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Representations of Lie superalgebras I

Extensions of representations of the even part

By

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Introduction.

Lie superalgebras are becoming important both in mathematics and in physics. The classification of finite-dimensional simple Lie superalgebras was done by Kac in [8] and also by Kaplansky in [11]. Kac also studied the finite-dimensional representations, especially character formulas for them, in [9] and [10]. The infinite-dimensional representations are much more interesting as in the case of usual Lie groups. Unitary (or unitarizable) representations are of particular interest and importance, dominantly in physical applications. As is well known, the classification and the construction of irreducible unitary representations of Lie groups are of great importance in the theory of infinite-dimensional representations. Therefore we intend to study similar problems for (infinite-dimensional) representations of Lie superalgebras from a general point of view.

In this paper we give a definition of unitarity of such representations, which is methematically natural. Then we give a method of constructing irreducible representations of Lie superalgebras. This method gives a standard approach to classifying irreducible (unitary) representations for any Lie superalgebras. In the second half of this paper, we take some simple Lie superalgebras and give the classification and the construction of their irreducible (unitary) representations.

Let $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ be a Lie superalgebra and (π, V) be its representation on a $\boldsymbol{Z}_2$-graded complex vector space $V=V_0+V_1$ in the sense of Kac [8]. Then, on the even part V_0 and also on the odd part V_1 of V, we have representations of a usual Lie algebra $\mathfrak{g}_0$. We consider the converse, expecting to utilize rich results on representations of $\mathfrak{g}_0$. More exactly, we take a representation (ρ, V_0) of $\mathfrak{g}_0$, and then try to construct a representation (π, V) of $\mathfrak{g}$ such that its even part is isomorphic to V_0 as $\mathfrak{g}_0$-modules. We call this (π, V) an extension of (ρ, V_0). We raise some problems concerning this extension.

Problem 1 (Extensions of irreducible representations of $\mathfrak{g}_0$). Take an irreducible representation ρ of $\mathfrak{g}_0$ on a complex vector space V_0. Then, do there exist any irreducible representations (π, V) of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ extending (ρ, V_0)? If they

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do exist, construct all of them.

Examining some types of simple Lie superalgebras, we recognize that for many irreducible representations (π, V), $V=V_0+V_1$, of $\mathfrak{g}$, the restriction of π to $\mathfrak{g}_0$ on V_0 or V_1 is not irreducible. As a matter of fact, when we take Lie superalgebras of type A as $\mathfrak{g}$, already the adjoint representation is in this case, even though it is finite-dimensional. So we generalize Problem 1 to Problem 1bis, where we start with a representation (ρ, V_0) of $\mathfrak{g}_0$ not necessarily irreducible. Requiring (ρ, V_0) and (π, V) to be unitary in Problems 1 and 1bis, we propose Problems 2 and 2bis. In this approach, if we can solve Problem 2bis, the unitary extension problem, then, as a result, all the irreducible unitary representations of a Lie superalgebra $\mathfrak{g}$ will be obtained.

To solve these extension problems, we introduce a bilinear map B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$ by means of π, as

$$B(\xi, \eta) = \pi(\xi)\,\pi(\eta)\,|\,V_0 \quad (\xi, \eta\in\mathfrak{g}_1)\,.$$

We see that irreducible π is determined uniquely, up to equivalence, by this map B. And then we give a necessary and sufficient condition (EXT1)–(EXT3) for B, and also a method of constructing (π, V) using (ρ, V_0) and B. Thus Problems 1 and 1bis are reduced to the following: find a bilinear map B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$ satisfying the system of equations (EXT1)–(EXT3). For Problems 2 and 2bis, a certain positive-definiteness condition (UNI) on B is required in addition. In many cases, the skew-symmetric bilinear map A: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$,

$$A(\xi, \eta) = B(\xi, \eta)-B(\eta, \xi) \quad (\xi, \eta\in\mathfrak{g}_1)\,,$$

is more convenient to treat with. So we rewrite (EXT1)–(EXT3) by means of A. Further we give a reduction of the system of equations (EXT1)–(EXT3).

After these general discussion of the problems, we give some examples in the latter half of this paper.

Let us explain the contents of this paper in more detail.

In §1, we give some basic definitions in 1.1∼1.2 and then define in 1.3 the (infinitesimal) unitarity for representations of Lie superalgebras, which is a natural extension of that for Lie algebras. Our unitarity is defined as follows. Let (π, V), $V=V_0+V_1$, be a representation of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$. We call (π, V) *unitary* if V is equipped with a positive definite inner product $\langle\cdot, \cdot\rangle$ satisfying the following:

(i) $V_0\perp V_1$ (orthogonal) with respect to $\langle\cdot, \cdot\rangle$, and

(ii) $\langle\cdot, \cdot\rangle$ is $\mathfrak{g}$-invariant in the sense that

$$\langle i\pi(X)\,v, v'\rangle = \langle v, i\pi(X)\,v'\rangle \quad (v, v'\in V, X\in\mathfrak{g}_0)\,,$$
$$\langle j\pi(\xi)\,v, v'\rangle = \langle v, j\pi(\xi)\,v'\rangle \quad (v, v'\in V, \xi\in\mathfrak{g}_1)\,,$$

where $i=\sqrt{-1}$ and j is a fixed forth root (depending only on π) of -1. We call j^2 the *associated constant* for π since the essential thing is not j but $j^2=\varepsilon i$ with $\varepsilon=1$ or -1.

If (π, V) is unitary, restrictions $\pi(\mathfrak{g}_0)|V_0$ and $\pi(\mathfrak{g}_0)|V_1$ are (infinitesimally) unitary representations of the Lie algebra $\mathfrak{g}_0$ in the usual sense.

In §2, we introduce the extension problems, Problems 1, 1bis, 2 and 2bis mentioned above.

In §3, we define the bilinear map B, and give a necessary and sufficient condition (PRO1)–(PRO3) for π to be irreducible in Lemma 3.1:

(PRO1) $\pi(\mathfrak{g}_1)V_0 = V_1$, where $\pi(\mathfrak{g}_1)V_0$ denotes the linear span of $\{\pi(\xi)v;\ \xi\in\mathfrak{g}_1, v\in V_0\}$;

(PRO2) an element $v_1\in V_1$ is equal to 0 if and only if $\pi(\xi)v_1=0$ for any $\xi\in\mathfrak{g}_1$;

(PRO3) the subalgebra $\langle\rho(\mathfrak{g}_0), B(\mathfrak{g}_1, \mathfrak{g}_1)\rangle$, generated by $\rho(\mathfrak{g}_0)$ and $B(\mathfrak{g}_1, \mathfrak{g}_1)$ of $\mathfrak{gl}(V_0)$, acts on V_0 irreducibly.

We give, in 3.2, a standard method of constructing (π, V) using (ρ, V_0) and B, and obtain, in Theorem 3.3, a necessary and sufficient condition (EXT1)–(EXT3) to get a representation π of $\mathfrak{g}$ with properties (PRO1)–(PRO2):

(EXT1) $B({}^X\xi, \eta)+B(\xi, {}^X\eta) = [\rho(X), B(\xi, \eta)]$,

(EXT2) $B(\xi, \eta)+B(\eta, \xi) = \rho([\xi, \eta])$,

(EXT3) $B(\tau, \xi)\,B(\eta, \zeta)+B(\tau, \eta)\,B(\xi, \zeta) = B(\tau, {}^{[\xi,\eta]}\zeta)+B(\tau, \zeta)\,\rho([\xi, \eta])$,

for $\tau, \xi, \eta, \zeta\in\mathfrak{g}_1$ and $X\in\mathfrak{g}_0$, where ${}^X\xi=[X, \xi]$.

In 3.4, the algebraic irreducibility is discussed. In 3.5, we define a skew-symmetric bilinear map A and rewrite (EXT1)–(EXT3) in terms of A. Further we give a reduction of the system of equations (EXT1)–(EXT3) in Theorem 3.11, taking into account its $\mathfrak{g}_1$-equivariance.

In §4, we concern Problems 2 and 2bis, and get a necessary and sufficient condition (UNI) on B for π to be unitary in Theorem 4.2:

$$\text{(UNI)} \qquad j^2\sum_{k,m}\langle B(\xi_m, \xi_k)\,v^k, v^m\rangle_0 \geqq 0 \qquad \text{for} \quad \xi_k\in\mathfrak{g}_1,\ v^k\in V_0,$$

where $\langle\cdot, \cdot\rangle_0$ is a $\mathfrak{g}_0$-invariant positive definte inner product on V_0. We give some remarks and examples for unitary representations in 4.2.

In the latter half of this paper, §§5∼8, some examples are discussed.

In §5, we classify and construct all the irreducible (unitary) representations of $\mathfrak{osp}(2/1)$. For our result for $\mathfrak{osp}(2n/1)$ with $n\geqq 2$, a real form of a simple Lie superalgebra $\mathfrak{osp}(1, 2n)$ of type $B(0, n)$, we refer the readers to [6].

In §§6∼8, we take $\mathfrak{gl}(m, n)$ or a real form of it as $\mathfrak{g}$ and study Problems 2 and 2bis. In §6, we first list up real forms of $\mathfrak{sl}(2, 1)$ up to isomorphism. There are three types of them up to transition to their duals: (a) $\mathfrak{sl}(2, 1; \boldsymbol{R})$, (b) $\mathfrak{su}(2, 1; 2, 1)$ and (c) $\mathfrak{su}(2, 1; 1, 1)$, for which the even parts are $\mathfrak{gl}(2; \boldsymbol{R})$, $\mathfrak{u}(2)$ and $\mathfrak{u}(1, 1)$ respectively. In 6.3, we get the solution of Problem 2 for (a). More generally, we get the solution of Problem 2bis for $\mathfrak{g}=\mathfrak{gl}(m, n; \boldsymbol{R})$ in Theorem 6.2, which says that $\mathfrak{g}$ has only a unique irreducible unitary representation, the trivial one. In 6.5, utilizing Theorem 3.11, we prepare some necessary conditions for existence of irreducible extensions by means of B when $\mathfrak{g}$ is one of real forms of $\mathfrak{sl}(2, 1)$.

In §7, the solution of Problem 2 for (b) is given as follows.

Theorem 7.1. *Let* $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$, *then* $\mathfrak{g}_0=\mathfrak{u}(2)$. *Take* (ρ, V_0) *a finite-dimensional irreducible unitary* $\mathfrak{g}_0$*-module with highest weight* Λ. *Put* $n=\Lambda(H)+1=\dim V_0$, *and* $m=\Lambda(C)$, *where* $H=\mathrm{diag}(1, -1, 0)$ *and* $C=\mathrm{diag}(1, 1, 2)$ *are elements of* $\mathfrak{h}_C$, *the complexification of a Cartan subalgebra* $\mathfrak{h}$. *Then there exist irreducible unitary extensions* (=*IUEs*) *of* ρ *if and only if one of the following three conditions holds*:

(i) $n=1$ *and* $m=-2, 0, 2$;

(ii) $n=2$ *and* $m\in\boldsymbol{R}, |m|\geqq 1$;

(iii) $n\geqq 3$ *and* $m=\pm(n-1), \pm(n+1)$.

Moreover IUEs are unique up to isomorphism, except the cases $n=2$ *and* $m=\pm 3$. *In these exceptional cases there exist exactly two IUEs up to isomorphism.*

The solution of Problem 2 for (c) is given as follows. Here $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$ and $\mathfrak{g}_0=\mathfrak{u}(1, 1)$, and irreducible (infinite-dimensional) unitary $(\mathfrak{g}_0, K_0)$-modules are well-known (cf. [15]).

Proposition 7.6. (i) *If* (ρ, V_0) *is in the principal continuous series or in the complementary series, then there does not exist any IUEs.*

(ii) *If* (ρ, V_0) *is trivial, then there exists an IUE if and only if* $\Lambda(C)=0$. *Actually an IUE is given by the trivial representation of* $\mathfrak{g}$.

Theorem 7.7. *Let* (ρ, V_0) *be in the holomorphic discrete series or its limit with highest weight* Λ. *Put* $l=-\Lambda(H)$ *and* $m=\Lambda(C)$. *Then there exist IUEs if and only if one of the following conditions holds*:

(i) $l=1$ *and* $m=\pm 1$;

(ii) $l=2$ *and* $m=0, \pm 2$;

(iii) $l\geqq 3$ *and* $m=\pm l, \pm(l-2)$.

Moreover IUEs are unique up to isomorphism except the case $l=2$ *and* $m=0$. *In this exceptional case there exist exactly two IUEs up to isomorphism. For all of these representations, their associated constants are given by* $j^2=i$ $(\varepsilon=1)$.

In case ρ is in the anti-holomorphic discrete series or its limit with lowest weight Λ, put $l=\Lambda(H)$ and $m=\Lambda(C)$. Then the same assertions as above hold except that $j^2=-i$ $(\varepsilon=-1)$ instead of $j^2=i$ $(\varepsilon=1)$.

In §8, we realize representations classified in §§6 and 7, that is, we give standard orthonormal bases for V_0 and V_1, and write down the actions of $\mathfrak{g}_0$ and $\mathfrak{g}_1$ on V explicitly with respect to these bases.

In a forthcoming paper, we study representations of real forms of $\mathfrak{sl}(n, 1)$ and give a complete classification of irreducible unitary representations in case of $\mathfrak{sl}(2, 1)$ and $\mathfrak{sl}(3, 1)$. Further the classification problems for $\mathfrak{osp}(2n/1)$ will be discussed in another paper.

§1. Unitary representations of a Lie superalgebra.

1.1. Basic definitions. A Lie superalgebra over a field $K=\boldsymbol{R}$ or $\boldsymbol{C}$ is defined as a $\boldsymbol{Z}_2$-graded algebra $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ over K whose product (or bracket) operation satisfies the super-antisymmetry and the so-called Jacobi identity. More exactly, it satisfies

(1.1) $$[\mathfrak{g}_s, \mathfrak{g}_t]\subset\mathfrak{g}_{s+t} \quad (s, t\in\boldsymbol{Z}_2),$$

and

(1.2) $$\begin{aligned}&[X, Y] = -(-1)^{d(X)d(Y)}[Y, X],\\ &[X, [Y, Z]] = [[X, Y], Z]+(-1)^{d(X)d(Y)}[Y, [X, Z]],\end{aligned}$$

for any X, Y, Z in $\mathfrak{g}_0$ or $\mathfrak{g}_1$. Here $s+t$ is calculated in $\boldsymbol{Z}_2=\boldsymbol{Z}/2\boldsymbol{Z}=\{0, 1\}$, and $d(X)$ denotes the degree of X: $d(X)=s$ if $X\in\mathfrak{g}_s$ $(s=0, 1)$. We call $\mathfrak{g}_0$ and $\mathfrak{g}_1$ the even part and the odd part of $\mathfrak{g}$ respectively, and call an element in $\mathfrak{g}_0$ or $\mathfrak{g}_1$ homogeneous. Note that $\mathfrak{g}_0$ is a usual Lie algebra over K.

Let us give a simplest and fundamental example of Lie superalgebra. Let $V=V_0+V_1$ be a $\boldsymbol{Z}_2$-graded vector space over K, that is, V is a direct sum of subspaces V_0 and V_1. Then we define the Lie superalgebra $\mathfrak{l}(V)$, with underlying associative algebra $\mathfrak{gl}(V)$ of all linear transformations of V, as follows: the subspaces $\mathfrak{l}(V)_s$ with degree $s=0, 1$ are given by

(1.3) $$\mathfrak{l}(V)_s=\{X\in\mathfrak{gl}(V);\ XV_t\subset V_{s+t} \quad \text{for} \quad t=0, 1\},$$

and the bracket operation is

(1.4) $$[X, Y] = XY-(-1)^{d(X)d(Y)}\,YX \qquad \text{for} \quad X, Y\in\mathfrak{g}_0 \text{ or } \mathfrak{g}_1.$$

According to V. G. Kac, we define a representation of $\mathfrak{g}$ as follows. Let $V=V_0+V_1$ be a $\boldsymbol{Z}_2$-graded complex vector space, possibly of infinite dimension. A representation π of $\mathfrak{g}$ on V is by definition a homomorphism of $\mathfrak{g}$ into $\mathfrak{l}(V)$ as Lie superalgebras over K. This means that

(1.5) $$\pi(X)\,V_t\subset V_{s+t} \qquad \text{for} \quad X\in\mathfrak{g}_s\,(s, t\in\{0, 1\}),$$

(1.6) $$\pi([X, Y]) = [\pi(X), \pi(Y)] \quad (X, Y\in\mathfrak{g}),$$

with

(1.7) $$[\pi(X), \pi(Y)] = \pi(X)\,\pi(Y)-(-1)^{d(X)d(Y)}\,\pi(Y)\,\pi(X) \qquad \text{for} \quad X, Y\in\mathfrak{g}_0 \text{ or } \mathfrak{g}_1.$$

We call π *irreducible* (resp. *algebraically irreducible*) if any graded invariant subspace (resp. any invariant subspace) of V is equal to (0) or V.

For π, we have naturally two representations of the even part $\mathfrak{g}_0$, one on V_0 and the other on V_1. They are usual representations of the usual Lie algebra $\mathfrak{g}_0$. We denote them by $\pi(\mathfrak{g}_0)\,|\,V_0$ and $\pi(\mathfrak{g}_0)\,|\,V_1$ respectively and call π an extension to $\mathfrak{g}$ of each of them.

If we change the grading of V by exchanging the roles of V_0 and V_1 (i.e., $V=V_0'$

$+V'_1$ with $V'_0=V_1$, $V'_1=V_0$), then the roles of $\pi(\mathfrak{g}_0)|V_0$ and $\pi(\mathfrak{g}_0)|V_1$ are exchanged. Taking this in mind, we refer usually only $\pi(\mathfrak{g}_0)|V_0$ and say π an extension of $\pi(\mathfrak{g}_0)|V_0$, in the following.

