h \sum_{j=1}^h \langle \sum_{x=(x_1, x_2) \in L_{ij} \oplus L_{ij}} P(x_1^* x_2)\boldsymbol{e}(\sigma(Q(x)z)), \tag{3.6}$$
$$\varphi_1(y_i) \otimes \varphi_2(y_j)/e_i e_j \rangle, \quad z \in \mathfrak{H}_2\,,$$

where $Q(x)=\begin{pmatrix} N(x_1) & Tr(x_1 x_2^*)/2 \\ Tr(x_1 x_2^*)/2 & N(x_2) \end{pmatrix}$ and Tr denotes the reduced trace. We denote this F by $F(\varphi_1, \varphi_2)$. If $\varphi_1 \in S^{\pm}(R, 0)$, $\varphi_2 \in S^{\mp}(R, 2n)$, then we have $F(\varphi_1, \varphi_2)=0$. Now we shall prove that $F(\varphi_1, \varphi_2)$ are θ_{11}-eingliedrig forms.

Theorem 3.2. *Let $\varphi_1 \in S^{\pm}(R, 0)$, $\varphi_2 \in S^{\pm}(R, 2n)$ and put $F=F(\varphi_1, \varphi_2)$, $k=n+2$. We set $\varepsilon=1$ (resp. -1) if $\varphi_1 \in S^{+}(R, 0)$ (resp. $S^{-}(R, 0)$. Then we have*

(1) $F|W=pF$.
(2) $F|H=\varepsilon F$.
(3) *If* $F\neq 0$, *then* $\rho_F \cong \theta_{11}$.
(4) $F \in S_k^0(\Gamma_P)$ *if* F *is a cusp form.*

Assume φ_1 and φ_2 are common eigenfunctions of Hecke operators $T'(l)$ for all primes $l\neq p$. Let $L(s, F)$ denote the Euler product attached to F as in [1], [15], §6. *Then F is a common eigenfunction of all operators in $\mathcal{L}$ and we have*

(5) $L(s, F)=L(s-n, \varphi_1)L(s-n, \varphi_2)$,

up to Euler p-factors.

Proof. First we shall prove (1). Let $w=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ as in (2.1). Since

$$F\Big|W=\sum_{u\in M(2,\mathbf{Z}),\, u={}^t u,\, u \bmod p} F\Big|[w]_k\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}_k,$$

it suffices to show

$$\sum_{u\in M(2,\mathbf{Z}),\, u={}^t u,\, u \bmod p} \pi_p(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} w) f_p = p f_p, \tag{3.7}$$

by Lemma 3.1. We get, by the definition of the Weil representation π_p (cf. [14], p. 403 and Remark 1), $\pi_p(w)f_p=f_p^*$ where f_p^* denotes the Fourier transformation of f_p with respect to the self dual measure. Set $\check{R}_p=\varpi_p^{-1}R_p=R_p\varpi_p^{-1}$ (the dual lattice of R_p). We have $f_p^*=p^{-2}\times$the characteristic function of $\check{R}_p\oplus\check{R}_p$, since $\mathrm{vol}(R_p\oplus R_p)=[\check{R}_p\oplus\check{R}_p : R_p\oplus R_p]^{-1/2}=p^{-2}$. As

$$\pi_p(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}) f_p^*(x)=\psi_p(\sigma(u\,{}^t xSx))f_p^*(x)$$

and $u\to\psi_p(\sigma(u\,{}^t xSx))$ defines the trivial character of $\{u\in M(2, \mathbf{Z}_p) \mid {}^t u=u\}$ if and only if $x\in R_p\oplus R_p$ for $x\in \check{R}_p\oplus\check{R}_p$, we obtain (3.7). This prove (1).

Now we shall prove (2). We have $F|[w]_k=\Psi(\Phi_{f'})$ with $f'=\prod_{v\neq p} f_v\times f_p^*$. From this, we get by a little computation that

$$(F\Big|[w]_k)(z)=p^{-2}\sum_{i=1}^{h}\sum_{j=1}^{h}\langle \sum_{x=(x_1,x_2)\in L'_{ij}\oplus L'_{ij}} P(x_1^* x_2)e(\sigma(Q(x)z)), \tag{3.8}$$

$$\varphi_1(y_i)\otimes\varphi_2(y_j)/e_i e_j\rangle,$$

where $L'_{ij}=D_Q\cap y_i(\prod_{l\neq p} R_l\times\check{R}_p)y_j^{-1}$. For each i, $1\leqq i\leqq h$, let $y_i\varpi_p^{-1}=\gamma_i y_s k_s$ with

$1 \leqq s \leqq h$, $\gamma_i \in D_{\mathbf{Q}}^{\times}$, $k_s \in K$. Then we find $N(\gamma_i)=p^{-1}$ and $L'_{ij}=\gamma_i L_{sj}$; so the map $L_{sj} \ni x \to \gamma_i x \in L'_{ij}$ is a bijection. As $(F|H)(z)=p^{-k}(F|[w]_k)(pz)$, we get

$$(F|H)(z)=p^{k-2}\sum_{s=1}^{h}\sum_{j=1}^{h}\langle \sum_{x=(x_1,x_2)\in L_{sj}\oplus L_{sj}} P(x_1^*\gamma_1^*\gamma_1 x_2)e(\sigma(Q(x)z)),$$
$$\varphi_1(y_s\varpi_p)\otimes\varphi_2(y_j)/e_ie_j\rangle .$$

We have $P(x_1^*\gamma_1^*\gamma_1 x_2)=P(p^{-1}x_1^* x_2)=p^{-n}P(x_1^* x_2)$. Hence we get $F|H=\varepsilon F$, which is (2).

By Lemma 2.2 and (1), we have $\rho_F \cong \theta_{11}$ if $F \neq 0$; hence we get (3). To prove (4), it suffices to show $(f, F)=0$ for any $f \in S_k(Sp(2, \mathbf{Z}))$. But this is clear since $f|W=p^3 f$, $F|W=pF$ and W is hermitian. The assertion (5), except for Euler 2-factors, is proved in [15]; the results of [16] show that this holds also for Euler 2-factors. This completes the proof of Theorem 3.2.

The restrictions (3.2) and (3.5) made on the choices of y_i and P are sometimes inconvenient for numerical computations. Drop the assumption (3.5) and define the lattice L_{ij} by the same formula; also assume simply that P is a (scalar valued) homogeneous harmonic polynomial of degree n on $\mathbf{H}$ of three variables depending on the pure quaternion part. Put

$$\theta_{ij}^{P}=\sum_{(x_1,x_2)\in L_{ij}\oplus L_{ij}} P(x_1^* x_2)e(\sigma(Q(x)z)),$$

where $Q(x)=\dfrac{1}{N(L_{ij})}\begin{pmatrix} N(x_1) & Tr(x_1x_2^*)/2 \\ Tr(x_1x_2^*)/2 & N(x_2)\end{pmatrix}$ with the norm $N(L_{ij})$ of the lattice L_{ij}. Then we have

Proposition 3.3. *Let Θ be the space spanned by θ_{ij}^{P} for $1 \leqq i, j \leqq h$ and all P as above. Then $\Theta \subset G_{n+2}(\Gamma_P)$ ($S_{n+2}(\Gamma_P)$ if $n>0$), and is invariant under $\mathcal{L}$. As a basis of Θ, we can take functions of the form $F(\varphi_1, \varphi_2)$, where φ_i ($i=1, 2$) may be assumed to be a common eigenfunction of all $T'(l)$, $l \neq p$.*

This is an easy consequence of Theorem 3.2, (5); we omit the proof.