Let (π', V'), $V'=V'_0+V'_1$, be another representation of $\mathfrak{g}$. Then (π, V) and (π', V') are said to be mutually *equivalent* if there exists a bijective linear map T of V onto V' such that

$$T\,V_s = V'_{s+t} \quad (s=0,\,1) \qquad \text{for a fixed } t;$$
$$T\circ\pi(X)\circ T^{-1} = \pi'(X) \quad (X\in\mathfrak{g})\,.$$

Note. For a representation (π, V), define $\pi_-(X)=(-1)^{d(X)}\,\pi(X)$ for any homogeneous $X\in\mathfrak{g}$. Then (π_-, V) is again a representation of $\mathfrak{g}$ which is equivalent to (π, V), without change of grading of V.

1.2. Representations contragradient or conjugate to π. Let (π, V), $V=V_0+V_1$, and (π', V'), $V'=V'_0+V'_1$, be two representations of $\mathfrak{g}$. Then π is said to be *contragradient* to π' if there exists a non-degenerate bilinear form $(\cdot,\cdot)$ on $V\times V'$ satisfying

$$(v, v') = 0 \quad \text{if} \quad v\in V_s,\ v'\in V'_t \qquad \text{for} \quad s\neq t;$$
$$(\pi(X)\,v, v')+i^{d(X)}\,(v, \pi'(X)\,v') = 0\,,$$

for any homogeneous element $X\in\mathfrak{g}$, where $i=\sqrt{-1}$. Note that, if we multiply $(\cdot,\cdot)|V_1\times V'_1$ by -1, then (π_-, V) is contragradient to (π', V') under this new pairing of V and V'.

Let (π, V), $V=V_0+V_1$, be as above. We define the conjugate vector space $\bar{V}=\bar{V}_0+\bar{V}_1$ changing only the scalar multiplication on V by its conjugate: put $\bar{V}=\{\bar{v};\ v\in V\}$ a set of symbols and define addition and scalar multiplication by

$$\bar{v}+\bar{v}' \equiv (v+v')^-\,, \quad \lambda\circ\bar{v} \equiv (\bar{\lambda}v)^-\,,$$

where $v, v'\in V$, $\lambda\in\boldsymbol{C}$, and $\bar{v}$ is denoted also by v^-. For a linear operator T on V, there corresponds uniquely that on $\bar{V}$, denoted by $\bar{T}$, as

$$\bar{T}\bar{v} \equiv (Tv)^- \quad (\bar{v}\in\bar{V} \quad \text{with} \quad v\in V)\,.$$

Then, as we see easily, the addition of two such operators and the scalar multiplication, denoted by $\bar{T}\to\lambda\circ\bar{T}$, are given as

$$(T+S)^- = \bar{T}+\bar{S}\,, \quad \lambda\circ\bar{T} = (\bar{\lambda}T)^-\,.$$

Thus, introducing canonically a $\boldsymbol{Z}_2$-gradation in $\bar{V}$ as $\bar{V}=\bar{V}_0+\bar{V}_1$, we get a representation $(\bar{\pi}, \bar{V})$ by $\bar{\pi}(X)=\overline{\pi(X)}$ $(X\in\mathfrak{g})$, which is said to be *conjugate* to (π, V).

1.3. Unitary representations. In accordance with some physicists, we call a representation (π, V) of $\mathfrak{g}$ *unitary* if V is equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$ satisfying the following:

(i) $V_0\perp V_1$ (orthogonal) under $\langle\cdot,\cdot\rangle$, and

(ii) $\langle\cdot,\cdot\rangle$ is $\mathfrak{g}$-invariant in the sense that

$$\langle i\pi(X)v, v'\rangle = \langle v, i\pi(X)v'\rangle \quad (v, v'\in V, X\in\mathfrak{g}_0), \tag{1.8}$$

$$\langle j\pi(Y)v, v'\rangle = \langle v, j\pi(Y)v'\rangle \quad (v, v'\in V, Y\in\mathfrak{g}_1), \tag{1.9}$$

where $i=\sqrt{-1}$ and j is a fixed forth root (depending only on π) of -1, i.e., $j^4=-1$. We call j^2 the *associated constant* for π since the essential thing is not j itself but $j^2=\varepsilon i$ with $\varepsilon=1$ or -1. Note that the second equality of the $\mathfrak{g}$-invariance of $\langle\cdot,\cdot\rangle$ is equivalent to the following:

$$j^2\langle\pi(Y)v, v'\rangle-\langle v, \pi(Y)v'\rangle = 0 \quad (Y\in\mathfrak{g}_1).$$

We call π *quasi-unitary* if, in the above definition, the condition "positive definite" is replaced by "positive semi-definite".

Let (π, V) be unitary, then its conjugate representation $(\bar\pi, \bar V)$ is again unitary if we introduce on $\bar V=\bar V_0+\bar V_1$ the inner product $\langle\cdot,\cdot\rangle^-$ canonically defined from $\langle\cdot,\cdot\rangle$: for $\bar v, \bar v'\in\bar V$ with $v, v'\in V$, put

$$\langle\bar v, \bar v'\rangle^- \equiv \langle v', v\rangle.$$

In this case, the associated constant for $\bar\pi$ is $\bar j^2$, because $\bar j\circ\bar\pi(X)=(j\pi(X))^-$ and so, for $X\in\mathfrak{g}_1$,

$$\begin{aligned}\langle\bar j\circ\bar\pi(X)\bar v, \bar v'\rangle^- &= \langle(j\pi(X)v)^-, \bar v'\rangle^- = \langle v', j\pi(X)v\rangle = \\ &= \langle j\pi(X)v', v\rangle = \langle\bar v, (j\pi(X)v')^-\rangle^- = \langle\bar v, \bar j\circ\bar\pi(X)\bar v'\rangle^-.\end{aligned}$$

Since $\bar j^2=-\varepsilon i$, there corresponds $-\varepsilon$ to $\bar\pi$, whereas ε to π.

Note that, in this case, we have a natural bilinear form on $V\times\bar V$ as

$$(v, \bar v') \equiv \langle v, v'\rangle \quad (v\in V, \bar v'\in\bar V \quad \text{with} \quad v'\in V),$$

and so, $(\bar\pi, \bar V)$ is contragradient, under $(\cdot,\cdot)$, to (π, V) or to (π_-, V) according as $\varepsilon=1$ or -1.

Remark 1.1. Physicists usually do not write down explicitly the multiplicative factors i or j in front of $\pi(X)$ or $\pi(Y)$, but they join them together with the latters, maybe because they are interested only in self-adjoint operators as physical objects under quantization, and so, to write down these factors explicitly is cumbersome for them. Thus what they call a representation of $\mathfrak{g}$ is usually something like $\tilde\pi$ given as follows: $\tilde\pi(X)=i\pi(X)$ for $X\in\mathfrak{g}_0$, and $\tilde\pi(Y)=j\pi(Y)$ for $Y\in\mathfrak{g}_1$. Therefore the property (1.6) of representation is rewritten as

$$\tilde\pi([X, Y]) = \frac{1}{i}[\tilde\pi(X), \tilde\pi(Y)]_- \tag{1.10}$$

$$\text{for homogeneous } X, Y\in\mathfrak{g} \text{ not both in } \mathfrak{g}_1,$$

$$\tilde\pi([X, Y]) = \varepsilon[\tilde\pi(X), \tilde\pi(Y)]_+ \qquad \text{for} \quad X, Y\in\mathfrak{g}_1, \tag{1.10'}$$

where $\varepsilon=\pm1$ with $j^2=\varepsilon i$. The essential thing is not j but j^2, as we remarked above, and it is not natural from mathematical point of view to discard the case $\varepsilon=-1$, as is done by some physicists (cf. [2, §3.2]). The second author discussed seriously on this point with C. Fronsdal when he was staying at RIMS, Kyoto Univ., 1983/84.

Remark 1.2. M. Duflo gave another formulation of unitarity in which there does not appear the forth root j. In discussion with him at Paris, May 1986, we found that our two different definitions of unitarity are equivalent to each other.

Remark 1.3. Let $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ be a Lie superalgebra. Its *dual* algebra $\mathfrak{g}^d=\mathfrak{g}_0^d+\mathfrak{g}_1^d$ is defined in [8, p. 98] as follows. We take $\mathfrak{g}_0^d=\mathfrak{g}_0$, $\mathfrak{g}_1^d=\mathfrak{g}_1$ as underlying vector spaces, and introduce the bracket operation $[\cdot,\cdot]^d$ as

$$[X, Y]^d=(-1)^{d(X)d(Y)}[X, Y]$$

for any homogeneous elements $X, Y\in\mathfrak{g}$. Note that $\mathfrak{g}^d$ is realized in the complexification $\mathfrak{g}_{\boldsymbol{C}}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}$ of $\mathfrak{g}$ as its real subalgebra $\mathfrak{g}_0+i\cdot\mathfrak{g}_1$, where $1\otimes X$ and $i\otimes Y$ are denoted by X and $i\cdot Y$ respectively.

Let (π, V) be a representation of $\mathfrak{g}$, then we get naturally a representation π^d of $\mathfrak{g}^d$ by putting

$$\pi^d(x)=i^{d(X)}\pi(X)\quad(i=\sqrt{-1})$$

for any homogeneous element $X\in\mathfrak{g}^d$. When π is unitary with the associated constant $j^2=\varepsilon i$, π^d is also unitary, with respect to the same inner product in V, and its associated constant is $\bar{j}^2=-\varepsilon i$.

Notation. For a vector space V, we put for $A, B\in\mathfrak{gl}(V)$, associative algebra,

$$[A, B]_{\pm}=AB\pm BA\,, \tag{1.11}$$

and we omit the suffices "$-$" if it does not cause any confusion.

§2. Problems of extensions of representations of the even part.

2.1. Extension problems. We propose some problems initially due to C. Fronsdal (cf. [2, §1.3]) and also to G. Zuckermann.

Problem 1 (Extensions of irreducible representations of $\mathfrak{g}_0$). Take an irreducible representation ρ of the even part $\mathfrak{g}_0$ on a complex vector space V_0. Then, do there exist any irreducible representations (π, V) of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ extending (ρ, V_0)? (More exactly, V_0 is imbedded into V as its subspace of degree 0, and ρ is equivalent to $\pi(\mathfrak{g}_0)|V_0$ under this embedding.) If they do exist, construct all of them.

When we study this problem for some types of simple Lie superalgebras, we recognize that the above extensions are not always possible or that the extensions become rather difficult to exist in general according as the dimension of $\mathfrak{g}$ increases. On the other hand, we encounter frequently an irreducible representations π of $\mathfrak{g}$ for which $\pi(\mathfrak{g}_0)|V_0$ is not irreducible. For instance, the adjoint representation of

$\mathfrak{g}$ itself is already in such a case for some simple $\mathfrak{g}$'s, eventhough it is finite-dimensional.

Thus, to study irreducible representations of $\mathfrak{g}$ in general or to classify all of them, we are forced to start from representations of $\mathfrak{g}_0$ not necessarily irreducible. Hence we come to the following problem generalizing Problem 1.

Problem 1bis. Take a representation (ρ, V_0) of $\mathfrak{g}_0$ not necessarily irreducible, and study its irreducible extensions to $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$. Especially, analyse the splitting of $\pi(\mathfrak{g}_0)\,|\,V_0$ into irreducibles for an irreducible representation (π, V), $V=V_0+V_1$, of $\mathfrak{g}$.

Note. From a physical point of view, a reducible (ρ, V_0) for an irreducible (π, V), $V=V_0+V_1$, means that many elementary particles live together in a closed physical world.

2.2. Unitary extension problems. As usual, we call a representation (ρ, V_0) of $\mathfrak{g}_0$ *(infinitesimally) unitary* if V_0 is equipped with a positive definite inner product $\langle\cdot, \cdot\rangle_0$ such that

$$\langle i\rho(X)\, v, v'\rangle_0 = \langle v, i\rho(X)\, v'\rangle_0 \quad (X\in\mathfrak{g}_0,\ v, v'\in V_0)\,. \tag{2.1}$$

It is a difficult problem to determine whether or not a unitary representation ρ of $\mathfrak{g}_0$ can be lifted up (or globalized) to a connected Lie group G_0 with Lie algebra $\mathfrak{g}_0$ (cf. [13, §9]). Putting this problem aside, we propose the following extension problem of unitary representations on the algebra level. For convenience of later references, we list up the problem for irreducible ρ separately.

Problem 2 (Extensions of irreducible unitary representations). Let (ρ, V_0) be an irreducible unitary $\mathfrak{g}_0$-module. Then do there exist any irreducible unitary extensions of (ρ, V_0) to $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$? If any, in which different ways can we extend it?

Problem 2bis. How about the case where ρ is no longer irreducible? In particular, study the branching rule for an irreducible representation π of $\mathfrak{g}$ into irreducibles when it is restricted to the even part $\mathfrak{g}_0$.

2.3. Case of reductive $\mathfrak{g}_0$. Now we restrict ourselves to more specialized situation which we will treat in the following. Assume that $\mathfrak{g}_0$ be a real reductive Lie algebra. Let G_0 be a connected Lie group corresponding to $\mathfrak{g}_0$, and K_0 the analytic subgroup of G_0 corresponding to a maximal compact subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$. We call a $\mathfrak{g}_0$-module (ρ, V_0) an admissible $(\mathfrak{g}_0, K_0)$-module (or Harish-Chandra module) if it satisfies the following conditions.

(i) $\rho(\mathfrak{k}_0)$ on V_0 is decomposed into a direct sum of finite-dimensional irreducible representations of $\mathfrak{k}_0$, which can be lifted up to K_0, with finite multiplicities.

(ii) V_0 is finitely generated as a $\mathfrak{g}_0$-module.

From the results in [13] and [1], we know that any unitarizable admissible $(\mathfrak{g}_0, K_0)$-module correspond canonically to a unitary representation of G_0, which is a finite direct sum of irreducible ones. Moreover, irreducible unitarizable $(\mathfrak{g}_0, K_0)$-

modules correspond one-to-one way (up to equivalence) to irreducible unitary representations of G_0. Here the equivalence for such $(\mathfrak{g}_0, K_0)$-modules is in purely algebraic sense. Furthermore, when the center of G_0 is finite, irreducible $(\mathfrak{g}_0, K_0)$-modules, not necessarily unitary, correspond one-to-one way (up to equivalence) to quasi-simple irreducible representations of G_0 on Hilbert spaces (cf. for instance, [1, §8]).

In later sections, we will study Problems 1bis and 2bis for a real form $\mathfrak{g}=\mathfrak{osp}(2/1)$ of $\mathfrak{osp}(1, 2)$ with $\mathfrak{g}_0=\mathfrak{sp}(2; \boldsymbol{R})$, and Problem 2 for those of $\mathfrak{sl}(2, 1)$ with $\mathfrak{g}_0=\mathfrak{u}(2)$ or $\mathfrak{u}(1, 1)$ respectively. For $\mathfrak{g}=\mathfrak{osp}(2n/1)$, a real form of $\mathfrak{osp}(1, 2n)$ of type $B(0, n)$ in the classification in [8], see also [6].

§3. Equations for an irreducible extensions.

In this section, we consider Problems 1 and 1bis and obtain a system of equations to solve these problems. Hereafter we use Greek letters $\xi, \eta, \cdots$, to denote elements in $\mathfrak{g}_1$ when it is convenient to distinguish them from elements in $\mathfrak{g}_0$.

3.1. Conditions for irreducibility. Let (π, V), $V=V_0+V_1$, be a representation of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$. We define for $\xi, \eta\in\mathfrak{g}_1$ a linear mapping $B(\xi, \eta)$ of V_0 into itself by

$$B(\xi, \eta)\, v = \pi(\xi)\, \pi(\eta)\, v \quad (v\in V_0)\,. \tag{3.1}$$

Then $B(\cdot, \cdot)$ is a bilinear mapping from $\mathfrak{g}_1\times\mathfrak{g}_1$ into $\mathfrak{gl}(V_0)$, which plays a decisive role in the following. We extend B by linearity to a complex bilinear map: $\mathfrak{g}_{1,\boldsymbol{C}}\times\mathfrak{g}_{1,\boldsymbol{C}}\to\mathfrak{gl}(V_0)$, where $\mathfrak{g}_{1,\boldsymbol{C}}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}_1$.

Let us first study the irreducibility of π. Denote by $\pi(\mathfrak{g}_1)\, V_0$ the subspace of V_1 spanned by $\{\pi(\xi)\, v;\, \xi\in\mathfrak{g}_1, v\in V_0\}$. Then $V_1'=\pi(\mathfrak{g}_1)\, V_0\subset V_1$ is a $\mathfrak{g}_0$-submodule, and $V'=V_0+V_1'$ is a $\mathfrak{g}$-submodule of V. Moreover put

$$M = \{v\in V_1;\, \pi(\xi)\, v = 0 \quad (\xi\in\mathfrak{g}_1)\}\,, \tag{3.2}$$

then it is $\mathfrak{g}_0$-invariant, and hence $V'=V_0'+M$ with $V_0'=(0)\subset V_0$ is a $\mathfrak{g}$-invariant subspace of V. Thus, we see that when (π, V) is irreducible, it necessarily has the following properties:

(PRO1) $\pi(\mathfrak{g}_1)\, V_0=V_1$, where $\pi(\mathfrak{g}_1)\, V_0$ denotes the linear span of $\{\rho(X)\, v;\, X\in\mathfrak{g}_1, v\in V_0\}$;

(PRO2) $M=(0)$, namely, an element $v_1\in V_1$ is equal to 0 if and only if $\pi(\eta)\, v_1=0$ for any $\eta\in\mathfrak{g}_1$.

Further put $\rho=\pi(\mathfrak{g}_0)\,|\, V_0$, and denote by $\langle\rho(\mathfrak{g}_0), B(\mathfrak{g}_1, \mathfrak{g}_1)\rangle$ the subalgebra of $\mathfrak{gl}(V_0)$ generated by $\{\rho(X), B(\xi, \eta);\, X\in\mathfrak{g}_0, \xi, \eta\in\mathfrak{g}_1\}$. Then we have the following criterion of irreducibility.

Lemma 3.1. *Let* (π, V), $V=V_0+V_1$, *be a representation of* $\mathfrak{g}$. *Then it is irreducible if and only if it has the properties* (PRO1), (PRO2) *and*

(PRO3) *the subalgebra* $\langle\pi(\mathfrak{g}_0), B(\mathfrak{g}_1, \mathfrak{g}_1)\rangle$ *of* $\mathfrak{gl}(V_0)$ *acts on* V_0 *irreducibly.*

Proof. The necessity of the property (PRO3) is easy to see. Hence we prove the sufficiency of these properties.

Let U be a non-zero graded invariant subspace of V: $U=U_0+U_1$ with $U_s=U\cap V_s$ $(s=0,\ 1)$. If $U_0\neq(0)$, then $U_0\supset\langle\rho(\mathfrak{g}_0),\ B(\mathfrak{g}_1,\ \mathfrak{g}_1)\rangle\, U_0=V_0$ by (PRO3) and so $U_0=V_0$. If $U_1\neq(0)$, then $(0)\neq\pi(\mathfrak{g}_1)\, U_1\subset U_0$ by (PRO2), whence $U_0\neq(0)$ and so we have $U_0=V_0$ as above. Thus in each case, $U_0=V_0$ and so $U_1\supset\pi(\mathfrak{g}_1)\, V_0=V_1$ by (PRO1), whence $U=V$. This means that π is irreducible. Q.E.D.

Corollary 3.2. *Let $(\pi,\ V)$ be a representation of $\mathfrak{g}$ as above. Assume that $\rho=\pi(\mathfrak{g}_0)\,|\,V_0$ is irreducible. Then π is irreducible if and only if it has the properties* (PRO1) *and* (PRO2).

Corollary 3.3. *Let $(\rho,\ V_0)$ be a representation of $\mathfrak{g}_0$, and $(\pi,\ V)$, $V=V_0+V_1$, its extension to $\mathfrak{g}$ with properties* (PRO1) *and* (PRO2), *or especially an irreducible extension. Then*

(i) *any element of V_1 is expressed as a linear combination $\sum_i \pi(\xi_i)\, v^i$ with $\xi_i\in\mathfrak{g}_1$, $v^i\in V_0$, and*

(ii) *this linear combination is equal to* 0 *if and only if*

$$\sum_i B(\eta,\ \xi_i)\, v^i = 0 \qquad \textit{for any} \quad \eta\in\mathfrak{g}_1\,. \tag{3.3}$$

3.2. Bilinear map *B*. Let us now prove that the bilinear map B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$, together with ρ on V_0, determines completely the extension π (up to equivalence) if π has the properties (PRO1) and (PRO2), hence especially if π is irreducible.

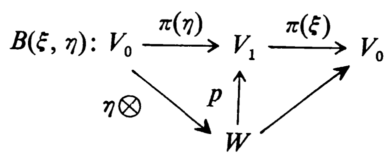
Consider the complexification $\mathfrak{g}_{1,C}$ of $\mathfrak{g}_1$ as $\mathfrak{g}_0$-module and put $W=\mathfrak{g}_{1,C}\otimes_C V_0$, the tensor product as $\mathfrak{g}_0$-module. Let p be the canonical $\mathfrak{g}_0$-homomophism of W into V_1 given by

$$p\colon\ W\ni\xi\otimes v\mapsto\pi(\xi)\, v\in V_1 \quad (\xi\in\mathfrak{g}_1,\ v\in V_0)\,. \tag{3.4}$$

Then p is surjective because of (PRO1). Further, by Corollary 3.3, the kernel $\mathfrak{m}$ of p is given by

$$\mathfrak{m} = \{\textstyle\sum_i \xi_i\otimes v^i;\ \xi_i\in\mathfrak{g}_1,\ v^i\in V_0 \ \text{ such that (3.3) holds}\}\ . \tag{3.5}$$

Let $\eta\in\mathfrak{g}_1$, then the map $\pi(\eta)$: $V_0\to V_1$, factors through $\eta\otimes$: $V_0\ni v\mapsto\eta\otimes v\in W$, as follows, and similarly the map $B(\xi,\ \eta)=\pi(\xi)\,\pi(\eta)\,|\,V_0\in\mathfrak{gl}(V_0)$:



Put $\tilde{W}=W/\mathfrak{m}$ and denote by $[w]$ the element in $\tilde{W}$ represented by $w\in W$. Then $\tilde{W}\cong V_1$ as $\mathfrak{g}_0$-modules through p. We define an action of $\mathfrak{g}_1$ on $\tilde{W}$ as follows: for $\xi\in\mathfrak{g}_1$, (3.6) $$\tilde{W}\ni[\eta\otimes v]\mapsto B(\xi,\eta)\,v\in V_0\quad(\eta\in\mathfrak{g}_1,\ v\in V_0)\,.$$

Then this is well-defined because the kernel $\mathfrak{m}$ is given by (3.5). Thus we get a canonical realization of an extension (π, V) with properties (PRO1), (PRO2), by the following method.

Method of construction using a bilinear map $B: \mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$.

(MET1) Take $W=\mathfrak{g}_{1,\boldsymbol{C}}\otimes_{\boldsymbol{C}} V_0$ as $\mathfrak{g}_0$-module, and determine its submodule $\mathfrak{m}$ by (3.5).

(MET2) Take $\tilde{W}=W/\mathfrak{m}$ as the subspace V_1 of degree 1, and put $V=V_0+V_1$. Define the action of $\xi\in\mathfrak{g}_1$ on $V_1=\tilde{W}$ by (3.6) and that on V_0 by

(3.7) $$V_0\ni v\mapsto[\xi\otimes v]\in\tilde{W}\,.$$

3.3. Equations for the map $B: \mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$. Let (ρ, V_0) be a not necessarily irreducible representation of $\mathfrak{g}_0$. We see above that an extension (π, V), $V=V_0+V_1$, of it is determined by a bilinear map $B: \mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$ if π has properties (PRO1) and (PRO2), or especially if π is irreducible. Let us study conditions for B to be satisfied.

First we list up the representation property (1.6) in three cases

(3.9) $$\pi([X,Y])=\pi(X)\,\pi(Y)-\pi(Y)\,\pi(X)\quad(X,Y\in\mathfrak{g}_0);$$

(3.10) $$\pi([X,\xi])=\pi(X)\,\pi(\xi)-\pi(\xi)\,\pi(X)\quad(X\in\mathfrak{g}_0,\ \xi\in\mathfrak{g}_1);$$

(3.11) $$\pi([\xi,\eta])=\pi(\xi)\,\pi(\eta)+\pi(\eta)\,\pi(\xi)\quad(\xi,\eta\in\mathfrak{g}_1)\,.$$

For simplicity, the cananical action of X on ξ is denoted as ${}^X\xi=[X,\xi]$. We write down the above equalities for $v\in V_0$ and $v_1=\pi(\zeta)\,v\in V_1$ $(\zeta\in\mathfrak{g}_1,\ v\in V_0)$.

(3.9.0) $$\rho([X,Y])=\rho(X)\,\rho(Y)-\rho(Y)\,\rho(X)\,,$$

(3.9.1) $$\pi([X,Y])\,\pi(\zeta)\,v=\pi(X)\,\pi(Y)\,\pi(\zeta)\,v-\pi(Y)\,\pi(X)\,\pi(\zeta)\,v;$$

(3.10.0) $$\pi({}^X\xi)\,v=\pi(X)\,\pi(\xi)\,v-\pi(\xi)\,\rho(X)\,v\,,$$

(3.10.1) $$B({}^X\xi,\zeta)\,v=\rho(X)\,B(\xi,\zeta)\,v-\pi(\xi)\,\pi(X)\,\pi(\zeta)\,v;$$

(3.11.0) $$\rho([\xi,\eta])=B(\xi,\eta)+B(\eta,\xi)\,,$$

(3.11.1) $$\pi([\xi,\eta])\,\pi(\zeta)\,v=\pi(\xi)\,\pi(\eta)\,\pi(\zeta)\,v+\pi(\eta)\,\pi(\xi)\,\pi(\zeta)\,v\,.$$

From (3.10.0), we have

(3.12) $$\pi(X)\,\pi(\zeta)\,v=\{\pi({}^X\zeta)+\pi(\zeta)\,\rho(X)\}\,v\,.$$

Apply this to the right hand side of (3.9.1), then we get

$$\pi([X,Y])\,\pi(\zeta)\,v=\{\pi({}^{[X,Y]}\zeta)+\pi(\zeta)\,\rho([X,Y])\}\,v\,.$$

Therefore (3.9.1) follows from (3.12) and (3.9.0). Again apply (3.12) to (3.10.1) and

(3.11.1), and further apply $\pi(\tau)$, $\tau\in\mathfrak{g}_1$, to the both sides of (3.11.1), then we get respectively

$$B({}^X\xi, \zeta) = \rho(X)\, B(\xi, \eta) - B(\xi, {}^X\zeta) - B(\xi, \zeta)\, \rho(X)\,, \tag{3.10.1'}$$

$$B(\tau, {}^{[\xi,\eta]}\zeta) + B(\tau, \zeta)\, \rho([\xi, \eta]) = \tag{3.11.1'}$$
$$= B(\tau, \xi)\, B(\eta, \zeta) + B(\tau, \eta)\, B(\xi, \zeta)\,.$$

Now we see that (3.12) shows how $X\in\mathfrak{g}_0$ operates on $v_1=\pi(\zeta)\, v\in V_1$ and that it corresponds exactly to the $\mathfrak{g}_0$-action on $\tilde{W}=W/\mathfrak{m}$, $W=\mathfrak{g}_{1,C}\otimes_C V_0$. Further we see that, under (3.12), the system of equations (3.9)-(3.11) for π with (PRO1) and (PRO2), is equivalent to the following one:

(EXT1) $\quad B({}^X\xi, \eta) + B(\xi, {}^X\eta) = [\rho(X), B(\xi, \eta)] \quad (X\in\mathfrak{g}_0,\ \xi, \eta\in\mathfrak{g}_1)\,,$

(EXT2) $\quad B(\xi, \eta) + B(\eta, \xi) = \rho([\xi, \eta]) \quad (\xi, \eta\in\mathfrak{g}_1)\,,$

(EXT3) $\quad B(\tau, \xi)\, B(\eta, \zeta) + B(\tau, \eta)\, B(\xi, \zeta) =$
$\qquad = B(\tau, {}^{[\xi,\eta]}\zeta) + B(\tau, \zeta)\, \rho([\xi, \eta]) \quad (\tau, \xi, \eta, \zeta\in\mathfrak{g}_1)\,,$

where in the right hand side of (EXT1)

$$[C, D] = CD - DC \qquad \text{for} \quad C, D\in\mathfrak{gl}(V_0)\,. \tag{3.13}$$

Note that $C\mapsto[\rho(X), C]$ $(C\in\mathfrak{gl}(V_0))$ gives a natural $\mathfrak{g}_0$-module structure on $\mathfrak{gl}(V_0)$. Then the condition (EXT1) says that the bilinear map B, extended by linearity,

$$B\colon \mathfrak{g}_{1,C}\times\mathfrak{g}_{1,C}\ni(\xi, \eta)\mapsto B(\xi, \eta)\in\mathfrak{gl}(V_0)\,,$$

is a $\mathfrak{g}_0$-homomorphism of $\mathfrak{g}_{1,C}\times\mathfrak{g}_{1,C}$ into $\mathfrak{gl}(V_0)$.

Now we can state a theorem which is fundamental for our later study.

Theorem 3.4. *Let (ρ, V_0) be a representation of the even part $\mathfrak{g}_0$ of $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$, not necessarily irreducible.*

(i) *Let (π, V), $V=V_0+V_1$, be an extension of (ρ, V_0) to $\mathfrak{g}$, having properties* (PRO1) *and* (PRO2). *Put for $\xi, \eta\in\mathfrak{g}_1$,*

$$B(\xi, \eta)\, v = \pi(\xi)\, \pi(\eta)\, v \quad (v\in V_0)\,. \tag{3.14}$$

Then B satisfies the system of equations (EXT1)-(EXT3).

(ii) *Conversely, assume that we are given a bilinear map B from $\mathfrak{g}_1\times\mathfrak{g}_1$ into $\mathfrak{gl}(V_0)$, which satisfies* (EXT1)-(EXT3). *Put $W=\mathfrak{g}_{1,C}\otimes_C V_0$ and define its $\mathfrak{g}_0$-submodule $\mathfrak{m}$ by* (3.5). *Take $\tilde{W}=W/\mathfrak{m}$ as the space V_1 of degree* 1, *and define $\mathfrak{g}_1$-action on $V=V_0+V_1$ by* (3.6)-(3.7). *Then we get an extension (π, V) of (ρ, V_0) with properties* (PRO1), (PRO2). *Moreover any such extension can be obtained in this way up to equivalence.*

Proof. The assertion (i) has been already proved. For the assertion (ii), it rests only to prove that we get from (EXT1)–(EXT3) the representation property (1.6). As an example, take (EXT3). Then by the definition of $\mathfrak{m}$, we get on V_0 the following equality

$$\pi(\xi)\, B(\eta, \zeta)+\pi(\eta)\, B(\xi, \zeta) = \pi({}^{[\xi,\eta]}\zeta)+\pi(\zeta)\,\rho([\xi, \eta])\,.$$

This, together with (3.12), gives the equality (3.11) on V_1. Other details are ommitted here because they are a kind of repetition of former arguments. Q.E.D.

Corollary 3.5. *Let* (ρ, V_0) *be an irreducible representation of* $\mathfrak{g}_0$. *Then, any irreducible extension* (π, V) *of it can be obtained, up to equivalence, canonically from a bilinear map* B: $\mathfrak{g}_1\times\mathfrak{g}_1\rightarrow\mathfrak{gl}(V_0)$, *which satisfies* (EXT1)–(EXT3). *Here "canonically" means "by the method* (MET1)–(MET2)".

3.4. Algebraic irreducibility. Let us give here some remarks about two kinds of irreducibility.

Lemma 3.6. *Let* (π, V), $V=V_0+V_1$, *be a representation of* $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$. *Then it is algebraically irreducible if* (1) *it is irreducible* (*as a representation of a Lie superalgebra*), *and* (2) *any intertwining operator from a* $\mathfrak{g}_0$*-invariant subspace of* V_s *into* V_{s+1}, *as* $\mathfrak{g}_0$*-modules, is trivial for* $s=0$ *or* 1.

Proof. Assume that (1) and (2) hold for (π, V). Let $U\subset V$ be a non-zero $\mathfrak{g}$-invariant subspace of V. Take a non-zero $u\in U$ and express it as $u=u_0+u_1$ $(u_s\in V_s)$. If $u_0=0$ or $u_1=0$, then $u\in V_1$ or $u\in V_0$, whence we get $U=V$ from (1). So we assume $u_0\neq 0$, $u_1\neq 0$. Then we see from (2) that there exists a $Z\in U(\mathfrak{g}_{0,C})$ such that $Zu_s=0$, $Zu_{s+1}\neq 0$ for the s in the lemma, where $U(\mathfrak{g}_{0,C})$ denotes the enveloping algebra of $\mathfrak{g}_{0,C}$. Hence $U\cap V_{s+1}$ contains a non-zero element $Zu=Zu_{s+1}$, and so we get $U=V$ by (1).

Thus we see that π is algebraically irreducible. Q.E.D.

Remark 3.7. The above sufficient condition for algebraic irreducibility is not so special but rather general. In fact, in many cases, V_0 and $W=\mathfrak{g}_{1,C}\otimes_C V_0$ have no irreducible components of $\mathfrak{g}_0$ in common, and so do V_0 and $V_1\cong\tilde{W}=W/\mathfrak{m}$ (see for instance later sections §§5–8).

3.5. Equations for the map A: $\mathfrak{g}_1\wedge\mathfrak{g}_1\rightarrow\mathfrak{gl}(V_0)$. When we apply the system of equations (EXT1)–(EXT3) to certain types of simple Lie superalgebras, it is more convenient to use, instead of $B(\cdot, \cdot)$, a skew-symmetric bilinear map $A(\cdot, \cdot)$: for $\xi, \eta\in\mathfrak{g}_1$,

$$A(\xi, \eta) = B(\xi, \eta)-B(\eta, \xi)\,. \tag{3.15}$$

We extend A by linearity to a complex linear map $\mathfrak{g}_{1,C}\times\mathfrak{g}_{1,C}\rightarrow\mathfrak{gl}(V_0)$, if necessary.

Let us rewrite the system of equations (EXT1)–(EXT3) on B by means of A. First of all, (EXT2) is equivalent to the skew-symmetricity of A and

$$B(\xi, \eta) = \frac{1}{2}(\rho([\xi, \eta])+A(\xi, \eta)) \quad (\xi, \eta\in\mathfrak{g}_1)\,. \tag{3.16}$$

Therefore (EXT2) is dissolved into the condition that the bilinear map A is skew-symmetric.