Remark. Let $f_1 \in G_2(\Gamma_0^{(1)}(p))$, $f_2 \in G_{2n+2}(\Gamma_0^{(1)}(p))$ be the modular forms which correspond to φ_1 and φ_2 respectively ([15], Prop. 7.1). We have $L(s, f_1)= L(s, \varphi_1)$, $L(s, f_2)=L(s-n, \varphi_2)$,

$$L(s, F)=L(s-n, f_1)L(s, f_2) \tag{3.9}$$

except for Euler p-factors. We also get

$$f_1\left|\begin{pmatrix} 0 & 1 \\ -p & 0\end{pmatrix}_2\right.=-\varepsilon f_1, \quad f_2\left|\begin{pmatrix} 0 & 1 \\ -p & 0\end{pmatrix}_{2n+2}\right.=-\varepsilon f_2.$$

§ 4. Numerical examples

Let $\boldsymbol{H}$ denote the Hamilton quaternion algebra and $1, i, j, k$ be the standard quaternion basis. We shall use the following harmonic polynomials P_n of degree n:

$$P_0(x)=1, \quad P_2(x)=c^2-d^2, \quad P_4(x)=c^4-6c^2d^2+d^4,$$
$$P_6^{(1)}(x)=c^6-15c^4d^2+15c^2d^4-d^6, \quad P_6^{(2)}(x)=b^6-15b^4c^2+15b^2c^4-c^6,$$

where $x=a+bi+cj+dk\in\boldsymbol{H}$. The data concerning the dimension of $S_k(\Gamma_0^{(2)}(p))$ are taken from K. Hashimoto [4]. Some of the formulas in this section are conjectural; we shall mark them by the subscript c. The equality of Euler products means the identity up to the Euler p-factor, where p is the level of modular forms.

(I) The case of level 3

The $(3, \infty)$-quaternion algebra D is given explicitly by $D=\boldsymbol{Q}+\boldsymbol{Q}i'+\boldsymbol{Q}j'+\boldsymbol{Q}k'$ with $i'^2=-1$, $j'^2=-3$, $i'j'=-j'i'=k'$. A maximal order R of D is given by $R=\boldsymbol{Z}\omega_1+\boldsymbol{Z}\omega_2+\boldsymbol{Z}\omega_3+\boldsymbol{Z}\omega_4$, where $\omega_1=(1+j')/2$, $\omega_2=(i'+k')/2$, $\omega_3=j'$, $\omega_4=k'$. We have $h=1$; so $S(R, 0)$ consists of constant functions. Put

$$x={}^t(x_1, x_2, x_3, x_4), \quad y={}^t(y_1, y_2, y_3, y_4)\in\boldsymbol{Z}^4,$$
$$\tilde{x}=x_1\omega_1+x_2\omega_2+x_3\omega_3+x_4\omega_4, \quad \tilde{y}=y_1\omega_1+y_2\omega_2+y_3\omega_3+y_4\omega_4.$$

We define a symmetric matrix S by $N(\tilde{x})={}^txSx$; we have

$$S=\begin{pmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 3/2 \\ 3/2 & 0 & 3 & 0 \\ 0 & 3/2 & 0 & 3 \end{pmatrix}. \quad \text{Set } Q(x, y)=\begin{pmatrix} {}^txSx & {}^txSy \\ {}^txSy & {}^txSy \end{pmatrix}.$$

Put $X=\tilde{x}^*\tilde{y}$ and let

$$b=2\times\text{the coefficient of } i' \text{ in } X,$$
$$c=2\times\text{the coefficient of } j' \text{ in } X,$$
$$d=2\times\text{the coefficient of } k' \text{ in } X.$$

The explicit forms are

$$b=(x_2y_1-y_2x_1)+3(x_2y_3-y_2x_3+x_4y_1-y_4x_1)+6(x_4y_3-y_4x_3),$$
$$c=(x_1y_3-y_1x_3)+(x_2y_4-y_2x_4),$$
$$d=(x_1y_4-y_1x_4)+(y_2x_3-x_2y_3)+(x_1y_2-y_1x_2).$$

We define five theta series by

$$\theta_2(z)=\sum_{(x, y)}\boldsymbol{e}(\sigma(Q(x, y)z)),$$
$$\theta_4(z)=\sum_{(x, y)}(c^2-d^2)\boldsymbol{e}(\sigma(Q(x, y)z)),$$
$$\theta_6(z)=\sum_{(x, y)}(c^4-6c^2d^2+d^4)\boldsymbol{e}(\sigma(Q(x, y)z)),$$

$$\theta_8^{(1)}(z)=\sum_{(x,y)}(c^6-15c^4d^2+15c^2d^4-d^6)e(\sigma(Q(x,y)z)),$$

$$\theta_8^{(2)}(z)=\sum_{(x,y)}(b^6-45b^4c^2+135b^2c^4-27c^6)e(\sigma(Q(x,y)z)),$$

where $z\in\mathfrak{H}_2$ and (x, y) extends over $\boldsymbol{Z}^4\oplus\boldsymbol{Z}^4$. By the results stated in § 3, we see $\theta_2\in G_2(\Gamma_0^{(2)}(3))$, $\theta_4\in S_4(\Gamma_0^{(2)}(3))$, $\theta_6\in S_6(\Gamma_0^{(2)}(3))$, $\theta_8^{(1)}$, $\theta_8^{(2)}\in S_8(\Gamma_0^{(2)}(3))$. We have

$$L(s,\theta_2)=\zeta(s)^2\zeta(s-1)^2, \tag{4.1}$$

$$L(s,\theta_4)=\zeta(s-2)\zeta(s-3)L(s,f_6), \tag{4.2}$$

$$L(s,\theta_6)=\zeta(s-4)\zeta(s-5)L(s,f_{10}^-), \tag{4.3}$$

where $f_6(\neq 0)\in S_6(\Gamma_0^{(1)}(3))=S_6^-(\Gamma_0^{(1)}(3))$ (one dimensional), $f_{10}^-(\neq 0)\in S_{10}^-(\Gamma_0^{(1)}(3))$ (one dimensional); (4.1) and (4.2) are given in [15], § 8; (4.3) follows from Proposition 3.3 since $\dim S^+(R, 8)=\dim S_{10}^-(\Gamma_0^{(1)}(3))=1$ and $\theta_6\neq 0$.

First we are going to decompose $S_6(\Gamma_0^{(2)}(3))$ into eigen spaces. We have $\dim S_6(\Gamma_0^{(2)}(3))=2$; θ_6 is a θ_{11}-eingliedrig form which satisfies $\theta_6|H=\theta_6$, $\theta_6|W=3\theta_6$. Take a non-zero cusp form ψ_6 from the orthogonal complement of $\langle\theta_6\rangle_{\boldsymbol{C}}$. Since the operators in $\mathcal{L}$, H and W are hermitian, ψ_6 must be an eingliedrig form. We find that the N-th Fourier coefficient of $\theta_2\times\theta_4\in S_6(\Gamma_0^{(2)}(3))$ for $N=\begin{pmatrix}1&0\\0&2\end{pmatrix}$ is 576. Therefore ψ_6 must be θ_9-eingliedrig by Proposition 2.10; we can take ψ_6 of the form $\psi_6=\theta_2\times\theta_4-c\theta_6$. The Fourier coefficients of $\theta_2\times\theta_4$ and θ_6 for $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ are both 48. Hence we get $c=1$ by Proposition 2.10. We have shown

$$\psi_6=\theta_2\times\theta_4-\theta_6,\quad \psi_6|H=\psi_6,\quad \psi_6|W=-3\psi_6.$$

Let $\psi_6(z)=\sum_N A(N)e(\sigma(Nz))$, $z\in\mathfrak{H}_2$, $f_{10}^+(z)=\sum_{n=1}^{\infty}a(n)e(nz)$, $a(1)=1$, $z\in\mathfrak{H}_1$ be Fourier expansions, where $f_{10}^+\in S_{10}^+(\Gamma_0^{(1)}(3))$ (one dimensional). In tables (I) and (II), we can observe the relation

$$\zeta(s-4)\zeta(s-5)L(s,\psi_6)=\zeta_{\boldsymbol{Q}(\sqrt{-2})}(s-4)\sum_{n=1}^{\infty}\left(A\left(n\begin{pmatrix}1&0\\0&2\end{pmatrix}\right)/48\right)n^{-s},$$

which suggests

$$L(s,\psi_6)=\zeta(s-4)\zeta(s-5)L(s,f_{10}^+). \tag{4.4$_c$}$$

It is almost certain that $(4.4)_c$ can be proven by the method of H. Maass [10] and D. Zagier. The eigenvalues of $T(3)$ on $S_6(\Gamma_0^{(2)}(3))$ are 3 and -3. Incidentally we get $S_6(\Gamma_0^{(2)}(3))=S_6^+(\Gamma_0^{(2)}(3))$.

Now we are going to decompose $S_8(\Gamma_0^{(2)}(3))$ into eigenspaces. We have $\dim S_8(\Gamma_0^{(2)}(3))=5$. Put

$$\theta_8^{(1)}(z)=\sum_N B(N)e(\sigma(Nz)),\quad \theta_8^{(2)}(z)=\sum_N C(N)e(\sigma(Nz)),$$

be the Fourier expansions. Put $V_0=\langle\theta_8^{(1)},\theta_8^{(2)}\rangle_{\boldsymbol{C}}$. By the table (II), we see that $\dim V_0=2$. Therefore, by Proposition 3.3, V_0 is stable under $\mathcal{L}$. Set

$$(\theta_8^{(1)}\Big|T(2))(z)=\sum_N B_2(N)e(\sigma(Nz)),\quad (\theta_8^{(2)}\Big|T(2))(z)=\sum_N C_2(N)e(\sigma(Nz)).$$

By tables (II) and (III), we find

$$\theta_8^{(1)}\Big|T(2)=-1224\theta_8^{(1)}-50\theta_8^{(2)},\tag{4.5}$$

$$\theta_8^{(2)}\Big|T(2)=38232\theta_8^{(1)}+1554\theta_8^{(2)}.\tag{4.6}$$

Put

$$\psi_8^{(1)}=(2535-33\sqrt{d})\theta_8^{(1)}+(87-\sqrt{d})\theta_8^{(2)},$$
$$\psi_8^{(2)}=(2535+33\sqrt{d})\theta_8^{(1)}+(87+\sqrt{d})\theta_8^{(2)},$$

where $d=11\times 179$. Then we have

$$\psi_8^{(1)}\Big|T(2)=(165+3\sqrt{d})\psi_8^{(1)},\quad \psi_8^{(2)}\Big|T(2)=(165-3\sqrt{d})\psi_8^{(2)}.$$

Let $f_{14}^-(z)=\sum_{n=1}^{\infty}b(n)e(nz)\in S_{14}^-(\Gamma_0^{(1)}(3))$ be normalized so that $b(1)=1$, $b(2)=-27+3\sqrt{d}$. By Proposition 3.3, we have

$$L(s,\psi_8^{(1)})=\zeta(s-6)\zeta(s-7)L(s,f_{14}^-).\tag{4.7}$$

Comparing tables (I) and (II), we can observe the relation

$$\zeta(s-6)\zeta(s-7)L(s,f_{14}^-)=\zeta_{Q(\sqrt{-1})}(s-6)\sum_{n=1}^{\infty}D\left(n\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)n^{-s},$$

which is a consequence of (4.7), where we put

$$\psi_8^{(1)}(z)/(48\times 336)=\sum_N D(N)e(\sigma(Nz)).$$

By Theorem 3.2, $\psi_8^{(1)}$ and $\psi_8^{(2)}$ are θ_{11}-eingliedrig forms and we have $\psi_8^{(i)}|H=\psi_8^{(i)}$, $\psi_8^{(i)}|W=3\psi_8^{(i)}$, $i=1, 2$.

Next we consider θ_4^2, $\theta_2\times\theta_6\in S_8(\Gamma_0^{(2)}(3))$. Put

$$\theta_4^2(z)=\sum_N E(N)e(\sigma(Nz)),\quad (\theta_2\times\theta_6)(z)=\sum_N F(N)e(\sigma(Nz)),$$
$$(\theta_4^2|T(2))(z)=\sum_N E_2(N)e(\sigma(Nz)),\quad ((\theta_2\times\theta_6)|T(2))(z)=\sum_N F_2(N)e(\sigma(Nz)).$$

Some of these values are given in tables (IV) and (V). Put $V=\langle\theta_8^{(1)},\theta_8^{(2)},\theta_4^2,\theta_2\times\theta_6\rangle_C$. As the Fourier coefficients of $\theta_8^{(1)}$ and of $\theta_8^{(2)}$ vanish for $n\begin{pmatrix}1&0\\0&2\end{pmatrix}$ (cf. Prop. 2.10), we have $\dim V=4$. We see $V\subset S_8^+(\Gamma_0^{(2)}(3))$ by (2) of Theorem 3.2. Let $E_4\in G_4(Sp(2,\boldsymbol{Z}))$ be the Eisenstein series and put $\eta_8^{(1)}=\theta_4\times(E_4-E_4|H)$. Obviously $\eta_8^{(1)}\neq 0$ and $\eta_8^{(1)}\in S_8^-(\Gamma_0^{(2)}(3))$. Therefore we have $S_8^+(\Gamma_0^{(2)}(3))=V$, $S_8^-(\Gamma_0^{(2)}(3))=\langle\eta_8^{(1)}\rangle_C$. In particular, V is invariant under $\mathcal{L}$ and $\eta_8^{(1)}$ is a common eigenfunction of all operators in $\mathcal{L}$ (cf. Lemma 2.5). Using the tables (IV) and (V) for $n\cdot N_2$, we find

$$\theta_4^2|T(2)\equiv-72\theta_4^2+24\theta_2\theta_6 \bmod V_0,$$

$$(\theta_2\theta_6)|T(2)\equiv 756\theta_4^2+108\theta_2\theta_6 \bmod V_0.$$

Then by tables (V) for $n\cdot N_1$ and (II), we obtain

$$\theta_4^2|T(2)=-72\theta_4^2+24\theta_2\theta_6+216\theta_8^{(1)}+8\theta_8^{(2)}, \tag{4.8}$$

$$(\theta_2\theta_6)|T(2)=756\theta_4^2+108\theta_2\theta_6+756\theta_8^{(1)}+26\theta_8^{(2)}. \tag{4.9}$$

Put

$$\phi_8^{(3)}=3\theta_4^2+\theta_2\theta_6+\theta_8^{(1)},\ \eta_8^{(2)}=-21\theta_4^2+2\theta_2\theta_6+9\theta_8^{(1)}+(\theta_8^{(2)}/3).$$

By (4.5), (4.6), (4.8) and (4.9), we have

$$\phi_8^{(3)}|T(2)=180\phi_8^{(3)},\quad \eta_8^{(2)}|T(2)=-144\eta_8^{(2)}.$$

On the other hand, by table (IV), we have $\eta_8^{(1)}|T(2)=-144\eta_8^{(1)}$. Therefore $\phi_8^{(3)}$ must be an eingliedrig form by Theorem 2.6, (1). By table (IV), the Fourier coefficient of $\phi_8^{(3)}$ for $\begin{pmatrix}1&0\\0&2\end{pmatrix}$ is 576. Hence $\phi_8^{(3)}$ must be θ_9-eingliedrig. We have

$$\phi_8^{(3)}|H=\phi_8^{(3)},\quad \phi_8^{(3)}|W=-3\phi_8^{(3)}.$$

For $f_{14}^+(z)=\sum_{n=1}^{\infty}c(n)e(nz)\in S_{14}^+(\Gamma_0^{(1)}(3))$ (one dimensional), $c(1)=1$, we observe the relation (cf. table (I) and (IV))

$$L(s,\phi_8^{(3)})=\zeta(s-6)\zeta(s-7)L(s,f_{14}^+). \tag{4.10$_c$}$$

Now we turn to the most interesting part of $S_8(\Gamma_0^{(2)}(3))$. Let V_1 denote the subspace $\langle\eta_8^{(1)},\eta_8^{(2)}\rangle_C$. As an orthogonal complement of $\langle\phi_8^{(1)},\phi_8^{(2)},\phi_8^{(3)}\rangle_C$, V_1 is a two dimensional $\boldsymbol{C}[H,W]$-module. We shall show that V_1 is irreducible. For this purpose, it is sufficient to prove that the eigenvalues of $T(3)$ on V_1 are not real, by Lemma 2.4, (3). Let

$$\eta_8^{(1)}(z)=\sum_N G^{(1)}(N)e(\sigma(Nz)),\quad \eta_8^{(2)}(z)=\sum_N G^{(2)}(N)e(\sigma(Nz)),$$