Next, (EXT1) is equivalent to the condition that the map A gives a $\mathfrak{g}_0$-homomorphism of $\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}$, the exterior product of $\mathfrak{g}_{1,C}$ with $\mathfrak{g}_{1,C}$, into $\mathfrak{gl}(V_0)$:

$$\text{(EXT1A)} \qquad A({}^X\xi, \eta)+A(\xi, {}^X\eta) = [\rho(X), A(\xi, \eta)] \quad (X\in\mathfrak{g}_0,\ \xi, \eta\in\mathfrak{g}_1)\,.$$

Assuming (EXT1A) or equivalently (EXT1), we get from (EXT3) two equations (EXT3A$_+$), (EXT3A$_-$) as follows. First rewrite the right hand side of the equation (EXT3) by using (EXT1), then we get

$$\text{(3.17)} \qquad \begin{aligned} &B(\tau, \xi)\, B(\eta, \zeta)+B(\tau, \eta)\, B(\xi, \zeta) = \\ &= -B({}^{[\xi,\eta]}\tau, \zeta)+\rho([\xi, \eta])\, B(\tau, \zeta)\,. \end{aligned}$$

Then, exchanging τ and ζ in (EXT3) and (3.17) above, we get respectively

$$\text{(3.17}'\text{)} \qquad \begin{aligned} &B(\zeta, \xi)\, B(\eta, \tau)+B(\zeta, \eta)\, B(\xi, \tau) = \\ &= B(\zeta, {}^{[\xi,\eta]}\tau)+B(\zeta, \tau)\, \rho([\xi, \eta])\,, \end{aligned}$$

$$\text{(3.17}''\text{)} \qquad \begin{aligned} &B(\zeta, \xi)\, B(\eta, \tau)+B(\zeta, \eta)\, B(\xi, \tau) = \\ &= -B({}^{[\xi,\eta]}\zeta, \tau)+\rho([\xi, \eta])\, B(\zeta, \tau)\,. \end{aligned}$$

Adding four equations (EXT3), (3.17)–(3.17″), side by side, we get

$$\text{(EXT3A}_+\text{)} \quad \begin{aligned} &[A(\tau, \xi), A(\eta, \zeta)]_+ + [A(\tau, \eta), A(\xi, \zeta)]_+ \\ &+[\rho([\tau, \xi]), A(\eta, \zeta)]+[\rho([\tau, \eta]), A(\xi, \zeta)] \\ &+[\rho([\xi, \zeta]), A(\eta, \tau)]+[\rho([\eta, \zeta]), A(\xi, \tau)] \\ &+[\rho([\tau, \xi]), \rho([\eta, \zeta])]_+ + [\rho([\tau, \eta]), \rho([\xi, \zeta])]_+ \\ &= 2A(\tau, {}^{[\xi,\eta]}\zeta)+2A(\zeta, {}^{[\xi,\eta]}\tau)+2[\rho([\tau, \zeta]), \rho([\xi, \eta])]_+\,, \end{aligned}$$

where, for $C, D\in\mathfrak{gl}(V_0)$,

$$[C, D]_+ = CD+DC\,, \quad [C, D] = CD-DC\,.$$

Now, we add (EXT3) and (3.17), and deduct (3.17′), (3.17″) from it, side by side. Then we obtain

$$\text{(EXT3A}_-\text{)} \quad \begin{aligned} &[A(\tau, \xi), A(\eta, \zeta)]+[A(\tau, \eta), A(\xi, \zeta)] \\ &+[\rho([\tau, \xi]), A(\eta, \zeta)]_+ + [\rho([\tau, \eta]), A(\xi, \zeta)]_+ \\ &+[\rho([\eta, \zeta]), A(\tau, \xi)]_+ + [\rho([\xi, \zeta]), A(\tau, \eta)]_+ \\ &+[\rho([\tau, \xi]), \rho([\eta, \zeta])]+[\rho([\tau, \eta]), \rho([\xi, \zeta])] \\ &= 2[\rho([\xi, \eta]), A(\tau, \zeta)]_+ + 2\rho([\tau, {}^{[\xi,\eta]}\zeta])-2\rho([\zeta, {}^{[\xi,\eta]}\tau])\,. \end{aligned}$$

Note that the equation (EXT3A$_+$) is symmetric under the permutations $\xi\leftrightarrow\eta$ and $\tau\leftrightarrow\zeta$, and that the equation (EXT3A$_-$) is symmetric under $\xi\leftrightarrow\eta$ and skew-symmetric under $\tau\leftrightarrow\zeta$.

The system of equations (EXT1)–(EXT3) on B is now rewritten by means of A as follows.

Theorem 3.8. *Let* (ρ, V_0) *be a representation of the even part* $\mathfrak{g}_0$ *of* $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$.

(i) *Assume that a bilinear map* B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$, *satisfies the system of equations* (EXT1)–(EXT3). *Then a map* A: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$, *defined by* (3.15) *is skew-symmetric, and it satisfies the system of equations* (EXT1A), (EXT3A$_+$), (EXT3A$_-$).

(ii) *Conversely, assume that a skew-symmetric map* A: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$, *satisfies* (EXT1A), (EXT3A$_+$), (EXT3A$_-$). *Then the map* B *defined by* (3.16) *satisfies* (EXT1)–(EXT3).

Proof. It rests only to prove the assertion (ii). For this, it is enough to note that (3.17) is equivalent to (EXT3) if we assume (EXT1A) which is equivalent to (EXT1). Q.E.D.

Notation. The bilinear map A can be considered as a complex linear map from the exterior product $\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}$ into $\mathfrak{gl}(V_0)$. In the following, when we consider it in this way, we denote, by abuse of notation, $A(\xi, \eta)$ also by $A(\xi\wedge\eta)$, and further use the notation $A(z)$ for $z\in\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}$. Similarly we denote $B(\xi, \eta)$ also by $B(\xi\otimes\eta)$ and so on.

3.6. Reduction of (EXT3) by the $\mathfrak{g}_0$-equivariance. Let us reduce the system of equations (EXT1)–(EXT3) to more simpler one, using $\mathfrak{g}_0$-equivariance property. Here we take (EXT3).

Let B be a bilinear map from $\mathfrak{g}_1\times\mathfrak{g}_1$ to $\mathfrak{gl}(V_0)$. First, assuming (EXT1) for B, we reduce (EXT3). Taking into account the form of (EXT3), we define a linear map P_B from $\mathfrak{g}_1^{(4)}=\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}$ to $\mathfrak{gl}(V_0)$ as follows: for $\tau\otimes\xi\otimes\eta\otimes\zeta$ with $\tau, \xi, \eta, \zeta\in\mathfrak{g}_1$,

$$
\begin{aligned}
(3.18)\qquad P_B(\tau\otimes\xi\otimes\eta\otimes\zeta) &= B(\tau, \xi)\, B(\eta, \zeta)+B(\tau, \eta)\, B(\xi, \zeta) \\
&\quad -B(\tau, {}^{[\xi,\eta]}\zeta)-B(\tau, \zeta)\, \rho([\xi, \eta])\,.
\end{aligned}
$$

We denote by $\xi\mapsto{}^X\xi$ the natural action of $X\in\mathfrak{g}_0$ on $\xi\in\mathfrak{g}_{1,C}$, and similarly that on $u\in\mathfrak{g}_1^{(4)}$ by $u\mapsto{}^Xu$. Then we have the following

Lemma 3.9. *Assume that* B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$ *satisfies* (EXT1), *that is,* B *is* $\mathfrak{g}_0$-*equivariant. Then* P_B: $\mathfrak{g}_1^{(4)}\to\mathfrak{gl}(V_0)$ *is also* $\mathfrak{g}_0$-*equivariant*:

$$(3.19)\qquad P_B({}^Xu) = [\rho(X), P_B(u)] \quad (u\in\mathfrak{g}_1^{(4)}, X\in\mathfrak{g}_0)\,.$$

Moreover, denote by S_{pq} the automorphism of $\mathfrak{g}_1^{(4)}$ exchanging the p-th and q-th factors in decomposable vectors, for instance,

$$(3.20)\qquad S_{23}(\tau\otimes\xi\otimes\eta\otimes\zeta) = \tau\otimes\eta\otimes\xi\otimes\zeta \quad (\tau, \eta, \xi, \zeta\in\mathfrak{g}_1)\,.$$

Then we have from the definition of P_B that

$$(3.21) \qquad P_B(S_{23}\,u) = P_B(u) \quad (u \in \mathfrak{g}_1^{(4)}) .$$

Denote by $\langle U(\mathfrak{g}_{0,C}), S_{23}\rangle$ the direct product of algebras $U(\mathfrak{g}_{0,C})$ and $\langle S_{23}\rangle = \boldsymbol{C}+\boldsymbol{C}\cdot S_{23}$. We make S_{23} act on $\mathfrak{gl}(V_0)$ as the identical transformation, then (3.19) and (3.21) says that P_B is a $\langle U(\mathfrak{g}_{0,C}), S_{23}\rangle$-homomorphism from $\mathfrak{g}_1^{(4)}$ to $\mathfrak{gl}(V_0)$. Therefore we get the following

Lemma 3.10. *Assume that* B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$ *satisfies* (EXT1). *Let* $\{u_1, u_2, \cdots, u_M\}\subset\mathfrak{g}_1^{(4)}$ *be a subset which generates the whole space* $\mathfrak{g}_1^{(4)}$ *as* $\langle U(\mathfrak{g}_{0,C}), S_{23}\rangle$-*module. Then, under* (EXT1), *the equation* (EXT3) *on* B *is equivalent to the following system of equations on* B:

$$(\mathrm{EXT3}^*) \qquad P_B(u_j) = 0 \quad (1 \leqq j \leqq M) .$$

Proof. As is shown above, under the condition (EXT1) which says that B is $\mathfrak{g}_0$-equivariant, the map P_B is a $\langle U(\mathfrak{g}_{0,C}), S_{23}\rangle$-homomorphism from $\mathfrak{g}_1^{(4)}$ to $\mathfrak{gl}(V_0)$. This gives our assertion immediately. Q.E.D.

Similar reduction can be carried out for equations (EXT3A$_+$) and (EXT3A$_-$), this time using $\langle S_{14}, S_{23}\rangle$ instead of $\langle S_{23}\rangle$ (cf. a remark just before Theorem 3.8).

3.7. Reduction of (EXT1). We now reduce the equation (EXT1) to more simple one. First note that (EXT1) is equivalent to (EXT1A) which says that the map A: $\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}\to\mathfrak{gl}(V_0)$ is $\mathfrak{g}_0$-equivariant. Let us take a system of generators $\{z_1, z_2, \cdots, z_N\}$ of $\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}$ as $\mathfrak{g}_0$-module. Then the map A is uniquely determined by its values on these generators, that is, by the system of operators $\{A_k = A(z_k)\in \mathfrak{gl}(V_0);\ 1\leqq k\leqq N\}$.

Conversely we have the following

Lemma 3.11. *Assume that we are given a system of operators* $\{A_k\in\mathfrak{gl}(V_0);\ 1\leqq k\leqq N\}$. *Put* $A(z_k)=A_k$ *for* $1\leqq k\leqq N$. *Then it can be extended to a* $\mathfrak{g}_0$-*homomorphic linear map* A: $\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}\to\mathfrak{gl}(V_0)$ *if and only if it satisfies the following condition*:

(EXT1*) *if* $\sum_{1\leqq k\leqq N} {}^{x_k}z_k=0$ *with* $x_k\in U(\mathfrak{g}_{0,C})$, *then necessarily*

$$(3.22) \qquad \sum_{1\leqq k\leqq N} {}^{x_k}A_k = 0 ,$$

where the action of $x_k\in U(\mathfrak{g}_{0,C})$ *on* $A_k\in\mathfrak{gl}(V_0)$ *is canonically induced from the action of* $X\in\mathfrak{g}_0$: $\mathfrak{gl}(V_0)\ni C\mapsto[\rho(X), C]\in\mathfrak{gl}(V_0)$. *In particular, if* $\sum_{1\leqq k\leqq N} {}^{X_k}z_k=0$ *with* $X_k\in\mathfrak{g}_0$, *then*

$$(3.23) \qquad \sum_{1\leqq k\leqq N} [\rho(X_k), A_k] = 0 .$$

We note here that, when $A(z)$ is given for a $z\in\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C}$, the corresponding value of $B(\cdot, \cdot)$ is defined as follows: express z as $z=\sum_m \xi_m\wedge\eta_m$, and put $\underline{z}=\sum_m \xi_m\otimes\eta_m\in\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}$, then

$$(3.24) \qquad B(\underline{z}) = \frac{1}{2}\,(\rho(\underline{z})+A(z)) ,$$

where, by definition,

$$B(\underline{z}) = \sum_m B(\xi_m, \eta_m); \quad \rho(\underline{z}) = \sum_m \rho([\xi_m, \eta_m]) .$$

Thus, defining B and putting it into (EXT3*), we get a system of equations on $\{A_1, A_2, \cdots, A_M\}$ which is again denoted by (EXT3*).

After these reductions of (EXT1) and (EXT3), we get finally the following result.

Theorem 3.12. *Let $\{u_j \in \mathfrak{g}_1^{(4)};\ 1 \leqq j \leqq M\}$ be a system of generators of $\mathfrak{g}_1^{(4)} = \mathfrak{g}_{1,C} \otimes \mathfrak{g}_{1,C} \otimes \mathfrak{g}_{1,C} \otimes \mathfrak{g}_{1,C}$ as $\langle U(\mathfrak{g}_{0,C}), S_{23} \rangle$-module, and $\{z_k \in \mathfrak{g}_{1,C} \wedge \mathfrak{g}_{1,C};\ 1 \leqq k \leqq N\}$ that of $\mathfrak{g}_{1,C} \wedge \mathfrak{g}_{1,C}$ as $\mathfrak{g}_0$-module.*

(i) *Under the condition* (EXT1) *on B: $\mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{gl}(V_0)$, the equation* (EXT3) *on B is equivalent to the equation* (EXT3*).

(ii) *Put, for $\xi, \eta \in \mathfrak{g}_1$*

$$A(\xi \wedge \eta) = A(\xi, \eta) = B(\xi, \eta) - B(\eta, \xi) ,$$

and $A_k = A(z_k) \in \mathfrak{gl}(V_0)$, $1 \leqq k \leqq N$. Then the condition (EXT1) *on B is equivalent to the condition* (EXT1*) *on $\{A_1, A_2, \cdots, A_N\}$.*

(iii) *The system of equations* (EXT1)–(EXT3) *on B is equivalent to the system of equations* (EXT1*), (EXT3*), *under the skew-symmetricity of A.*

Note. In §§6–8, we treat irreducible unitary representations (=IURs) of real forms of $\mathfrak{sl}(2, 1)$, and there we encounter the cases where all $A_k = 0$ except only one $A_{k_0} = A(z_{k_0})$.

§4. Conditions for unitarity of representations.

4.1. Positive definiteness for unitarity. In this section, we study unitary extensions. Let (ρ, V_0) be an (infinitesimally) unitary representations of $\mathfrak{g}_0$, and denote by $\langle \cdot, \cdot \rangle_0$ a $\mathfrak{g}_0$-invariant positive definite inner product on V_0. Note that if ρ is not irreducible, $\langle \cdot, \cdot \rangle_0$ is not necessarily unique.

Let us first study a necessary condition for existence of unitary extentions. Let (π, V), $V = V_0 + V_1$, be a quasi-unitary extention of ρ, with properties (PRO1), (PRO2) in 3.1. Denote by $\langle \cdot, \cdot \rangle$ a $\mathfrak{g}$-invariant positive semi-definite inner product on V extending $\langle \cdot, \cdot \rangle_0$ on V_0. By (PRO1), any element $v_1 \in V_1$ is expressed as

$$v_1 = \sum_i \pi(\xi_i)\, v^i \quad \text{with} \quad \xi_i \in \mathfrak{g}_1,\ v^i \in V_0 . \tag{4.1}$$

Therefore, by (1.9),

$$\begin{aligned} \langle v_1, v_1 \rangle &= \langle \sum_k \pi(\xi_k)\, v^k, \sum_m \pi(\xi_m)\, v^m \rangle_0 \\ &= (j/\bar{j}) \sum_{k,m} \langle \pi(\xi_m)\, \pi(\xi_k)\, v^k, v^m \rangle_0 \\ &= j^2 \sum_{k,m} \langle B(\xi_m, \xi_k)\, v^k, v^m \rangle_0 \geqq 0 . \end{aligned}$$

Here j is the fixed forth root of -1 in (1.9): $j^4 = -1$, whence $j/\bar{j} = j^2 = \varepsilon i$.

Thus we get the following

Lemma 4.1. *Let* (π, V), $V=V_0+V_1$, *be a quasi-unitary extension with properties* (PRO1), (PRO2), *of a unitary representation* (ρ, V_0) *of* $\mathfrak{g}_0$, *not necessarily irreducible. Then the corresponding bilinear map B of* $\mathfrak{g}_1\times\mathfrak{g}_1$ *into* $\mathfrak{gl}(V_0)$ *satisfies*

$$\text{(UNI)} \qquad j^2\sum_{k,m}\langle B(\xi_m, \xi_k)\, v^k, v^m\rangle_0 \geqq 0 \quad (\xi_k\in\mathfrak{g}_1,\ v^k\in V_0)\,,$$

where j is a constant depending only on π, *and* $j^2=\varepsilon i$, $\varepsilon=\pm 1$. *In particular,*

$$B(\xi, \eta)^* = -B(\eta, \xi) \quad (\xi, \eta\in\mathfrak{g}_1)\,, \tag{4.2}$$

where D^* *denotes the adjoint operator of* $D\in\mathfrak{gl}(V_0)$ *with respect to* $\langle\cdot, \cdot\rangle_0$, *and moreover*

$$\text{(UNI}') \qquad j^2B(\xi, \xi)\geqq 0 \qquad \textit{for any} \quad \xi\in\mathfrak{g}_1\,,$$

where $D\geqq 0$ *means that* $D\in\mathfrak{gl}(V_0)$ *is positive semi-definite.*

Now consider the kernel N of $\langle\cdot, \cdot\rangle$, that is,

$$N = \{v\in V;\ \langle v, u\rangle = 0 \qquad \text{for any} \quad u\in V\}\,. \tag{4.3}$$

Since $\langle\cdot, \cdot\rangle_0$ is positive definite, N is contained in V_1. Let $v_1\in N\subset V_1$. Then, taking $u=\pi(\eta)\, v'$ with $\eta\in\mathfrak{g}_1$, $v'\in V_0$, we have

$$\langle v_1, u\rangle = j^2\langle\pi(\eta)\, v_1, v'\rangle_0 = 0\,.$$

Since $\langle\cdot, \cdot\rangle_0$ is definite, we get $\pi(\eta)\, v_1=0$ for any $\eta\in\mathfrak{g}_1$, and so $v_1=0$ by (PRO2), whence $N=(0)$. Hence we see that $\langle\cdot, \cdot\rangle$ must be definite, and so (π, V) is necessarily unitary.

Thus we get the first half of the following theorem.

Theorem 4.2. *Let* (ρ, V_0) *be a unitary representation of* $\mathfrak{g}_0$, *and* (π, V), $V=V_0+V_1$, *be its extension with* (PRO1), (PRO2), *which is given canonically by* $B(\cdot, \cdot)$ *satisfying* (EXT1)–(EXT3).

(i) *If* (π, V) *is quasi-unitary, then it is necessarily unitary. Moreover, for* $v_1=\sum_i \pi(\xi_i)\, v^i\in V_1$ *with* $\xi_i\in\mathfrak{g}_1$, $v^i\in V_0$,

$$\langle v_1, v_1\rangle = j^2\sum_{k,m}\langle B(\xi_m, \xi_k)\, v^k, v^m\rangle_0\,. \tag{4.4}$$

(ii) (π, V) *can be made unitary if and only if there exists a* $\mathfrak{g}_0$*-invariant positive definite inner product* $\langle\cdot, \cdot\rangle_0$ *on* V_0 *for which the condition* (UNI) *holds for* $B(\cdot, \cdot)$. *In particular, it is necessary that the operator* $j^2B(\xi, \xi)$ *on* V_0 *is positive semi-definite with respect to* $\langle\cdot, \cdot\rangle_0$ *for any* $\xi\in\mathfrak{g}_1$: $j^2B(\xi, \xi)\geqq 0$.

Proof. It rests only to prove the second assertion (ii). We must prove the equalities (1.8) and (1.9). Remark that $V_0\perp V_1$, then these equalities reduce to the following

$$\langle i\pi(X)\,v_1,\,v_1'\rangle=\langle v_1,\,i\pi(X)\,v_1'\rangle \quad (X\in\mathfrak{g}_0,\ v_1,\,v_1'\in V_1)\,,$$
$$\langle j\pi(\xi)\,v_0,\,v_1\rangle=\langle v_0,\,j\pi(\xi)\,v_1\rangle \quad (\xi\in\mathfrak{g}_1,\ v_s\in V_s)\,.$$

The first equality follows from (EXT1) for B and (4.4), and the second one from (4.2). Q.E.D.

Remark 4.3. Let π be a unitary representation extending a unitary (ρ, V_0), with properties (PRO1), (PRO2), and put $B(\xi, \eta)=\pi(\xi)\,\pi(\eta)\,|\,V_0$. Assume that B satisfies the condition (UNI). Then we can define on $W=\mathfrak{g}_{1,C}\otimes_C V_0$ a positive semi-definite inner product by

$$\langle \textstyle\sum_k \xi_k\otimes v^k,\ \sum_m \eta_m\otimes u^m\rangle_W = j^2 \sum_{k,m} \langle B(\eta_m, \xi_k)\,v^k,\ u^m\rangle_0\,, \tag{4.5}$$

where ξ_k, $\eta_m\in\mathfrak{g}_1$, v^k, $u^m\in V_0$. Then a calculation similar as above shows that the kernel $\mathfrak{n}$ of $\langle\cdot,\cdot\rangle_W$ coincides with the $\mathfrak{g}_0$-submodule $\mathfrak{m}$ of W determined by (3.5). This means that the method (MET1)–(MET2) in §3.2 is compatible with unitarity of extensions, since $V_1\cong\tilde{W}=W/\mathfrak{m}$, $\mathfrak{m}=\mathfrak{n}$.

4.2. A property of unitary representations. Let us give some remarks on a property, peculiar to unitary representations of Lie superalgebras. Let $\xi\in\mathfrak{g}_1$, then by (UNI′),

$$j^2\,\pi(\xi)\,\pi(\xi)\,|\,V_0=j^2B(\xi,\xi)\geqq 0\,.$$

Note that $[\xi,\xi]\in\mathfrak{g}_0$, $\rho([\xi,\xi])=\pi([\xi,\xi])\,|\,V_0$ and $\pi([\xi,\xi])=[\pi(\xi),\,\pi(\xi)]_+=2\pi(\xi)^2$. Then we see that $j^2\rho([\xi,\xi])\geqq 0$. Thus we get the following

Lemma 4.4. *Let* $\mathfrak{g}_0(+)$ *be the subset of* $\mathfrak{g}_0$ *consisting of linear combinations of* $[\xi,\xi]$, $\xi\in\mathfrak{g}_1$, *with non-negative real coefficients. If a unitary representation* (ρ, V_0) *of* $\mathfrak{g}_0$ *has a unitary extention with the associated constant* $j^2=\varepsilon i$ $(\varepsilon=\pm 1)$, *then*

$$\varepsilon\, i\, \rho(X)\geqq 0 \qquad \textit{for}\quad X\in\mathfrak{g}_0(+)\,. \tag{4.6}$$

Corollary 4.5. *Assume that* $\mathfrak{g}_0$ *be reductive and* (ρ, V_0) *be an admissible* $(\mathfrak{g}_0, K_0)$-*module as in* 2.3. *Assume further that a compact Cartan subalgebra* $\mathfrak{h}_0'$ *of* $\mathfrak{g}_0'=[\mathfrak{g}_0,\mathfrak{g}_0]$ *is spanned by* $\mathfrak{h}_0'\cap\mathfrak{g}_0(+)$. *Then, if* ρ *has a unitary extension to* $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$, *then* ρ *is a highest weight module or a lowest weight module according as* $\varepsilon=1$ *or* -1, *with respect to a lexicographic order comming from a basis* $\{X_k\}$ *of* $\mathfrak{h}_0'$ *such that* $X_k\in\mathfrak{h}_0'\cap\mathfrak{g}_0(+)$.

Note. When we turn to conjugate representations, we see that the condition corresponding to (4.6) for the conjugate $\bar{\rho}$ of ρ is

$$((-\varepsilon)\,i)\circ\bar{\rho}(X)\geqq 0 \quad (X\in\mathfrak{g}_0(+))\,,$$

that is, $(\varepsilon\, i\,\rho(X))^-\geqq 0$, which is equivalent to (4.6) itself (cf. §§1.2∼1.3). Further, if ρ has a unitary extension π, then $\bar{\rho}$ has $\bar{\pi}$ as its unitary extension, and vice versa. Note also that, if ρ is a highest weight module, then $\bar{\rho}$ is a lowest weight module.

Corollary 4.6. *Let $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$ be a complex Lie superalgebra. Assume that a unitary representation ρ of $\mathfrak{g}_0$ has a unitary extension to $\mathfrak{g}$. Then necessarily,*

$$\rho(X)=0 \quad \textit{for any} \quad X\in\langle\mathfrak{g}_0(+)\rangle, \tag{4.7}$$

where $\langle\mathfrak{g}_0(+)\rangle$ denotes the subalgebra generated by $\mathfrak{g}_0(+)$. Moreover if $\mathfrak{g}_0(+)$ generates $\mathfrak{g}_0$, then $\rho=0$, or (dim V_0)*-multiple of the* 1*-dimensional trivial representation.*

Proof. Let $\xi\in\mathfrak{g}_1$. Then $\pm[\xi,\xi]\in\mathfrak{g}_0(+)$ because $i\xi\in\mathfrak{g}_1$ and $[i\xi, i\xi]=-[\xi,\xi]$. Therefore $\pm\varepsilon i\,\rho([\xi,\xi])\geqq 0$, whence $\rho([\xi,\xi])=0$. This proves the assertion.

Q.E.D.

Example 4.7. Let $U=U_0+U_1$ be a $\boldsymbol{Z}_2$-graded vector space over a field K. Let $U_0\cong K^m$, $U_1\cong K^n$, then the Lie superalgebra $\mathfrak{l}(U)$ is denoted by $\mathfrak{l}(m, n; K)$. We define its subalgebra $\mathfrak{sl}(m, n; K)$ as

$$\mathfrak{sl}(m,n;K)=\{X\in\mathfrak{l}(m,n;K);\ \mathrm{str}(X)=0\}.$$

Here str(X), the supertrace of X, is defined as follows: express X as $X=X_{00}\oplus X_{01}\oplus X_{10}\oplus X_{11}$ with $X_{st}\in\mathrm{Hom}_K(U_t, U_s)$, then

$$\mathrm{str}(X)=\mathrm{tr}(X_{00})-\mathrm{tr}(X_{11}). \tag{4.8}$$

We denote $\mathfrak{l}(m, n; \boldsymbol{C})$ and $\mathfrak{sl}(m, n; \boldsymbol{C})$ also by $\mathfrak{l}(m, n)$ and $\mathfrak{sl}(m, n)$ respectively. Take here $\mathfrak{g}=\mathfrak{l}(m, n; \boldsymbol{R})$ or $\mathfrak{sl}(m, n; \boldsymbol{R})$. Then the subset $\mathfrak{g}_0(+)$ of $\mathfrak{g}_0$ contains the canonical basis for $\{X\in\mathfrak{g}_0;\ \mathrm{str}(X)=0\}$, and actually it is so big that, for instance for the latter $\mathfrak{g}$, we can prove in Theorem 6.2 that if a unitary representation ρ can be extended to an irreducible unitary representation of $\mathfrak{g}$, then $\rho=0$, and that $\rho=0$ has a unique irreducible extension, the trivial representation of $\mathfrak{g}$ (dim $V_1=0$), if and only if dim $V_0=1$. The latter part is proved by using Theorem 3.4 (ii).

Example 4.8. Take $\mathfrak{g}=\mathfrak{l}(m, n)$ or $\mathfrak{sl}(m, n)$. Then $\mathfrak{g}=\langle\mathfrak{g}_0(+)\rangle$. Therefore we are just in the last case of Corollary 4.6. Thus we conclude that if a unitary representation ρ of $\mathfrak{g}_0$ has a unitary extension, then $\rho=0$. Moreover, using Theorem 3.4 (ii), we see that we have a unique irreducible unitary extension, trivial one (dim $V_1=0$), if and only if dim $V_0=1$.

In §§5–8, we study some special cases. In §5, we study Problems 1bis and 2bis for $\mathfrak{g}=\mathfrak{osp}(2/1)$, a real form of $\mathfrak{osp}(1, 2)$ (type $B(0, 1)$), and in §§6–8, Problem 2 mainly for $\mathfrak{g}$=real forms of $\mathfrak{sl}(2, 1)$ (type $A(1, 0)$).

4.3. A remark on Wakimoto's definition of unitarity. In [16], Wakimoto defined and constructed unitarizable representations of complex Lie superalgebras $\mathfrak{gl}(p|q)$. Let us explain that his definition coincides essentially with ours if we introduce one of two real forms of $\mathfrak{gl}(p|q)$, naturally attached to the definition.

His definition of unitarity is as follows. Let $\mathfrak{g}=\mathfrak{gl}(p|q)$ $(=\mathfrak{l}(p, q)$ in our notation, but not necessarily finite-dimensional) and ω a certain involutive conjugate-linear map from $\mathfrak{g}$ into itself for which

$$\omega([X, Y]) = (-1)^{d(X)d(Y)}[\omega(X), \omega(Y)] \tag{4.9}$$

for any homogeneous elements $X, Y \in \mathfrak{g}$. Then a representation π of $\mathfrak{g}$ on V is called unitarizable if V has a positive definite inner product $\langle \cdot, \cdot \rangle$ such that for any $X \in \mathfrak{g}$,

$$\langle \pi(X)v, v' \rangle + \langle v, \pi(\omega(X))v' \rangle = 0 \quad (v, v' \in V). \tag{4.10}$$

Now define, for $\varepsilon = \pm 1$, a real form of $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ as $\mathfrak{g}(\omega, \varepsilon) = \mathfrak{g}(\omega, \varepsilon)_0 + \mathfrak{g}(\omega, \varepsilon)_1$ with

$$\mathfrak{g}(\omega, \varepsilon)_s = \{X \in \mathfrak{g}_s;\ \omega(X) = (\varepsilon i)^s X\}. \tag{4.11}$$

We see easily that $\mathfrak{g}(\omega, \varepsilon)$ is actually a real subalgebra of $\mathfrak{g}$, and $\mathfrak{g} = \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{g}(\omega, \varepsilon)$ as Lie superalgebras. Moreover, put $\varphi_\varepsilon = \omega | \mathfrak{g}(\omega, \varepsilon)$, then it is real linear and maps $\mathfrak{g}(\omega, \varepsilon)$ to $\mathfrak{g}(\omega, -\varepsilon)$ bijectively, and we have

$$\begin{aligned} &\varphi_\varepsilon = \text{the identity, on} \quad \mathfrak{g}(\omega, \varepsilon)_0 \ (= \mathfrak{g}(\omega, -\varepsilon)_0), \\ &\varphi_\varepsilon([X, Y]) = (-1)^{d(X)d(Y)}[\varphi_\varepsilon(X), \varphi_\varepsilon(Y)] \\ &\qquad\qquad \text{for homogeneous } X, Y \in \mathfrak{g}(\omega, \varepsilon). \end{aligned}$$

This means that $\mathfrak{g}(\omega, -\varepsilon) = \varphi_\varepsilon(\mathfrak{g}(\omega, \varepsilon))$ is dual to $\mathfrak{g}(\omega, \varepsilon)$.

For $X \in \mathfrak{g}(\omega, \varepsilon)_0$ and $\xi \in \mathfrak{g}(\omega, \varepsilon)_1$, the equality (4.10) takes the following form:

$$\begin{aligned} &\langle \pi(X)v, v' \rangle + \langle v, \pi(X)v' \rangle = 0 \qquad (X \in \mathfrak{g}(\omega, \varepsilon)_0), \\ &(\varepsilon i)\langle \pi(\xi)v, v' \rangle - \langle v, \pi(\xi)v' \rangle = 0 \quad (\xi \in \mathfrak{g}(\omega, \varepsilon)_1). \end{aligned} \tag{4.12}$$

This means that $\pi | \mathfrak{g}(\omega, \varepsilon)$ is a unitary representation of the real Lie superalgebra $\mathfrak{g}(\omega, \varepsilon)$ in our sense with the associated constant $j^2 = \varepsilon i$. Therefore his definition coincides essentially with ours modulo the ambiguity of real forms: which of $\mathfrak{g}(\omega, \varepsilon)$, $\varepsilon = \pm 1$, should be taken. Further note that the orthogonality between V_0 and V_1 is not demanded apriori, contrary to (i) in our definition of unitarity in 1.3, whereas actually in his construction, $V_0 \perp V_1$ is satisfied.

Let us consider in the converse way. Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be a real Lie superalgebra and (π, V) a unitary representation of $\mathfrak{g}$ in our sense. Take the complexification $\mathfrak{g}_{\boldsymbol{C}} = \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{g}$ of $\mathfrak{g}$ and extend π to $\mathfrak{g}_{\boldsymbol{C}}$ by linearity. Define an involutive conjugate-linear map ω_κ for a fixed $\kappa = \pm 1$ as

$$\begin{aligned} \omega_\kappa(X + i\cdot Y + \xi + i\cdot\eta) &= X - i\cdot Y + \kappa i\cdot(\xi - i\cdot\eta) \\ &\qquad (X, Y \in \mathfrak{g}_0,\ \xi, \eta \in \mathfrak{g}_1), \end{aligned} \tag{4.13}$$

where $i\cdot Y = i \otimes Y$ etc. Then $\omega = \omega_\kappa$ for a fixed κ satisfies (4.9), and

$$\mathfrak{g}(\omega, \varepsilon) = \begin{cases} \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{g} & \text{if} \quad \varepsilon = \kappa, \\ \mathfrak{g}_0 + i\cdot\mathfrak{g}_1 \equiv \mathfrak{g}^d & \text{if} \quad \varepsilon = -\kappa. \end{cases}$$

Note that $\mathfrak{g}^d$ is dual to $\mathfrak{g}$ under the correspondence $X + \xi \to X + i\cdot\xi$ $(X \in \mathfrak{g}_0, \xi \in \mathfrak{g}_1)$.

Finally we remark that in Wakimoto's case the real forms $\mathfrak{g}(\omega, \varepsilon)$ of $\mathfrak{gl}(p|q)$

are respectively equal to $\mathfrak{u}(p, q; p, q-1)$ and $\mathfrak{u}(p, q; p, 1)$ with even parts isomorphic to $\mathfrak{u}(p)\times\mathfrak{u}(1, q-1)$.

§5. Some examples of irreducible representations.

In this section, we take a simple Lie superalgebra $\mathfrak{osp}(2n/1)$ with $n=1$ as an example and study Problems 1bis and 2bis. The cases $n\geqq 2$ will be treated in another paper.