be the Fourier expansions. By table (IV) and (2.7), we find

$$\eta_8^{(1)}|T(3)=-\eta_8^{(1)}+(\eta_8^{(2)}/5), \tag{4.11}$$

$$\eta_8^{(2)}|T(3)=-40\eta_8^{(1)}-\eta_8^{(2)}. \tag{4.12}$$

Hence the characteristic roots of $T(3)$ are $-1\pm2\sqrt{-2}$. Therefore V_1 is irreducible. Put $\phi_8^{(4)}=20\eta_8^{(1)}-\sqrt{-2}\,\eta_8^{(2)}$, $\phi_8^{(5)}=20\eta_8^{(1)}+\sqrt{-2}\,\eta_8^{(2)}$. By (4.11) and (4.12), we see

$$\phi_8^{(4)}|T(3)=(-1+2\sqrt{-2})\phi_8^{(4)}. \tag{4.13}$$

Thus $\phi_8^{(4)}$ and $\phi_8^{(5)}$ are zweigliedrig forms. This completes the decomposition of $S_8(\Gamma_0^{(2)}(3))$. The eigenvalues of $T(3)$ on this space are 3, 3, -3 and $-1\pm2\sqrt{-2}$. Incidentally we can prove

Proposition 4.1. $\dim S_2(\Gamma_0^{(2)}(3))=0$ *and* $\dim S_4(\Gamma_0^{(2)}(3))=1$.

Proof. Considering the injection

$$S_4(\Gamma_0^{(2)}(3))\ni f \longrightarrow f\times\theta_2\in S_6(\Gamma_0^{(2)}(3)),$$

we see $\dim S_4(\Gamma_0^{(2)}(3))\leqq 2$. Asscme $\dim S_4(\Gamma_0^{(2)}(3))=2$. As we have shown $S_6(\Gamma_0^{(2)}(3))=S_6^+(\Gamma_0^{(2)}(3))$ and $\theta_2|H=\theta_2$, we must have $S_4(\Gamma_0^{(2)}(3))=S_4^+(\Gamma_0^{(2)}(3))$. Now consider another injection

$$S_4(\Gamma_0^{(2)}(3))\ni f \longrightarrow f\times(E_4-E_4|H)\in S_8^-(\Gamma_0^{(2)}(3)).$$

Since $\dim S_8^-(\Gamma_0^{(2)}(3))=1$ as we have shown, we get a contradiction. We obtain $\dim S_4(\Gamma_0^{(2)}(3))=1$ since it contains θ_4. If $f(\neq 0)\in S_2(\Gamma_0^{(2)}(3))$, then f^2 and $f\times\theta_2$ are linearly independent cusp forms in $S_4(\Gamma_0^{(2)}(3))$. This is a contradiction. Hence we obtain $\dim S_2(\Gamma_0^{(2)}(3))=0$.

(II) The case of level 11

The $(11, \infty)$-quaternion algebra D is given by $D=\boldsymbol{Q}+\boldsymbol{Q}i'+\boldsymbol{Q}j'+\boldsymbol{Q}k'$ with $i'^2=-1$, $j'^2=-11$, $i'j'=-j'i'=k'$. We have $h=2$ and $S(R, 0)=S^+(R, 0)$. We are going to construct theta series from lattices $L_{ij}(1\leqq i, j\leqq 2)$.

We use the following notation. The lattice L_{ij} is written as $L_{ij}=\boldsymbol{Z}\omega_1+\boldsymbol{Z}\omega_2+\boldsymbol{Z}\omega_3+\boldsymbol{Z}\omega_4$, and $N(L_{ij})$ denotes the norm of L_{ij}; an explicit form is due to A. Pizer. We put

$$x={}^t(x_1, x_2, x_3, x_4),\ y={}^t(y_1, y_2, y_3, y_4)\in\boldsymbol{Z}^4,$$

and set $\tilde{x}=\sum_{i=1}^{4} x_i\omega_i$, $\tilde{y}=\sum_{i=1}^{4} y_i\omega_i$. Define $S_{ij}\in M(4, \boldsymbol{Q})$, ${}^tS_{ij}=S_{ij}$ by $N(\tilde{x})/N(L_{ij})={}^txS_{ij}x$ and set $Q_{ij}(x, y)=\begin{pmatrix}{}^txS_{ij}x & {}^txS_{ij}y\\ {}^txS_{ij}y & {}^txS_{ij}y\end{pmatrix}$. Put $X_{ij}=\tilde{x}^*\tilde{y}$ and let

$$b=\text{the coefficient of } i' \text{ in } X_{ij},$$

$$c=\text{the coefficient of } j' \text{ in } X_{ij},$$

$$d=\text{the coefficient of } k' \text{ in } X_{ij}.$$

Thus b, c and d are polynomials of x and y which may depend on L_{ij}.

First we consider L_{11}. We have $\omega_1=(1+j')/2$, $\omega_2=(i'+j')/2$, $\omega_3=j'$, $\omega_4=k'$ and $N(L_{11})=1$. We set

$$\theta_2^{(1)}(z)=\sum_{(x, y)} \boldsymbol{e}(\sigma(Q_{11}(x, y)z)),$$

$$\theta_4^{(1)}(z)=\frac{1}{8}\sum_{(x, y)}(b^2-11c^2)\boldsymbol{e}(\sigma(Q_{11}(x, y)z)),$$

$$\theta_4^{(2)}(z)=\frac{1}{4}\sum_{(x, y)}(c^2-d^2)\boldsymbol{e}(\sigma(Q_{11}(x, y)z)),$$

where (x, y) extends over $\boldsymbol{Z}^4\oplus\boldsymbol{Z}^4$ and $z\in\mathfrak{H}_2$.

We consider L_{22}. We have $\omega_1=(1+3j')/2$, $\omega_2=(i'+16j'+3k')/6$, $\omega_3=3j'$, $\omega_4=k'$ and $N(L_{22})=1$. We set

$$\theta_2^{(2)}(z)=\sum_{(x,y)} e(\sigma(Q_{22}(x, y)z)),$$

$$\theta_4^{(3)}(z)=\frac{3}{4}\sum_{(x,y)}(c^2-d^2)e(\sigma(Q_{22}(x, y)z)).$$

We consider L_{12}. We have $\omega_1=(1+3j'+4k')/12$, $\omega_2=(i'+4j'+3k')/12$, $\omega_3=j'/2$, $\omega_4=k'/2$, $N(L_{12})=1/12$. We set

$$\theta_2^{(3)}(z)=\sum_{(x,y)} e(\sigma(Q_{12}(x, y)z)),$$

$$\theta_4^{(4)}(z)=18\sum_{(x,y)}(b^2-11c^2)e(\sigma(Q_{12}(x, y)z)),$$

$$\theta_4^{(5)}(z)=9\sum_{(x,y)}(c^2-d^2)e(\sigma(Q_{12}(x, y)z)).$$

We consider L_{21}. We have $\omega_1=1+3j'+2k'$, $\omega_2=i'+4j'+3k'$, $\omega_3=6j'$, $\omega_4=6k'$, $N(L_{21})=12$. We set

$$\theta_4^{(6)}(z)=\frac{1}{64}\sum_{(x,y)}(c^2-d^2)e(\sigma(Q_{21}(x, y)z)).$$

We have $\theta_2^{(i)}\in G_2(\Gamma_0^{(2)}(11))(1\leqq i\leqq 3)$, $\theta_4^{(i)}\in S_4(\Gamma_0^{(2)}(11))(1\leqq i\leqq 6)$; $\dim G_2(\Gamma_0^{(1)}(11))=\dim S(R, 0)=\dim G_2^-(\Gamma_0^{(1)}(11))=\dim S^+(R, 0)=2$, $\dim S_6^-(\Gamma_0^{(1)}(11))=\dim S^+(R, 4)=3$, $\dim S_6^+(\Gamma_0^{(1)}(11))=\dim S^-(R, 4)=1$. Let E_2 and f_2 denote the Eisenstein series and the normalized cusp forms in $G_2^-(\Gamma_0^{(1)}(11))$ respectively.