5.1. Structure of $\mathfrak{osp}(2n/1)$. By definition, $\mathfrak{g}=\mathfrak{osp}(2n/1)=\mathfrak{g}_0+\mathfrak{g}_1$ are given as follows:

$$\mathfrak{g}_0=\mathfrak{sp}(2n; \boldsymbol{R}), \quad \mathfrak{g}_1=\boldsymbol{R}^{2n},$$

with $\mathfrak{sp}(2n; \boldsymbol{R})=\{X\in\mathfrak{gl}(2n; \boldsymbol{R}); {}^tXJ+JX=0\}$ and the bracket operation

$$[X, \xi]\equiv X\xi \qquad (X\in\mathfrak{g}_0, \xi\in\mathfrak{g}_1),$$
$$[\xi, \eta]\equiv -(\xi{}^t\eta+\eta{}^t\xi)J \quad (\xi, \eta\in\mathfrak{g}_1),$$

where J is a $2n\times 2n$ matrix given by $J=\begin{bmatrix}0_n & -1_n\\ 1_n & 0_n\end{bmatrix}$ with $n\times n$ zero matrix 0_n and identity matrix 1_n. The algebra $\mathfrak{osp}(2n/1)$ is a real form of a complex Lie superalgebra $\mathfrak{osp}(1, 2n)$ of type $B(0, \mathrm{n})$.

Introduce a canonical basis $\{\xi_a, \xi_{\bar{a}}; 1\leqq a\leqq n, \bar{a}=a+n\}$ for $\mathfrak{g}_1$, and denote by E_{ab} an $n\times n$ matrix with entries 1 at (a, b) and 0 elsewhere. Put $X_{pq}=[\xi_p, \xi_q]$ for $1\leqq p, q\leqq 2n$, then they span $\mathfrak{g}_0$ and, for $1\leqq a, b\leqq n$,

$$X_{ab}=\begin{bmatrix}0_n & E_{ab}+E_{ba}\\ 0_n & 0_n\end{bmatrix}, \quad X_{\bar{a}\bar{b}}=\begin{bmatrix}0_n & 0_n\\ -E_{ab}-E_{ba} & 0_n\end{bmatrix}, \quad X_{a\bar{b}}=X_{\bar{b}a}=\begin{bmatrix}-E_{ab} & 0_n\\ 0_n & E_{ba}\end{bmatrix}.$$

Let $\mathfrak{g}_0(+)$ be as in Lemma 4.4 the subset of $\mathfrak{g}_0$ consisting of linear combinations of $[\xi, \xi]$, $\xi\in\mathfrak{g}_1$, with non-negative real coefficients. Then $\mathfrak{g}_0(+)$ contains a basis $\{X_{aa}+X_{\bar{a}\bar{a}}; 1\leqq a\leqq n\}$ of a compact Cartan subalgebra of $\mathfrak{g}_0$. Therefore, when we consider unitary extension problems, we are exactly in the case of Corollary 4.5. Thus, to get an irreducible unitary representation of $\mathfrak{g}$, we should start from unitarizable highest or lowest weight modules (ρ, V_0) of $\mathfrak{g}_0$.

5.2. Equations for extensions. Let (ρ, V_0) be an admissible $(\mathfrak{g}_0, K_0)$-module, where K_0 is a maximal compact subgroup of $G_0=Sp(2n; \boldsymbol{R})$. To study the extension problems for (ρ, V_0), we have to treat a $\mathfrak{g}_0$-equivariant map B: $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$, which satisfies the system of equations (EXT1)–(EXT3). We apply the reduction of these equations, given in Theorem 3.12. Let A be the map $\mathfrak{g}_{1,\boldsymbol{C}}\wedge\mathfrak{g}_{1,\boldsymbol{C}}\to\mathfrak{gl}(V_0)$ given by

$$A(\xi\wedge\eta)=A(\xi, \eta)=2B(\xi, \eta)-\rho([\xi, \eta]) \qquad \text{for} \quad \xi, \eta\in\mathfrak{g}_1,$$

which is again $\mathfrak{g}_0$-equivariant. Put $A_{pq}=A(\xi_p\wedge\xi_q)$ for $1\leqq p, q\leqq 2n$.

For a reduced form of the equations, we refer [6] for general n, and here we treat only the simplest case $n=1$.

Hereafter we put always $n=1$. Then, at first, the $\mathfrak{g}_0$-module $\mathfrak{g}_{1,C}\wedge\mathfrak{g}_{1,C1}$ is spanned by one element $z_1=\xi_1\wedge\xi_{\bar{1}}$ and carries the trivial representation. This means that $A_{1\bar{1}}=A(z_1)\in\mathfrak{gl}(V_0)$ intertwines the representation ρ with itself. From this fact, we get

Lemma 5.1. *The representation* (ρ, V_0) *of* $\mathfrak{g}_0=\mathfrak{sp}(2; \boldsymbol{R})$ *should be irreducible to get an irreducible extension of it to* $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1=\mathfrak{osp}(2/1)$. *Hence the operator* $A_{1\bar{1}}$ *on* V_0 *should be a scalar operator.*

Proof. This follows from (PRO3) in the criterion of irreducibility in Lemma 3.1. Q.E.D.

Thus, in particular, for $\mathfrak{g}=\mathfrak{osp}(2/1)$, Problems 1bis and 2bis are equivalent to Problems 1 and 2 respectively.

Now, examining $\mathfrak{g}_0$-module structure of $\mathfrak{g}_1^{(4)}=\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}$, we get the following system of equations for irreducible extensions. This will lead to the classification of all the irreducible representations.

Lemma 5.2. *For* $\mathfrak{g}=\mathfrak{osp}(2/1)$, *the system of equations* (EXT1)–(EXT3) *and* (PRO1)–(PRO3) *is reduced to the following*:

(1) (ρ, V_0) *is irreducible and* $A_{1\bar{1}}\in\mathfrak{gl}(V_0)$ *is a scalar operator*;

(2) *put* $A=A_{1\bar{1}}$, *and* $\rho_{ab}=\rho(X_{ab})$, *then*

$$[A, A]_+-4A = [\rho_{1\bar{1}}, \rho_{1\bar{1}}]_+-[\rho_{11}, \rho_{\bar{1}\bar{1}}]_+ . \tag{5.2}$$

We note that the above equation (5.2) comes from (EXT3A$_+$) for some ξ, η, ζ, $\tau\in\mathfrak{g}_1$. It is rewritten as

$$(A-I)^2 = \rho(\Delta)+I , \tag{5.3}$$

where I denotes the identity operator on V_0 and $\Delta\in U(\mathfrak{g}_{0,C})$ denotes a constant multiple of the Casimir element given by

$$\Delta = (X_{1\bar{1}})^2-\frac{1}{2}(X_{11}X_{\bar{1}\bar{1}}+X_{\bar{1}\bar{1}}X_{11}) = (X_{1\bar{1}})^2-\frac{1}{2}[X_{11}, X_{\bar{1}\bar{1}}]_+ , \tag{5.4}$$

with $$X_{1\bar{1}} = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, X_{11} = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}, X_{\bar{1}\bar{1}} = \begin{bmatrix}0 & 0\\ -2 & 0\end{bmatrix}.$$

5.3. Irreducible $(\mathfrak{g}_0, K_0)$-modules. We list up here irreducible $(\mathfrak{g}_0, K_0)$-modules for $\mathfrak{g}_0=\mathfrak{sp}(2; \boldsymbol{R})=\mathfrak{sl}(2; \boldsymbol{R})$, $K_0\subset G_0=Sp(2; \boldsymbol{R})$. Let $\{v_m; m\in\Omega\}$ be a basis of a vector space V_0 over $\boldsymbol{C}$, where $\Omega\subset\boldsymbol{Z}$ will be specified later. Fix a complex number $c\in\boldsymbol{C}$ and a $\nu\in\boldsymbol{Z}_2=\{0, 1\}$. Put

$$\boldsymbol{Z}(\nu) = \{m\in\boldsymbol{Z};\ m\equiv\nu \pmod 2\} . \tag{5.5}$$

Depending on the parameter (c, ν), we determine Ω and so V_0, and define $\mathfrak{g}_0$-action ρ on V_0 as

$$\rho(X_{1\bar{1}})\,v_m = -ic_m\,v_{m+2} - ic_{m-2}\,v_{m-2}\,,$$

(5.6)
$$\rho(X_{11})\,v_m = c_m\,v_{m+2} + imv_m - c_{m-2}\,v_{m-2}\,,$$
$$\rho(X_{\bar{1}\bar{1}})\,v_m = -c_m\,v_{m+2} + imv_m + c_{m-2}\,v_{m-2}\,,$$

where $m \in \Omega$, $i = \sqrt{-1}$ and

(5.7)
$$c_m = \frac{1}{2}\sqrt{(m+1)^2 - c^2} \quad (\arg(c_m) \text{ arbitrary but fixed}).$$

Therefore we have

(5.8)
$$\rho(\Delta) = (c^2 - 1)\,I\,.$$

We list up the sets Ω of weights, and the symbols for the irreducible $(\mathfrak{g}_0, K_0)$-modules thus obtained. Note that we may assume that $0 \leqq \arg(c) < \pi$, if necessary.

Case 1. Assume that $c \not\equiv \nu+1 \bmod 2$. Then $\Omega = \boldsymbol{Z}(\nu)$, the $\mathfrak{g}_0$-module $\mathcal{D}_{c,\nu} = (\rho, V_0)$ corresponds to the representation of G_0 induced from a character of its minimal parabolic subgroup.

Case 2. Assume that $c \in \boldsymbol{Z}$, $c \geqq 0$ and $c \equiv \nu+1 \bmod 2$. Then there exist three kinds of $\Omega \subset \boldsymbol{Z}(\nu)$:

$$\Omega_+ = \{m \in \boldsymbol{Z}(\nu);\ m \geqq c+1\}\,,$$
$$\Omega_- = \{m \in \boldsymbol{Z}(\nu);\ m \leqq -(c+1)\}\,,$$
$$\Omega_f = \{m \in \boldsymbol{Z}(\nu);\ -(c-1) \leqq m \leqq c-1\}\,.$$

Note that $\Omega_f = \emptyset$ if $(c, \nu) = (0, 1)$.

The corresponding representations, denoted by D_μ^+, D_μ^- and F_N with $\mu = (c+1)/2$ and $N = c \geqq 1$, are in the discrete series (or in its limit if $c = 0$ and so $\mu = 1/2$) and N-dimensional representations respectively.

We summarize known facts in the following two lemmas.

Lemma 5.3. *Irreducible $(\mathfrak{g}_0, K_0)$-modules for $\mathfrak{g}_0 = \mathfrak{sp}(2; \boldsymbol{R})$, $K_0 \subset G_0 = Sp(2; \boldsymbol{R})$, are isomorphic to one of the following modules*:

$\mathcal{D}_{c,\nu} = \mathcal{D}_{-c,\nu}$ *with* $c \in \boldsymbol{C}$, $0 \leqq \arg(c) < \pi$, $c \not\equiv \nu+1 \pmod 2$;

D_μ^+ *and* D_μ^- *with* $\mu \in (1/2)\boldsymbol{Z} \equiv \{p/2;\ p \in \boldsymbol{Z}\}$, $\mu \geqq 1/2$;

F_N *with* $N \in \boldsymbol{Z}$, $\geqq 1$.

Lemma 5.4. *Unitarizable modules among the above modules are given as follows*:

$\mathcal{D}_{i\eta,\nu}$ *with* $i = \sqrt{-1}$, $\eta \in R$, $\eta \geqq 0$ *for* $\nu = 0$, *and* $\eta > 0$ *for* $\nu = 1$;

$\mathcal{D}_{\sigma,0}$ *with* $0 < \sigma < 1$;

all $D_\mu^\pm$, *and the trivial representation* F_1.

For each of them, an invariant positive definite inner product is introduced in V_0 *by setting the standard basis* $\{v_m;\ m \in \Omega\}$ *as a complete orthonormal system.*

For later convenience, we rewrite the formula for $\mathfrak{g}_0$-action as follows. Let H, X_+ and X_- be elements of $\mathfrak{g}_{0,C}$ given by

$$(5.9)\qquad H=\frac{1}{2}(X_{11}+X_{\bar{1}\bar{1}}),\quad X_\pm=iX_{1\bar{1}}\pm\frac{1}{2}(X_{11}-X_{\bar{1}\bar{1}}).$$

Then H spans a compact Cartan subalgebra of $\mathfrak{g}_0$ and

$$[H, X_+]=2iX_+,\quad [H, X_-]=-2iX_-,\quad [X_+, X_-]=4iH,$$

with $i=\sqrt{-1}$, and the formula (5.6) takes the form

$$(5.10)\qquad \rho(H)\,v_m=imv_m,\ \rho(X_+)\,v_m=2c_m\,v_{m+2},\ \rho(X_-)\,v_m=2c_{m-2}\,v_{m-2}.$$

with $2c_m=\sqrt{(m+1)^2-c^2}$. Moreover we have

$$(5.11)\qquad \Delta=-H^2-\frac{1}{2}(X_+X_-+X_-X_+)=-H^2-\frac{1}{2}[X_+, X_-]_+.$$

5.4. The $\mathfrak{g}_0$-module $W=\mathfrak{g}_{1,C}\otimes_C V_0$. Here we determine the structure of $\mathfrak{g}_0$-module W. First note that the $\mathfrak{g}_0$-module $\mathfrak{g}_{1,C}$ is F_2 and has the highest weight 1 and that the character of the corresponding representation of G_0 is equal to trace(g), $g\in G_0$. We see, as a matter of fact, that the simplest way to decompose W into irreducibles, is to use the character of V_0, more exactly that of the corresponding irreducible representation of G_0. These characters are listed up, for instance, in [7, pp. 50–51]. Thus we get

Lemma 5.5. *The $\mathfrak{g}_0$-module $W=\mathfrak{g}_{1,C}\otimes V_0$ is decomposed into a direct sum of mutually inequivalent irreducible ones, except when $(\rho, V_0)=\mathcal{D}_{0,0}$, $D^+_{1/2}$, $D^-_{1/2}$ or F_1:*

(ρ, V_0)	$W=\mathfrak{g}_{1,C}\otimes V_0$
$\mathcal{D}_{c,\nu}$ $((c,\nu)\neq(0,0))$	$\mathcal{D}_{c+1,\nu+1}\oplus\mathcal{D}_{-c+1,\nu+1}$
D^α_μ $(\alpha=\pm,\ \mu\in 1)$	$D^\alpha_{\mu+1/2}\oplus D^\alpha_{\mu-1/2}$
F_N $(N\geqq 2)$	$F_{N+1}\oplus F_{N-1}$
F_1	F_2

Proof. For an admissible $(\mathfrak{g}_0, K_0)$-module U, denote by $\chi(U)$ the corresponding character on G_0, which is an invariant eigendistribution given by a locally summable function on G_0. We know that for the tensor product $W=\mathfrak{g}_{1,C}\otimes V_0$, $\chi(W)=\chi(\mathfrak{g}_{1,C})\cdot\chi(V_0)$. Then, by simple calculations, we get character identities

$$\chi(W)=\chi(\mathcal{D}_{c+1,\nu+1})+\chi(\mathcal{D}_{-c+1,\nu+1})$$

etc. corresponding to the right hand side of the above list. These character identities give irreducible components of W as subquotients.

The direct sum property is proved as follows. First, calculate the infinitesimal character of each irreducible component. Since $\mathfrak{g}_0$ is of rank 1, it is determined by the scalar corresponding to $\Delta+1$. In this sense, the infinitesimal characters of two irreducible components are exactly $(c+1)^2$ and $(-c+1)^2$. (The parameters μ and N connect with c as above.) These two values are different from each other if and only if $c\neq 0$.

Then we apply the general fact for Harish-Chandra modules that any such module is a direct sum of its submodules with different infinitesimal characters.

Q.E.D.

Of course, an elementary proof, not using characters, is possible.

For the case $(\rho, V_0)=\mathcal{D}_{0,0}$, $D^+_{1/2}$ or $D^-_{1/2}$, we need more direct and detailed calculations, and then get the following

Lemma 5.6. *The* $\mathfrak{g}_0$*-modules* $W=\mathfrak{g}_{1,c}\otimes V_0$ *for* $\rho=\mathcal{D}_{0,0}$, $D^\alpha_{1/2}$ $(\alpha=+$ *or* $-)$ *are not semisimple.* (i) *For* $\mathcal{D}_{0,0}$, *W has only one non-trivial invariant subspace* W_1 *such that both* W_1 *and* W/W_1 *carries* $\mathcal{D}_{1,1}$. (ii) *For* $D^\alpha_{1/2}$, *W has exactly two non-trivial invariant subspaces* W_1, W_2 *such that* $(0)\subset W_1\subset W_2\subset W$, *and that* W_1 *and* W/W_2 *carry* D^α_1, *and* W_2/W_1 *carries* F_1. *Thus we have*

(ρ, V_0)	$W=\mathfrak{g}_{1,c}\otimes V_0$
$\mathcal{D}_{0,0}$	$(0)\subset W_1\subset W$ *with* $W_1\cong W/W_1\cong\mathcal{D}_{1,1}$
$D^\alpha_{1/2}$ $(\alpha=\pm)$	$(0)\subset W_1\subset W_2\subset W$ *with* $W_1\cong W/W_2\cong D^\alpha_1$, $W_2/W_1\cong F_1$

Proof. (i) One invariant subspace W_1 of W is easy to find, for instance, by a kind of "analytic continuation" from the case $(c, 0)$ with small $c\neq 0$. The uniqueness of W_1 is proved by checking the weight subspace $W(1)$ of W with weight 1 $(=i^{-1}\cdot$ (eigenvalue of H)).