As is shown in [15], §8, we have

$$L(s, \theta_2^{(1)}+\theta_2^{(2)}-2\theta_2^{(3)})=L(s, f_2)^2,$$

$$L(s, 3\theta_2^{(1)}-2\theta_2^{(2)}-\theta_2^{(3)})=\zeta(s)\zeta(s-1)L(s, f_2)=L(s, E_2)L(s, f_2),$$

$$L(s, 9\theta_2^{(1)}+4\theta_2^{(2)}+12\theta_2^{(3)})=\zeta(s)^2\zeta(s-1)^2=L(s, E_2)^2,$$

and $3\theta_2^{(1)}-2\theta_2^{(2)}-\theta_2^{(3)}\in S_2(\Gamma_0^{(2)}(11))$; these are modular forms which correspond to the pairing $G_2^-(\Gamma_0^{(1)}(11))\times G_2^-(\Gamma_0^{(1)}(11))$.

Now let us consider the pairing $G_2^-(\Gamma_0^{(1)}(11))\times S_6^-(\Gamma_0^{(1)}(11))$. Corresponding to the six pairs, we can obtain six linearly independent cusp forms in $S_4^+(\Gamma_0^{(2)}(11))$ by linear combinations of $\theta_4^{(i)}(1\leqq i\leqq 6)$. These are given explicitly as follows. Let $f_6^-(z)=\sum_{n=1}^{\infty} a_n e(nz)$ be any normalized eigen cusp form of all Hecke operators in $S_6^-(\Gamma_0^{(1)}(11))$. First determine α_4, α_2, α_6, α_3 and α_5 successively by

$$\alpha_4=\frac{1}{264}\{50+8(a_2+4)+9a_3-a_5\},$$

$$\alpha_2=55\alpha_4-\frac{11}{2}-a_3-\frac{7}{3}(a_2+4),$$

$$\alpha_6=4\alpha_4-\frac{1}{3}(a_2+4),$$

$$12\alpha_3=45+6\alpha_2+48\alpha_4-12\alpha_6-(a_2+8)^2+8a_2+32,$$

$$27\alpha_5=2758-116\alpha_2+252\alpha_3+4266\alpha_4+908\alpha_6$$
$$-4(a_3+27)(a_3+36)+108a_3+972,$$

and set $F=\theta_4^{(1)}+\sum_{i=2}^{6}\alpha_i\theta_4^{(i)}$. Then we have $F\neq 0$ and

$$L(s, F)=\zeta(s-2)\zeta(s-3)L(s, f_6^-)=L(s-2, E_2)L(s, f_6^-). \tag{4.13}$$

Next determine β_4, β_2, β_6, β_3 and β_5 successively by

$$\beta_4=\frac{1}{264}\{50+8(a_2-16)+9(a_3-45)-(a_5-125)\},$$

$$\beta_2=55\beta_4-\frac{11}{2}-(a_3-45)-\frac{7}{3}(a_2-16),$$

$$\beta_6=4\beta_4-\frac{1}{3}(a_2-16),$$

$$12\beta_3=45+6\beta_2+48\beta_4-12\beta_6-(a_2-12)(a_2-8)+8a_2+64,$$

$$27\beta_5=2758-116\beta_2+252\beta_3+4266\beta_4+908\beta_6-4(a_3-9)^2$$
$$-36a_3+2268.$$

and set $G=\theta_4^{(1)}+\sum_{i=2}^{6}\beta_i\theta_4^{(i)}$. Then we have $G\neq 0$ and

$$L(s, G)=L(s-2, f_2)L(s, f_6^-). \tag{4.14}$$

According to Hashimoto [4], it is very plausible that $\dim S_4(\Gamma_0^{(2)}(11))=7$. Hereafter in this example, we shall assume this value of the dimension. As we have constructed six θ_{11}-eingliedrig forms, a form $\psi_4(\neq 0)$ in the orthogonal complement of $\langle\theta_4^{(i)}\,|\,1\leqq i\leqq 6\rangle_C$ must be an eingliedrig form. Put

$$\eta_4=(3\theta_2^{(1)}-2\theta_2^{(2)}-\theta_2^{(3)})(9\theta_2^{(1)}+4\theta_2^{(2)}+12\theta_2^{(3)})\in S_4(\Gamma_0^{(2)}(11)).$$

The Fourier coefficient of η_4 for $\begin{pmatrix}1 & 1/2\\ 1/2 & 2\end{pmatrix}$ is -1440. Hence ψ_4 must be θ_9-eingliedrig. As the Fourier coefficients of ψ_4 for $n\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ must vanish, we obtain

$$\psi_4=\eta_4+\frac{25}{2}\left\{-2\theta_4^{(1)}+\frac{585}{2}\theta_4^{(2)}+100\theta_4^{(3)}+\frac{632}{33}\theta_4^{(4)}+888\theta_4^{(5)}+\frac{388}{11}\theta_4^{(6)}\right\}$$

up to a constant multiple. Thus $S_4(\Gamma_0^{(2)}(11))=S_4^+(\Gamma_0^{(2)}(11))$ and this space is spanned by six θ_{11}-eingliedrig forms and by one θ_9-eingliedrig form. By computing more Fourier coefficients of ψ_4, we can observe the relation

$$L(s, \psi_4)=\zeta(s-2)\zeta(s-3)L(s, f_6^+), \tag{4.15$_c$}$$

where f_6^+ is the normalized cusp form in $S_6^+(\Gamma_0^{(1)}(11))$. Here the following observation seems very interesting: *There is no form whose Euler product corresponds to the pair* (f_2, f_6^+); i.e. *no form* $H\in S_4(\Gamma_0^{(2)}(11))$ *such that* $L(s, H)=L(s-2, f_2)L(s, f_6^+)$. We shall take into account of this fact when we shall formulate Conjecture 5.2.

(III) The case of level 7

The $(7, \infty)$-quaternion algebra D is given explicitly by $D=\boldsymbol{Q}+\boldsymbol{Q}i'+\boldsymbol{Q}j'+\boldsymbol{Q}k'$ with $i'^2=-1$, $j'^2=-7$, $i'j'=-j'i'=k'$. A maximal order R of D is given by

$R=\boldsymbol{Z}\omega_1+\boldsymbol{Z}\omega_2+\boldsymbol{Z}\omega_3+\boldsymbol{Z}\omega_4$, where $\omega_1=(1+j')/2$, $\omega_2=(i'+k')/2$, $\omega_3=j'$, $\omega_4=k'$. We have $h=1$. Put

$$x={}^t(x_1, x_2, x_3, x_4),\ y={}^t(y_1, y_2, y_3, y_4)\in\boldsymbol{Z}^4,$$

$\tilde{x}=\sum_{i=1}^4 x_i\omega_i$, $\tilde{y}=\sum_{i=1}^4 y_i\omega_i$. Define a symmetric matrix S by $N(\tilde{x})={}^txSx$. Set $Q(x, y)=\begin{pmatrix} {}^txSx & {}^txSy \\ {}^txSy & {}^tySy \end{pmatrix}$, $X=\tilde{x}^*\tilde{y}$ and let

$$b=2\times\text{the coefficient of } i' \text{ in } X,$$
$$c=2\times\text{the coefficient of } j' \text{ in } X,$$
$$d=2\times\text{the coefficient of } k' \text{ in } X.$$

We define six theta series by

$$\theta_2(z)=\sum_{(x,y)} e(\sigma(Q(x, y)z)),$$
$$\theta_4^{(1)}(z)=\sum_{(x,y)}(b^2-7c^2)e(\sigma(Q(x, y)z)),$$
$$\theta_4^{(2)}(z)=\sum_{(x,y)}(c^2-d^2)e(\sigma(Q(x, y)z)),$$
$$\theta_6^{(1)}(z)=\sum_{(x,y)}(b^4-42b^2c^2+49c^4)e(\sigma(Q(x, y)z)),$$
$$\theta_6^{(2)}(z)=\sum_{(x,y)}(b^4-42b^2d^2+49d^4)e(\sigma(Q(x, y)z)),$$
$$\theta_6^{(3)}(z)=\sum_{(x,y)}(c^4-6c^2d^2+d^4)e(\sigma(Q(x, y)z)).$$