(ii) According as $\alpha=+$ or $-$, we determine explicitly all the highest or lowest weight vectors in W. Further, a little more detailed calculation shows that the only proper submodules are W_1 and W_2. To do this, we can apply for instance the realization of ρ in [5]. Cf. also explicit calculations in the next subsection. Thus we get the assertion.

Q.E.D.

5.5. Explicit determination of module structure for ***W*****.** To give explicitly $\mathfrak{g}$-action on $V=V_0+V_1$, $V_1=W/\mathfrak{m}$, it is necessary to write down $\mathfrak{g}_0$-action on $W=\mathfrak{g}_{1,c}\otimes V_0$ with respect to its certain standard basis. First take a basis of the space $\mathfrak{g}_{1,c}$ as $\mathfrak{g}_0$-module. Put

$$u_1=\xi_1+i\xi_{\bar{1}}\,,\quad u_{-1}=i\xi_1+\xi_{\bar{1}}\quad (i=\sqrt{-1})\,. \tag{5.12}$$

Then,

$$\begin{aligned}&[u_1, u_1]=2X_+\,,\quad [u_{-1}, u_{-1}]=2X_-\,,\\ &[u_1, u_{-1}]=[u_{-1}, u_1]=2iH\,,\end{aligned} \tag{5.13}$$

and the $\mathfrak{g}_{0,C}$-action on $\mathfrak{g}_{1,C}$ is given by

$$
(5.14)\qquad \begin{aligned} &Hu_l = ilu_l \quad (l=\pm 1);\\ &X_+u_1 = 0\,, \qquad X_+u_{-1} = 2u_1;\\ &X_-u_1 = -2u_{-1}\,, \quad X_-u_{-1} = 0\,. \end{aligned}
$$

Next, taking $\{u_l v_m = u_l \otimes v_m;\ l=\pm 1,\ m\in\Omega\}$ as a basis of W, we get from (5.10) and (5.14)

$$
(5.15)\qquad \begin{aligned} &H(u_l v_m) = i(l+m)\, u_l v_m\,,\\ &X_+(u_1 v_m) = 2c_m u_1 v_{m+2}\,, \qquad X_+(u_{-1}v_m) = 2u_1 v_m + 2c_m u_{-1} v_{m+2}\,,\\ &X_-(u_1 v_m) = 2c_{m-2}\, u_1 v_{m-2} - 2u_{-1}v_m,\ X_-(u_{-1}v_m) = 2c_{m-2} u_{-1} v_{m-2}\,. \end{aligned}
$$

with $2c_m = \sqrt{(m+1)^2 - c^2}$. Thus, the weight space $W(m+1)$ for weight $m+1$ is spanned by $\{u_1 v_m,\ u_{-1} v_{m+2}\}$, and we have by (5.11)

$$
(5.16)\qquad \begin{aligned} &\Delta(u_1 v_m) = [2(m+1)+c^2]\, u_1 v_m + 4c_m u_{-1} v_{m+2}\,,\\ &\Delta(u_{-1}v_{m+2}) = -4c_m u_1 v_m + [-2(m+1)+c^2]\, u_{-1} v_{m+2}\,. \end{aligned}
$$

CASE $c \neq 0$. We decompose each $W(m+1)$, for $m\in\Omega$ such that $m+2\in\Omega$, into two eigenspaces $W_{m+1}[c+1]$, $W_{m+1}[-c+1]$ of $\Delta+1$ with eigenvalues $(c+1)^2$, $(-c+1)^2$ respectively, then

$$
(5.17)\qquad W = W[c+1] \oplus W[-c+1]\,, \quad W[\pm c+1] = \textstyle\sum_m W_{m+1}[\pm c+1]\,,
$$

gives the irreducible decomposition of $\mathfrak{g}_0$-module W, given in Lemma 5.5 in several cases. Note that, in the special case $\rho = F_1$, we have $W = W[2]$, $W[0] = (0)$. We see easily that $W_{m+1}[kc+1]$, $k=\pm 1$, is spanned by a vector

$$
(5.18)\qquad w_{m+1}[kc+1] = d_{m+1,k}\, u_1 v_m + d_{m+1,-k}\, u_{-1} v_{m+2}\,,
$$

where $d_{p,k} = \sqrt{p+kc}$ for $p=m+1$, $k=\pm 1$ with

$$
(5.19)\qquad \arg(d_{m+1,1}) \arg(d_{m+1,-1}) = \arg(d_{m+1,1}\, d_{m+1,-1}) = \arg(c_m)\,.
$$

Hence we have $d_{m+1,1}\, d_{m+1,-1} = 2c_m$, and moreover

$$
(5.20)\qquad \begin{aligned} &H_+\, w_{m+1}[kc+1] = i(m+1)\, w_{m+1}[kc+1]\,,\\ &X_+\, w_{m+1}[kc+1] = 2c_{m+1,kc+1}\, w_{m+3}[kc+1]\,,\\ &X_-\, w_{m+1}[kc+1] = 2c_{m-1,kc+1}\, w_{m-1}[kc+1]\,, \end{aligned}
$$

where $2c_{m+1,kc+1} = \sqrt{(m+2)^2 - (kc+1)^2}$ corresponds to $2c_{m+1}$ for $kc+1$ instead of c, and $\arg(c_{m+1,kc+1})$ is determined so as to hold

$$
(5.21)\qquad 2c_{m+1,kc+1} = d_{m+3,k}\, d_{m+1,-k}\,.
$$

The subspace $W[kc+1]$ carries the following representation: in case $c \not\equiv \nu+1$ (2), $\mathcal{D}_{kc+1,\nu+1} = \mathcal{D}_{c+k,\nu+1}$; in case $c \equiv \nu+1$ (2) and $c \geqq 1$, $D^{\alpha}_{\mu+k/2}$ or F_{N+k} ($F_0 = (0)$).

Further the $\mathfrak{g}_{1,C}$-action $V_0 \to W$ is given with respect to $\{w_{m+1}[kc+1]\}$ as

$$
(5.22)\qquad
\begin{aligned}
u_1 v_m &= \frac{1}{2c}\{d_{m+1,1}\, w_{m+1}[c+1] - d_{m+1,-1}\, w_{m+1}[-c+1]\}\,,\\
u_{-1} v_{m+2} &= \frac{1}{2c}\{-d_{m+1,-1}\, w_{m+1}[c+1] + d_{m+1,1}\, w_{m+1}[-c+1]\}\,.
\end{aligned}
$$

CASE $c=0$. In this case, we can put $2c_m = m+1$, $d_{m+1,1} = d_{m+1,-1} = \sqrt{m+1}$. The $\mathfrak{g}_0$-invariant subspaces $W[c+1]$ and $W[-c+1]$ coincide with each other to get an invariant subspace

$$
(5.23)\qquad W_1 = \textstyle\sum_m C w_{m+1}[1]\,,\quad w_{m+1}[1] = \sqrt{m+1}\,(u_1 v_m + u_{-1} v_{m+2})\,,
$$

on which $\mathfrak{g}_0$ acts according to (5.20)–(5.21) with $c=0$. W_1 carries $\mathcal{D}_{1,1}$ or D_1^{α} according as ρ is $\mathcal{D}_{0,0}$ or $D_{1/2}^{\alpha}$. Note that, in case of $D_{1/2}^{\alpha}$ with $\alpha=\pm$, the vector $w_0[1] = 0\cdot u_{\mp 1} v_{\pm 1} = 0$ by the factor 0 in front of, and that the space $W_2 = W_1 + \boldsymbol{C}(u_{\mp 1} v_{\pm 1})$ carries $D_1^{\alpha} + F_1$ which is not a direct sum since

$$
\begin{aligned}
&X_+(u_{-1} v_1) = 2(u_1 v_1 + u_{-1} v_3)\,,\quad X_-(u_{-1} v_1) = 0\,, &&\text{in case}\quad \alpha = +\,,\\
&X_+(u_1 v_{-1}) = 0\,,\quad X_-(u_1 v_{-1}) = -2(u_1 v_{-3} + u_{-1} v_{-1})\,, &&\text{in case}\quad \alpha = -\,.
\end{aligned}
$$

Moreover, for instance, for $\alpha=+$, since $\dim W(0) = \dim(W(2)\cap W_1) = 1$, $\dim W(2) = 2$, $X_- W(2) = W(0)$, $\mathrm{Ker}(X_- \mid W(2)) = X_+ W(0) = W(2)\cap W_1$, there exist no proper submodules except W_1 and $W_2 = W_1 + W(0)$.

5.6. Complete description of irreducible extensions. To solve Problem 1bis or to get all the irreducible extensions of $\mathfrak{g} = \mathfrak{osp}(2/1)$, it is now sufficient to determine the scalar operator $A = A_{1\bar{1}} \in \mathfrak{gl}(V_0)$, and then the corresponding submodule $\mathfrak{m} \subset W$ defined in (3.5). Thus we get a $\mathfrak{g}_0$-module $\tilde{W} = W/\mathfrak{m}$. Put $V_1 = \tilde{W}$, $V = V_0 + V_1$, then the action of $\mathfrak{g}_1$ on V is given by (3.6)–(3.7), and more explicitly using (5.22) above.

First, it follows from (5.3) and (5.8) that

$$
(5.24)\qquad (A-I)^2 = c^2 I\,.
$$

Therefore there exist two choices of A except when $c=0$:

$$
(5.25)\qquad A = (\gamma c + 1)\, I \quad\text{with}\quad \gamma = \pm 1\,.
$$

Second, we have by (3.16) the following:

$$
\begin{aligned}
B(\xi_1, \xi_1) &= \frac{1}{2}\rho(X_{11})\,, & B(\xi_{\bar{1}}, \xi_{\bar{1}}) &= \frac{1}{2}\rho(X_{\bar{1}\bar{1}})\,,\\
B(\xi_1, \xi_{\bar{1}}) &= \frac{1}{2}(\rho(X_{1\bar{1}}) + A)\,, & B(\xi_{\bar{1}}, \xi_1) &= \frac{1}{2}(\rho(X_{1\bar{1}}) - A)\,.
\end{aligned}
$$

By means of the basis $u_1 = \xi_1 + i\xi_{\bar{1}}$, $u_{-1} = i\xi_1 + \xi_{\bar{1}}$ for $\mathfrak{g}_{1,C}$, this is rewritten as

$$(5.26)\qquad \begin{aligned} &B(u_1, u_1) = \rho(X_+), \quad B(u_{-1}, u_{-1}) = \rho(X_-), \\ &B(u_1, u_{-1}) = \rho(iH)+A, \ B(u_{-1}, u_1) = \rho(iH)-A. \end{aligned}$$

Using the formula (5.10) for $\rho(H)$, $\rho(X_\pm)$, we rewrite the defining equation (3.5) for $\mathfrak{m}$. Since $\mathfrak{m}=\sum_m(\mathfrak{m}\cap W(m+1))$, it is enough to determine $w=xu_1v_m+yu_{-1}v_{m+2}\in \mathfrak{m}\cap W(m+1)$ for each m, where $x, y\in \boldsymbol{C}$. Then, for any $\eta\in\mathfrak{g}_{1,\boldsymbol{C}}$,

$$B(\eta, xu_1)\, v_m+B(\eta, yu_{-1})\, v_{m+2}=0.$$

This is equivalent to

$$(5.27)\qquad \begin{aligned} &xB(u_1, u_1)\, v_m+yB(u_1, u_{-1})\, v_{m+2}=0, \\ &xB(u_{-1}, u_1)\, v_m+yB(u_{-1}, u_{-1})\, v_{m+2}=0. \end{aligned}$$

By (5.10) and (5.25)–(5.26), this, in turn, is written as

$$\begin{aligned} &\{2c_m x+[-(m+2)+(\gamma c+1)]\, y\}\ v_{m+2}=0, \\ &\{[-m-(\gamma c+1)]\, x+2c_m y\}\ v_m=0. \end{aligned}$$

Hence we get

$$(5.28)\qquad \begin{pmatrix} 2c_m & -m-1+\gamma c \\ -m-1-\gamma c & 2c_m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=0.$$

Note that $2c_m=d_{m+1,1}\, d_{m+1,-1}$ and $d_{m+1,k}=\sqrt{m+1+kc}$, then we get the following result.

Case $c\neq 0$. We have $(x, y)=\lambda(d_{m+1,-\gamma}, d_{m+1,\gamma})$ with a constant $\lambda\in\boldsymbol{C}$, whence $w=\lambda w_{m+1}[-\gamma c+1]$, and therefore $\mathfrak{m}=W[-\gamma c+1]$. In the special case $\rho=F_1\ (c=1)$, $\mathfrak{m}=W[0]=(0)$ or $=W[2]=W$ according as $\gamma=+1$ or -1.

Case $c=0$. If $m+1\neq 0$, we get similarly as above, $w=\lambda w_{m+1}[1]$, whence $\mathfrak{m}\cap W(m+1)\subset\mathfrak{m}\cap W_1$. For $\rho=\mathcal{D}_{0,0}$, we conclude from this that $\mathfrak{m}=W_1$. On the other hand, for $\rho=D^{\alpha}_{1/2}$ with $\alpha=\pm$, we should take into account $m+1=0$, and then get $\mathfrak{m}=W_1+\boldsymbol{C}(u_{\mp 1}v_{\pm 2})=W_2$.

Summarizing these results, we get the following

Lemma 5.7. *For every irreducible* $(\mathfrak{g}_0, K_0)$*-module* (ρ, V_0), *the submodule* $\mathfrak{m}$ *of* $W=\mathfrak{g}_{1,\boldsymbol{C}}\otimes V_0$ *defined by* (3.5) *and the quotient module* $\tilde{W}=W/\mathfrak{m}$ *are given as follows.*

Case $c\neq 0$. *For* $A=(\gamma c+1)I$ *with* $\gamma=\pm 1$, $\mathfrak{m}=W[-\gamma c+1]$ *and* $\tilde{W}\cong W[\gamma c+1]$, *another direct sum component.* (*For* $\rho=F_1$, $W[0]=(0)$, $W[2]=W$.)

Case $c=0$. *In this case*, $A=I$. *For* $\rho=\mathcal{D}_{0,0}$, $\mathfrak{m}=W_1\cong\mathcal{D}_{1,1}$ *and* $\tilde{W}\cong\mathcal{D}_{1,1}$. *For* $\rho=D^{\alpha}_{1/2}$, $\mathfrak{m}=W_2\cong D^{\alpha}_1+F_1$ *and* $\tilde{W}\cong D^{\alpha}_1$.

Now we put $V_1=\tilde{W}$ as $\mathfrak{g}_0$-module and put $V=V_0+V_1$.

Case $c\neq 0$. Denote by $[w_{m+1}]$ the element in $\tilde{W}$ represented by $(2c)^{-1}\cdot(w_{m+1}[c+1]-w_{m+1}[-c+1])$. Then, using the above lemma, we see that the $\mathfrak{g}_0$-action on $V_1=\tilde{W}$ comes from (5.20) for $k=\gamma$: put $\Omega(V_1)=\{m+1;\ [w_{m+1}]\neq 0\}$, the set of all weights for V_1, then for $m\in\Omega(V_1)$,

$$(5.29)\qquad \begin{aligned} &H[w_m] = im[w_m]\,, \\ &X_+[w_m] = 2c_{m,\gamma c+1}[w_{m+2}]\,,\quad X_-[w_m] = 2c_{m-2,\gamma c+1}[w_{m-2}]\,. \end{aligned}$$

CASE $c=0$. In this case, $[w_{m+1}]$ is defined as the element in $\tilde{W}$ represented by $(2\sqrt{m+1})^{-1}(u_1 v_m - u_{-1}v_{m+2})$. Then we get from (5.15) and (5.23) that for $m\in \Omega(V_1)$,

$$(5.30)\qquad \begin{aligned} &H[w_m] = im[w_m]\,, \\ &X_+[w_m] = 2c_{m,1}[w_{m+2}]\,,\quad X_-[w_m] = 2c_{m-2,1}[w_{m-2}]\,, \end{aligned}$$

where $2c_{m,1}=\sqrt{(m+1)^2-1} = \sqrt{m}\cdot\sqrt{m-2}$.

When the $\mathfrak{g}_0$-module $V_1=\tilde{W}$ is unitarizable, an invariant positive definite inner product is introduced if we make the basis $\{[w_m];\, m\in\Omega(V_1)\}$ an orthonormal system, thanks to Lemma 5.4.

Furthermore the $\mathfrak{g}_1$-action on V is given by means of $u_1, u_{-1}\in\mathfrak{g}_{1,C}$ as follows.

CASE $c\neq 0$. The map $\mathfrak{g}_{1,C}\times V_0\to V_1$ is given by (5.22) as

$$(5.31)\qquad u_1\cdot v_m = d_{m+1,\gamma}[w_{m+1}]\,,\quad u_{-1}\cdot v_m = -d_{m-1,-\gamma}[w_{m-1}]\quad (m\in\Omega)\,.$$

The map $\mathfrak{g}_{1,C}\times V_1\to V_0$ is given by (5.18), (5.26) as

$$(5.32)\qquad u_1\cdot[w_m] = d_{m,-\gamma}\, v_{m+1}\,,\quad u_{-1}\cdot[w_m] = -d_{m,\gamma}\, v_{m-1}\quad (m\in\Omega(V_1))\,.$$

CASE $c=0$. The maps $\mathfrak{g}_{1,C}\times V_0\to V_1$ and $\mathfrak{g}_{1,C}\times V_1\to V_0$ can be calculated as above and given respectively as follows.

$$(5.33)\qquad u_1\cdot v_m = \sqrt{m+1}\,[w_{m+1}]\,,\quad u_{-1}\cdot v_m = -\sqrt{m-1}\,[w_{m-1}]\quad (m\in\Omega)\,,$$

$$(5.34)\qquad u_1\cdot[w_m] = \sqrt{m}\; v_{m+1}\,,\qquad u_{-1}\cdot[w_m] = -\sqrt{m}\; v_{m-1}\quad (m\in\Omega(V_1))\,.$$

Note that $d_{m,\gamma}=\sqrt{m}$ for any $\gamma=\pm1$ if $c=0$, and so the formulas (5.33) and (5.34) take the same form as (5.31) and (5.32) respectively.