We have $\theta_2\in G_2(\Gamma_0^{(2)}(7))$, $\theta_4^{(i)}\in S_4(\Gamma_0^{(2)}(7))(1\leqq i\leqq 2)$, $\theta_6^{(i)}\in S_6(\Gamma_0^{(2)}(7))(1\leqq i\leqq 3)$. Put

$$\phi_4^{(1)}=\theta_4^{(1)}+(8-\sqrt{57})\theta_4^{(2)},\ \phi_4^{(2)}=\theta_4^{(1)}+(8+\sqrt{57})\theta_4^{(2)}.$$

Let $f_6^-(z)=\sum_{n=1}^{\infty}a_n e(nz)\in S_6^-(\Gamma_0^{(1)}(7))$(two dimensional) be the normalized eigen cusp form such that $a_1=1$, $a_2=(9+\sqrt{57})/2$. Then we have $\phi_4^{(1)}\neq 0$ and

$$L(s, \phi_4^{(1)})=\zeta(s-2)\zeta(s-3)L(s, f_6^-). \tag{4.16}$$

Now we are going to consider the decomposition of $S_6(\Gamma_0^{(2)}(7))$. We have $\dim S_6(\Gamma_0^{(2)}(7))=11$. First the modular forms which correspond to the pairing

$$S^+(R, 0)\times S^+(R, 8)\cong G_2^-(\Gamma_0^{(1)}(7))\times S_{10}^-(\Gamma_0^{(1)}(7))$$

can be constructed as follows. Put $V_0=\langle\theta_6^{(1)}, \theta_6^{(2)}, \theta_6^{(3)}\rangle_C$; we find $\dim V_0=3$, $\dim S_{10}^-(\Gamma_0^{(1)}(7))=3$. Let $f_{10}^-(z)=\sum_{n=1}^{\infty}b_n e(nz)\in S_{10}^-(\Gamma_0^{(1)}(7))$ be a normalized eigen cusp form; b_2 satisfies the irreducible equation $X^3-21X^2-1326X+19080=0$. Put

$$\phi_6^{(1)}=\frac{1}{8}\theta_6^{(2)}+(b_2+15)\theta_6^{(3)}$$
$$+\frac{1}{5824}(14b_2+b_3+208)(\theta_6^{(1)}-\theta_6^{(2)}-329\theta_6^{(3)}).$$

Then $\phi_6^{(1)} \neq 0$ and we can prove

$$L(s, \phi_6^{(1)}) = \zeta(s-4)\zeta(s-5)L(s, f_{10}^{-}), \tag{4.17}$$

by Proposition 3.3. Put

$$\eta_6^{(1)} = \theta_2\theta_4^{(1)},\ \eta_6^{(2)} = \theta_2\theta_4^{(2)},\ \eta_6^{(3)} = \eta_6^{(1)}\Big| T(2),\ \eta_6^{(4)} = \eta_6^{(2)}\Big| T(2),$$

$$\eta_6^{(5)} = \eta_6^{(2)}\Big| T(3),\ V_1 = \langle \eta_6^{(1)}, \eta_6^{(2)}, \eta_6^{(3)}, \eta_6^{(4)}, \eta_6^{(5)} \rangle_C,$$

$$V_2 = V_0 + V_1.$$

By computing Fourier coefficients of $\eta_6^{(i)}$ for $n \cdot \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, we find $\dim V_1 = 5$, $\dim V_2 = 8$. Applying $T(7)$ and computing Fourier coefficients using (2.7), we get $S_6(\Gamma_0^{(2)}(7)) = V_2 + V_2 \Big| T(7)$. More precisely, by computing the Fourier coefficients for $n \cdot \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, we can see that V_1 and $\sum_{i=1}^{3} C\eta_6^{(i)} \Big| T(7)$ generate a 8-dimensional space. Then, by the theory in § 2, we can conclude that $S_6(\Gamma_0^{(2)}(7))$ contains at least 3-pairs of independent zweigliedrig forms. As $V_2 \subset S_6^{+}(\Gamma_0^{(2)}(7))$, $\dim S_6^{-}(\Gamma_0^{(2)}(7)) \leqq 3$, it must contain exactly 3-pairs of independent zweigliedrig forms. Thus we obtain

$$V_2 = S_6^{+}(\Gamma_0^{(2)}(7)),\ \dim S_6^{+}(\Gamma_0^{(2)}(7)) = 8,\ \dim S_6^{-}(\Gamma_0^{(2)}(7)) = 3.$$

In particular, V_2 is invariant under $\mathcal{L}$. It must contain two more eingliedrig forms. The characteristic polynomial of $T(2)$ on V_2 is $(X^3 - 165X^2 + 7602X - 76248)(X^2 - 90X + 1832)(X^3 + 4X^2 - 636X + 4656)$. Put $d = 193$ and

$$\begin{aligned}\phi_6^{(4)} = {} & 2(8013 - 861\sqrt{d})\eta_6^{(1)} + 2(475195 - 5915\sqrt{d})\eta_6^{(2)} + (9969 + 51\sqrt{d})\eta_6^{(3)} \\ & + (36785 - 973\sqrt{d})\eta_6^{(4)} + 2(1515 - 51\sqrt{d})\eta_6^{(5)} + (66711 + 1461\sqrt{d})\theta_6^{(2)} \\ & + 4(411143 + 284645\sqrt{d})\theta_6^{(3)} + \left(-\frac{6224}{7} + 3088\sqrt{d}\right)(\theta_6^{(1)} - \theta_6^{(2)} - 329\theta_6^{(3)}).\end{aligned}$$

Then we can prove that $\phi_6^{(4)}$ and its conjugate $\phi_6^{(5)}$ by $\mathrm{Gal}(\boldsymbol{Q}(\sqrt{d})/\boldsymbol{Q})$ are θ_9-eingliedrig forms. We have observed the relation

$$L(s, \phi_6^{(4)}) = \zeta(s-4)\zeta(s-5)L(s, f_{10}^{+}), \tag{4.18$_c$}$$

where $f_{10}^{+}(z) = \sum_{n=1}^{\infty} c_n e(nz) \in S_{10}^{+}(\Gamma_0^{(1)}(7))$(two dimensional) is the normalized eigen cusp form such that $c_1 = 1$, $c_2 = -3 + \sqrt{d}$.

Finally we have calculated eigenvalues of $T(7)$ using Fourier coefficients of $\eta_6^{(i)}$, $1 \leqq i \leqq 6$.

Proposition 4.2. *The eigenvalues of* $7 \cdot T(7)$ *on* $S_6(\Gamma_0^{(2)}(7))$ *are* $7^2, 7^2, 7^2, -7^2, -7^2$ *and the six roots of the equation* $X^6 + 10X^5 + 2303X^4 - 57428X^3 + 5529503X^2 + 57648010X + 13841287201 = 0$.

Table (I)

p	$a(p)$	$b(p)$, $d=11\times 179$	$c(p)$
2	-36	$-27+3\sqrt{d}$	-12
3	-81	729	-729
5	-1314	$20385+384\sqrt{d}$	-30210
7	-4480	$-10504-10368\sqrt{d}$	235088
11	1476	$336204-109824\sqrt{d}$	-11182908
13	-151522	$8766302+269568\sqrt{d}$	8049614

In the Table below, we set $N_1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $N_2=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$.

Table (II)

n	$A(n\ N_2)/576$	$B(n\ N_1)/48$	$c(n\ N_1)/48$
1	1	-1	33
2	-4	-362	10938
3	-81	-729	24057
4	656	-42500	1327908
5	2436	-116266	3707754
6	324	-263898	7973802
7	14728	-28672	4429824
8	11200	-5347048	163352616
9	6561	-531441	17537553
10	-9744	-35637092	1079471748
11	147886	-12040248	434229048
12	-53136	-30982500	968044932
13	248332	-90140426	2884059210
14	-58912	-184561664	5504249856
15	-197316	-84757914	2702952666
16	309504	-516486288	16038071568

Table (III)

n	$B_2(n\ N_1)/288$	$C_2(n\ N_1)/288$
1	-71	2175
2	-17302	526278
3	-51759	1585575

Table (IV)

n	$\frac{E(n\ N_1)}{576\times 6}$	$\frac{F(n\ N_1)}{48}$	$\frac{E(n\ N_2)}{576\times 24}$	$\frac{F(n\ N_2)}{576}$
1	0	1	0	1
2	1	146	1	44
3	4	−135	−3	−513
4	42	33428	−28	8816
5	224	67882	110	55620
6	−103	286146	−105	−77004
7	960	−178688	4086	882088
8	9860	3217288	4944	709312
9	972	321489	3645	269001
10	55338	23684089	40660	4443120
11	29392	5691576	28534	4478254
12	94106	106555604	−27204	−2998512
13	183264	50555402	223596	59526028
14	216064	137891840	326392	112948256
15	149896	52380378	−163890	−34520580
16	824296	338438352	518208	162635520

Table (V)

n	$\frac{E_2(n\ N_1)}{576\times 6}$	$\frac{F_2(n\ N_1)}{48\times 2}$	$\frac{E_2(n\ N_2)}{576\times 24}$	$\frac{F_2(n\ N_2)}{576}$
1	1	105	1	108
2	106	40458	−28	22898
3	153	138753	−297	−109836

Table (VI)

n	$\frac{G^{(1)}(n\ N_1)}{192}$	$\frac{G^{(2)}(n\ N_1)}{3840}$	$\frac{G^{(1)}(n\ N_2)}{576}$	$\frac{G^{(2)}(n\ N_2)}{3\times 3840}$
1	1	1	1	1
2	−208	−208	−208	−208
3	−1215	243	243	−1215
4	15872	15872	15872	15872
5	−3350	−3350	27900	27900
6	252720	−50544	−50544	252720
7	−147584	−147584	−147584	−147584
8	−536576	−536576	−536576	−536576
9	59049	−649539	−649539	59049
10	696800	696800	−5803200	−5803200
11	830808	830808	−2712314	−2712314
12	−19284480	3856896	3856896	−19284480
13	−6473782	−6473782	3179836	3179836
14	30697472	30697472	30697472	30697472
15	4070250	−814050	6779700	−33898500
16	32047104	32047104	32047104	32047104

5. Conjectures

We now come to the place to meditate upon our experiments and to formulate conjectures.

Conjecture 5.1. *Let $F\in S_k^0(\Gamma_P)$ be a θ_{11}-eingliedrig form. Let $F|H=\varepsilon F$, $\varepsilon=\pm 1$. Then $F=F(\varphi_1, \varphi_2)$ with $\varphi_1\in S^{\pm}(R, 0)$, $\varphi_2\in S^{\pm}(R, 2n)$, where $n=k-2$ and $\pm$ is the signature of ε. The Euler product of F corresponds to a pair of forms in $G_2^-(\Gamma_0^{(1)}(p))\times S_{2k-2}^-(\Gamma_0^{(1)}(p))$ or forms in $S_2^+(\Gamma_0^{(1)}(p))\times S_{2k-2}^+(\Gamma_0^{(1)}(p))$ as (3.9).*

This conjecture, which accords with all experiments that we have made, characterizes θ_{11}-eingliedrig forms. One may expect similar characterization for θ_9-eingliedrig forms. However, by the data

$$\dim S_8(\Gamma_0^{(2)}(7))=26,\quad \dim S_{14}(\Gamma_0^{(1)}(7))=7,\quad \text{or}$$

$$\dim S_6(\Gamma_0^{(2)}(13))=46,\quad \dim S_{10}(\Gamma_0^{(1)}(13))=9,$$

we see that this expectation cannot be satisfied, since $26\not\equiv 7 \bmod 2$, etc. To formulate more accurate conjectures, we need information about representations of a certain Hecke algebra. We use the notation of § 2. Consider three Hecke operators

$$S_1=\Gamma_B w_1 \Gamma_B,\qquad S_2=\Gamma_B w_2 \Gamma_B,\qquad S_\rho=\Gamma_B w_\rho \Gamma_B\,,$$

where w_1 and w_2 are given by (2.1) and $w_\rho=w_1\begin{pmatrix} 0 & 1\\ -p & 0\end{pmatrix}$. Then S_1, S_2 and S_ρ satisfy the relations

$$\left\{\begin{array}{l} S_i^2=(p-1)S_i+p,\quad i=1, 2\,,\\ (S_1S_2)^2=(S_2S_1)^2\,,\\ S_\rho^2=1,\quad S_1S_\rho=S_\rho S_1,\quad (S_2S_\rho)^2=(S_\rho S_2)^2\,. \end{array}\right. \tag{5.1}$$

These relations follow from Iwahori-Matsumoto [8]. In fact, set

$$\tilde{G}=GSp(2, \boldsymbol{Q}_p)/\text{the center},\quad \tilde{K}=GSp(2, \boldsymbol{Z}_p)/\text{the center},$$

$$\tilde{B}=\left\{k\in\tilde{K}\mid k \bmod p=\begin{pmatrix} * & 0 & * & *\\ * & * & * & *\\ 0 & 0 & * & *\\ 0 & 0 & 0 & * \end{pmatrix}\right\}.$$

Then $\mathcal{H}(\tilde{G}, \tilde{B})$, the Hecke algebra of $\tilde{G}$ with respect to $\tilde{B}$, is isomorphic to $\boldsymbol{C}[S_1, S_2, S_\rho]$. We have $\mathcal{H}(\tilde{K}, \tilde{B})\cong\mathcal{H}(G, B)\cong\boldsymbol{C}[S_1, S_2]$. We set $\mathcal{H}=\boldsymbol{C}[S_1, S_2, S_\rho]\cong\mathcal{H}(\tilde{G}, \tilde{B})$, $\mathcal{H}_1=\boldsymbol{C}[S_1, S_2]\cong\mathcal{H}(\tilde{K}, \tilde{B})$.

As S_1, S_2 and S_ρ act on $S_k(\Gamma_B)$, we get a representation of $\mathcal{H}$ on $S_k(\Gamma_B)$; this representation decomposes into a direct sum of irreducible representations, since S_1, S_2, S_ρ are hermitian with respect to the Petersson inner product. Assume $F\in S_k^0(\Gamma_P)$ $(\subset S_k(\Gamma_B))$ is a θ_{11}-eingliedrig form. Then we see easily that F generates the one dimensional representation $S_1\to p$, $S_2\to -1$, $S_\rho\to\varepsilon$

($\varepsilon=\pm 1$) of $\mathcal{H}$. The point is that, if F is θ_9-eingliedrig, there are three possibilities for the representations of $\mathcal{H}$ generated by F.

To determine irreducible representations of $\mathcal{H}$ explicitly is a somewhat laborious task, although there is a work of Kazhdan-Lusztig [9] on general case. In the Appendix, we list explicit realizations of representations π of $\mathcal{H}$; if $\dim \pi>2$, we assumed that $\pi|\mathcal{H}_1$ does not contain the "trivial representations π_1": $S_1\to p$, $S_2\to p$. This condition is satisfied if π is generated by a form in $S_k^0(\Gamma_P)$.

Assume $F\in S_k^0(\Gamma_P)$ is a θ_9-eingliedrig form which generates an irreducible representation π of $\mathcal{H}$. Then there are three possibilities: $\pi\cong\Pi_3$, or Π_4, or the two dimensional representation

$$\Pi_0: S_1 \longrightarrow \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix}p & 0\\ 0 & -1\end{pmatrix}, \quad S_\rho \longrightarrow \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

of $\mathcal{H}$ (cf. Appendix). By the examples in §4, we see that the "lifted" θ_9-eingliedrig forms generate the representation Π_0. We can now formulate the following (cf. §4. (II))

Conjecture 5.2. *Let $F\in S_k^0(\Gamma_P)$ be a θ_9-eingliedrig form which generates the irreducible two dimensional representation Π_0 of $\mathcal{H}$. Then the Euler product of F corresponds to a pair of the Eisenstein series in $G_2^-(\Gamma_0^{(1)}(p))$ and of a form in $S_{2k-2}^+(\Gamma_0^{(1)}(p))$.*

Appendix

(I) The one dimensional representations of $\mathcal{H}$.
These are

$$S_1 \longrightarrow p \text{ or } -1. \qquad S_2 \longrightarrow p \text{ or } -1, \qquad S_\rho \longrightarrow 1 \text{ or } -1\,.$$

According as the above choices, we get eight irreducible representations of $\mathcal{H}$.

Let $\alpha, \beta, \gamma, \delta$ be given by (2.4).

(II) The two dimensional representations of $\mathcal{H}$.
These are

$$S_1 \longrightarrow \begin{pmatrix}p & 0\\ 0 & -1\end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad S_\rho \longrightarrow \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

or

$$S_1 \longrightarrow p\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ or } \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$,\quad S_2 \longrightarrow \begin{pmatrix}(p-1)/2 & (p+1)/2\\ (p+1)/2 & (p-1)/2\end{pmatrix}, \quad S_\rho \longrightarrow \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

According as the above choices, we get four irreducible representations of $\mathcal{H}$.

(III) The irreducible representations of $\mathcal{H}$ whose dimensions are higher than 2 and whose restriction to $\mathcal{H}_1$ does not contain the "trivial representation": $S_1\to p$, $S_2\to p$.

We label the representations of $\mathscr{H}_1$ by the corresponding representations of $Sp(2, \boldsymbol{F}_p)$ (cf. (2.3)). Set

(5.2) $$\lambda=-\frac{(p-1)^2}{2p}, \qquad \mu=\frac{(p-1)^2}{p^2+1}.$$

$$\Pi_1: S_1 \longrightarrow \begin{pmatrix} \alpha & 0 & \beta \\ 0 & -1 & 0 \\ \gamma & 0 & \delta \end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_p \longrightarrow \pm \begin{pmatrix} -\lambda & 1 & -1/2p \\ -\lambda^2\mu^{-1} & \lambda\mu^{-1} & -\lambda/2p \\ -(p^2+1) & -2p & 0 \end{pmatrix}, \quad \Pi_1 | \mathscr{H}_1 \cong \theta_9 \oplus \theta_{12}.$$

$$\Pi_2: S_1 \longrightarrow \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & p \end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix} p & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_p \longrightarrow \pm \begin{pmatrix} 0 & -1/2p & 1/(p^2+1) \\ -(p^2+1) & \lambda & 1 \\ (p^2+1)\lambda & -\lambda^2\mu^{-1} & -\lambda\mu^{-1} \end{pmatrix}, \quad \Pi_2 | \mathscr{H}_1 \cong \theta_9 \oplus \theta_{11}.$$

$$\Pi_3: S_1 \longrightarrow \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix} p & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$S_p \longrightarrow \pm \begin{pmatrix} 0 & 1/(p^2+1) & -1/2p \\ 2p & \mu & 1 \\ -2p\mu & -\mu^2\lambda^{-1} & -\mu\lambda^{-1} \end{pmatrix}, \quad \Pi_3 | \mathscr{H}_1 \cong \theta_9 \oplus \theta_{13}.$$

$$\Pi_4: S_1 \longrightarrow \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & -1 & 0 & 0 \\ \gamma & 0 & \delta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$S_p \longrightarrow \pm \begin{pmatrix} 1-\mu^{-1}t & (1-\mu^{-1}t)/2p & t/(p-1)^2 & -1/2p \\ 2p\mu^{-1}t & \mu^{-1}t & \lambda^{-1}t & 1 \\ 2p\mu^{-1}t & -(1-\mu^{-1}t) & 1+\lambda^{-1}t & 1 \\ -2p\mu^{-1}t(1+\lambda^{-1}t) & (1-\mu^{-1}t)(1+\lambda^{-1}t) & -\lambda^{-1}t(1+\lambda^{-1}t) & -\lambda^{-1}t \end{pmatrix},$$

$t \in C$, $t \neq 0$, μ, $-\lambda$, $\Pi_4 | \mathscr{H}_1 \cong \theta_9 \oplus \theta_{12} \oplus \theta_{13}$.

$$\Pi_5: S_1 \longrightarrow \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_2 \longrightarrow \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$S_p \longrightarrow \begin{pmatrix} 0 & t/(p-1)^2 & 1/(p^2+1) & -1/2p \\ 2p\mu^{-1}t & t & 1 & 1 \\ 2p\nu[1-(\mu^{-1}t)^2] & \nu[1-\mu^{-1}t)^2] & -\mu^{-1}t & 0 \\ -2p\nu[1-(\lambda^{-1}t)^2] & \eta[1-(\lambda^{-1}t)^2] & 0 & -\lambda^{-1}t \end{pmatrix},$$

where $\nu=\frac{p^2+1}{(p+1)^2}$, $\eta=\frac{2p}{(p+1)^2}$, $t\in\boldsymbol{C}$, $t\neq 0$, $\pm\mu$, $\pm\lambda$, $\Pi_5|\mathcal{H}_1\cong\theta_9\oplus\theta_{11}\oplus\theta_{13}$.

Let π be an irreducible finite dimensional representation of $\mathcal{H}$ on a vector space V over $\boldsymbol{C}$. For π to occur in the representation of $\mathcal{H}$ on $S_k(\Gamma_B)$, there must exist a positive hermitian inner product (,) on V such that S_1, S_2 and S_ρ become hermitian with respect to (,). When this is the case, let us call π *unitarizable.* We obtain

Proposition A.1. *All representation π such that* $\dim\pi\leqq 2$ *are unitarizable. Π_1 and Π_2 are not unitarizable. Π_3 is unitarizable. Π_4 is unitarizable if and only if $0<t<\mu$. Π_5 is unitarizable if and only if $-\mu<t<\mu$.*

The proof, which is not difficult, is omitted.

Example A.2. The zweigliedrig form $20\,\eta_8^{(1)}+\sqrt{-2}\,\eta_8^{(2)}\in S_8(\Gamma_0^{(2)}(3))$ generates the irreducible representation Π_5 with $t=\mu/3=2/15$.

DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY

References

[1] A.N. Andrianov, Euler products corresponding to Siegel modular forms of genus 2, Uspekhi Math. Nauk, **29** (1974), 43-110.
[2] A. Borel, Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahari subgroup, Inv. Math., **35** (1976), 233-259.
[3] W. Casselman, An assortment of results on representations of $GL_2(k)$, Lecture notes in Mathematics 349, Springer Verlag, 1973.
[4] K. Hashimoto, The dimension of the spaces of cusp forms on Siegel upper half plane of degree two (I), J. Fac. Sci. Univ. Tokyo, Sec. IA, **30** (1983), 403-488.
[5] K. Hashimoto, Representations of the finite symplectic group $Sp(4, \boldsymbol{F}_p)$ in the space of Siegel modular forms, Contemporary Mathematics, **53** (1986), 253-276.
[6] E. Hecke, Mathematische Werke, Zweite Auflage, Vandenhoeck, 1970.
[7] N. Iwahori, On the structure of a Hecke ring of a Chevalley group over a finite field, J. Fac. of Sci. Univ. Tokyo, **10** (1964), 215-236.
[8] N. Iwahori and H. Matsumoto, On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups, Publ. I.H.E.S., **25** (1965), 5-48.
[9] D. Kazhdan and D. Lusztig, Proof of the Deligne—Langlands conjecture for Hecke algebras, Inv. Math., **87** (1987), 153-215.
[10] H. Maass, Über eine Spezialschar von Modulformen zweiten Grades III, Inv. Math., **53** (1979), 255-265.
[11] G. Shimura, On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields, Nagoya Math. J., **43** (1971), 198-208.
[12] G. Shimura, On the factors of the jacobian variety of a modular function field, J. Math. Soc. Japan, **25** (1973), 523-544.
[13] B. Srinivasan, The characters of the finite symplectic group $Sp(4, q)$, Trans. Amer. Math. Soc., **131** (1968), 488-525.
[14] H. Yoshida, Weil's representations of the symplectic groups over finite fields, J. Math. Soc. Japan, **31** (1979), 399-426.
[15] H. Yoshida, Siegel's modular forms and the arithmetic of quadratic forms, Inv. Math., **60** (1980), 193-248.
[16] H. Yoshida, The action of Hecke operators on theta series, Algebraic and Topological Theories, 197-238, Kinokuniya, 1985.