Thus we get a complete answer for Problem 1bis of irreducible extensions as follows, since we know Lemma 5.1.

Theorem 5.8. (i) *Let* (ρ, V_0) *be an irreducible* $(\mathfrak{g}_0, K_0)$*-module of* $\mathfrak{g}_0=\mathfrak{sp}(2;\boldsymbol{R})$, $K_0\subset Sp(2;\boldsymbol{R})$. *Then,* ρ *has exactly two inequivalent irreducible extensions except for* $\rho=\mathcal{D}_{0,0}$ *and* $D^{\alpha}_{1/2}$ $(\alpha=\pm)$, *each of which has only a unique such extension.*

(ii) *Assume that* $\rho(\Delta)=(c^2-1)\,I$ *with* $c\in\boldsymbol{C}$. *Then such an extension* (π, V), $V=V_0+V_1$, *corresponds canonically one to one way to the choice of the operator* $A=A_{1\bar{1}}=B(\xi_1,\xi_{\bar{1}})-B(\xi_{\bar{1}},\xi_1)$ *as* $A=(\gamma c+1)\,I$, $\gamma=\pm1$. *The odd part* V_1 *of* V *as* $\mathfrak{g}_0$*-module is given by the formula* (5.29). *The* $\mathfrak{g}_1$*-actions* $V_0\to V_1$ *and* $V_1\to V_0$ *are given respectively by the formulas* (5.31) *and* (5.32). *The case* $c=0$ *can be included in this statements.*

To illustrate these results, we summarize them in a table.

TABLE 5.1. The operator $A=A_{1\bar{1}}=(\gamma c+1)\,I$ with $\gamma=\pm1$ for $c\neq0$; $A=I$ for

$c=0$, where we put $c=2\mu-1$ and $c=N$ for D_{μ}^{α} and F_N respectively.

(ρ, V_0)	$W=\mathfrak{g}_{1,C}\otimes V_0$	$\mathfrak{m}$	V_1
$\mathcal{D}_{c,\nu}$ $(c\not\equiv\nu+1(2), (c,\nu)\neq(0,0))$	$\mathcal{D}_{c+1,\nu+1}\oplus\mathcal{D}_{-c+1,\nu+1}$	$\mathcal{D}_{-\gamma c+1,\nu+1}$	$\mathcal{D}_{\gamma c+1,\nu+1}$
D_{μ}^{α} $(\mu\in(1/2)\boldsymbol{Z}, \geqq 1)$	$D_{\mu+1/2}^{\alpha}\oplus D_{\mu+1/2}^{\alpha}$	$D_{\mu-\gamma/2}^{\alpha}$	$D_{\mu+\gamma/2}^{\alpha}$
F_N $(N\in\boldsymbol{Z}, \geqq 2)$	$F_{N+1}\oplus F_{N-1}$	$F_{N-\gamma}$	$F_{N+\gamma}$
F_1	F_2	(0) or F_2 $(\gamma=+1$ or $-1)$	F_2 or (0)
$\mathcal{D}_{0,0}$	$2\cdot\mathcal{D}_{1,1}$ (not direct)	$\mathcal{D}_{1,1}$	$\mathcal{D}_{1,1}$
$D_{1/2}^{\alpha}$ $(\alpha=\pm)$	$F_1+2\cdot D_1^{\alpha}$ (not direct)	$F_1+D_1^{\alpha}$	D_1^{α}

Further, we have a unified formula for the $\mathfrak{g}_{1,C}$-action on V_0 and V_1, and that for the key bilinear map B: $\mathfrak{g}_{1,C}\times\mathfrak{g}_{1,C}\to\mathfrak{gl}(V_0)$ as follows.

FORMULA 5.2. Formula for $\mathfrak{g}_{1,C}$-action:

$$(5.35)\qquad \begin{aligned} u_l\cdot v_m &= l d_{m+l,l\gamma}[w_{m+l}] \quad (l=\pm 1, m\in\Omega),\\ u_l\cdot[w_m] &= l d_{m,-l\gamma}\, v_{m+l} \quad (l=\pm 1, m\in\Omega(V_1)). \end{aligned}$$

Formula for B: for $l, l'=\pm 1$, $m\in\Omega$,

$$(5.36)\qquad B(u_{l'}, u_l)\, v_m = ll' d_{m+l,l\gamma}\, d_{m+l,-l'\gamma}\, v_{m+l+l'}.$$

5.7. Classification of irreducible representations. Now we get directly from the above results the following classification theorem of irreducible representations of $\mathfrak{osp}(2/1)$. Note that $\mathcal{D}_{c,\nu}=\mathcal{D}_{-c,\nu}$ in our convension.

Theorem 5.9. *Any irreducible representation (π, V), $V=V_0+V_1$, of the real Lie superalgebra $\mathfrak{g}=\mathfrak{osp}(2/1)$ is equivalent, modulo exchange of the roles of the even part V_0 and the odd part V_1, to one of $V=V_0+V_1$ in the following list.*

V_0 *as* $\mathfrak{g}_0$-*module*	V_1 *as* $\mathfrak{g}_0$-*module*
$\mathcal{D}_{c,\nu}$ $(c\not\equiv\nu+1 \pmod 2)$	$\mathcal{D}_{-c+1,\nu+1}$
D_{μ}^{α} $(\mu\in(1/2)\boldsymbol{Z}, \geqq 1/2)$	$D_{\mu+1/2}^{\alpha}$
F_N $(N\in\boldsymbol{Z}, \geqq 1)$	F_{N-1} $(F_0=(0))$

5.8. Irreducible unitary representations. As for unitary extension problem,

Problem 2bis, we see easily from Theorem 5.8 and Table 5.1 that if an irreducible unitary $(\mathfrak{g}_0, K_0)$-module ρ has an irreducible unitary extension (=IUE) to $\mathfrak{g}$, then ρ should belong to the discrete series or its limit D_μ^α $(\mu\geqq 1/2)$ or be the trivial representation F_1 (cf. also the remark at the end of 5.1).

Let $\rho=D_\mu^\alpha$ and check if it actually has IUEs. For this, it is necessary and sufficient to verify the invariance (1.9) and the positive definiteness (UNI) for the $\mathfrak{g}_0$-invariant inner products on V_0 and V_1. However the positive definiteness comes from the very definition of inner products.

The invariance can be proved by using the formulas (5.35) for $\mathfrak{g}_1$-actions and (5.36) for the bilinear map B as follows. First we see easily that, for the invariance (1.9), it is enough to prove

$$j^2\langle\pi(u)\, v^0,\, v^1\rangle+\langle v^0,\, \pi(\bar{u})\, v^1\rangle = 0\,, \tag{5.37}$$

for any $u\in\mathfrak{g}_{1,C}$, $v^p\in V_p (p=0, 1)$, where $u\mapsto\bar{u}$ denotes the conjugation of $\mathfrak{g}_{1,C}$ with respect to $\mathfrak{g}_1$. Since $\mathfrak{g}_{1,C}$, V_0 and V_1 are respectively spanned by $\{u_1, u_{-1}\}$, $\{v_{m'};\ m'\in\Omega\}$ and $\{\pi(u_l)\, v_m;\ l=\pm 1,\ m\in\Omega\}$, we put $u=u_{l'}$, $v^0=v_{m'}$ and $v^1=\pi(u_l)\, v_m=u_l\cdot v_m$. Then, the 2nd term of (5.37) equals to

$$\langle v_{m'},\, \pi(\bar{u}_{l'})\, \pi(u_l)\, v_m\rangle = \langle v_{m'},\, B(\bar{u}_{l'},\, u_l)\, v_m\rangle = i\langle v_{m'},\, B(u_{-l'},\, u_l)\, v_m\rangle$$

because $\bar{u}_{l'}=-iu_{-l'}$ for $l'=\pm 1$. Put $j^2=\varepsilon i$, then the equation (1.9) turns out finally to

$$\varepsilon\langle u_{l'}\cdot v_{m'},\, u_l\cdot v_m\rangle+\langle v_{m'},\, B(u_{-l'},\, u_l)\, v_m\rangle = 0\,. \tag{5.38}$$

Now apply the formulas (5.35)–(5.36), then we get the equation

$$\begin{aligned}\varepsilon ll' d_{m'+l',l'\gamma}\, \bar{d}_{m+l,l\gamma}\, &\langle [w_{m'+l'}],\, [w_{m+l}]\rangle - \\ &-ll'\bar{d}_{m+l,l\gamma}\, \bar{d}_{m+l,l'\gamma}\, \langle v_{m'},\, v_{m+l-l'}\rangle = 0\,.\end{aligned} \tag{5.39}$$

Both sides equal to zero unless $m'+l'=m+l$. So, assume $m'+l'=m+l$, then we come to

$$(\varepsilon d_{m+l,l'\gamma}-\bar{d}_{m+l,l'\gamma})\, \bar{d}_{m+l,l\gamma} = 0\,. \tag{5.40}$$

Note that $\bar{d}_{m+l,k}=\operatorname{sgn}(m+l+kc)\, d_{m+l,k}$ if c is real, and that, for $\rho=D_\mu^\alpha$, we have $c=2\mu-1$ and so

$$d_{m+l,k} = \sqrt{m+l+kc} = \sqrt{m+l+k(2\mu-1)} \quad (l=\pm 1,\ k=\pm 1)\,,$$

On the other hand, for $\rho=D_\mu^\alpha$, the set of weights Ω is given by

$$\Omega = \{m;\ m\geqq 2\mu,\ m\equiv 2\mu\ (2)\} \qquad \text{for} \quad \alpha=+\,,$$

$$\Omega = \{m;\ m\leqq -2\mu,\ m\equiv -2\mu\ (2)\} \qquad \text{for} \quad \alpha=-\,.$$

Therefore we see that (5.40) is satisfied with $\varepsilon=1$ or -1 according as $\alpha=+$ or $-$, whence the invariance (1.9) holds with the associated constants $j^2=i$ and $-i$ for D_μ^+ and D_μ^- respectively.

Thus we get a complete solution for Problem 2bis as follows.

Theorem 5.10. *Let* (ρ, V_0) *be an irreducible unitary representation* $(=IUR)$ *of* $\mathfrak{g}_0=\mathfrak{sp}(2;\boldsymbol{R})$.

(i) (ρ, V_0) *has an irreducible unitary extension* $(=IUE)$ *if and only if* ρ *is a highest or a lowest weight module.*

(ii) *For* $\rho=D_\mu^\alpha$ *with* $\alpha=\pm$, $\mu\in(1/2)\boldsymbol{Z}$, $\geqq 1$, *there exist exactly two IUEs* (π, V), $V=V_0+V_1$, *up to equivalence, for which* $V_1=D_{\mu+1/2}^\alpha$ *or* $D_{\mu-1/2}^\alpha$ *(as* $\mathfrak{g}_0$*-modules) in Table* 5.1.

(iii) *For* $\rho=D_{1/2}^\alpha$ $(\alpha=\pm)$ *or* $\rho=F_1$, *the trivial representation, there exists exactly one IUE, up to equivalence, for which* $V_1=D_1^\alpha$ *or* (0) *respectively, in Table* 5.1.

From this result for IUEs, we get directly the classification of IURs of $\mathfrak{g}=\mathfrak{osp}(2/1)$ as follows.

Theorem 5.11. *For* $\mathfrak{g}=\mathfrak{osp}(2/1)$, *any IUR* (π, V), $V=V_0+V_1$, *is equivalent, up to exchange of the roles of* V_0 *and* V_1, *to one of irreducible representations in the list in Theorem* 5.9 *for which*

$$(V_0, V_1)=(D_\mu^\alpha, D_{\mu+1/2}^\alpha) \quad \textit{with} \quad \mu\in(1/2)\boldsymbol{Z}, \geqq 1/2, \quad \textit{or}$$

$$(V_0, V_1)=(F_1, (0)).$$

5.9. Irreducible representations with invariant inner products. There exist many irreducible representations (π, V), apart from unitary ones, which has a (non-degenerate, hermitian) inner product $\langle\cdot,\cdot\rangle$ on V such that $V_0\perp V_1$ and with the invariance property (1.8)–(1.9). Let us make some remarks about this kind of representations.

First we know all the irreducible $(\mathfrak{g}_0, K_0)$-modules of $\mathfrak{g}_0=\mathfrak{sp}(2;\boldsymbol{R})$ admitting an invariant inner product as follows.

Lemma 5.12. *An irreducible* $(\mathfrak{g}_0, K_0)$*-module* (ρ, V_0) *has a (non-degenerate) invariant inner product if and only if it is equivalent to one of the following*:

$$\mathcal{D}_{c,\nu} \quad \textit{with} \quad c\in\boldsymbol{R} \quad \textit{or} \quad \in\sqrt{-1}\,\boldsymbol{R},\ \nu=\pm1,\ c\not\equiv\nu+1(2);$$

$$D_\mu^\alpha \quad \textit{with} \quad \alpha=\pm,\ \mu\in(1/2)\boldsymbol{Z},\ \geqq 1/2;$$

$$F_N \quad \textit{with} \quad N\in\boldsymbol{Z},\ \geqq 1.$$

Moreover such an inner product on V_0 is given by making the standard basis $\{v_m;\, m\in\Omega\}$ for (5.10) an orthonormal system such that

$$\langle v_m, v_{m'}\rangle=\kappa_m\,\delta_{m,m'}, \tag{5.41}$$

where $\delta_{m,m'}$ is the Kronecker's symbol and $\kappa_m=\kappa_m(V_0)=\pm1$ is determined so as to hold

$$\kappa_{m+2}=\operatorname{sgn}((m+1)^2-c^2)\,\kappa_m \quad \text{if} \quad m, m+2\in\Omega. \tag{5.42}$$

Put $n_+(V_0)=\#\{m;\ \kappa_m>0\}$, $n_-(V_0)=\#\{m;\ \kappa_m<0\}$, and call $(n_+(V_0), n_-(V_0))$ the index of the inner product on V_0.

Remark 5.13. The set of irreducible $(\mathfrak{g}_0, K_0)$-modules in the above lemma correspond canonically to the totality of irreducible components of the representations of $G_0=Sp(2;\boldsymbol{R})$ induced from real or unitary characters of a minimal parabolic subgroup.

Now consider irreducible extensions (π, V), $V=V_0+V_1$, of a (ρ, V_0) in the above list. Then we get from Table 5.1 the following.

Lemma 5.14. *In Table* 5.1, *the pairs* (V_0, V_1) *for which both* V_0 *and* V_1 *admit* $\mathfrak{g}_0$*-invariant inner products, are those with* $V_0=\mathcal{D}_{c,\nu}$ $(c\in\boldsymbol{R},\ c\neq\nu+1(2))$, D^{α}_{μ} $(\alpha=\pm,$ $\mu\in(1/2)\boldsymbol{Z},\ \geqq 1/2)$ *and* F_N $(N\in\boldsymbol{Z},\ \geqq 1)$.

Let us study the $\mathfrak{g}_1$-invariance (1.9) for the pairs (V_0, V_1) in Lemma 5.14. Then, similarly as in the unitary case, we see that (1.9) is equivalent to (5.39). Put

$$\kappa_m(V_0)=\langle v_m, v_m\rangle \quad (m\in\Omega), \quad \kappa_m(V_1)=\langle [w_m], [w_m]\rangle \quad (m\in\Omega(V_1)).$$

Then, $\kappa_m=\kappa_m(V_0)$ satisfy (5.42) and $\kappa_m(V_1)$ satisfy

$$\kappa_{m+2}(V_1)=\mathrm{sgn}\,((m+1)^2-(\gamma c+1)^2)\,\kappa_m(V_1)\,. \tag{5.43}$$

Taking into account (5.41), we see that, in the present case, (5.39) is equivalent to the following for $m'+l'=m+l$ $(m, m'\in\Omega,\ l, l'=\pm 1)$:

$$(\varepsilon d_{m+l,l'\gamma}\,\kappa_{m+l}(V_1)-\bar{d}_{m+l,l'\gamma}\,\kappa_{m+l-l'}(V_0))\,d_{m+l,l\gamma}=0\,. \tag{5.44}$$

Recall that $d_{p,k}=\sqrt{p+kc}$, and $\bar{d}_{p,k}=\mathrm{sgn}(p+kc)\,d_{p,k}$ if c is real. Then the above equations for $l=\pm 1$, $l'=\pm 1$, are in total equivalent to the following:

$$\varepsilon\kappa_{m+1}(V_1)=\mathrm{sgn}\,(m+1+\gamma c)\,\kappa_m(V_0) \qquad (m\in\Omega)\,, \tag{5.45}$$

$$\varepsilon\kappa_{m+1}(V_1)=\mathrm{sgn}\,(m+1-\gamma c)\,\kappa_{m+2}(V_0) \quad (m+2\in\Omega)\,, \tag{5.45'}$$

where we understand that if $m+1+\gamma c=0$ or $m+1-\gamma c=0$ (each very rare), then the corresponding equation does not exist. Using (5.42) for $\kappa_m=\kappa_m(V_0)$, and (5.43) for $\kappa_m(V_1)$, we can prove that, when we choose $\varepsilon=\pm 1$ so that (5.45) holds for an $m=m_0\in\Omega$ such that $m_0+1\in\Omega(V_1)$, then, for this choice of ε, (5.45) and (5.45′) hold for any possible m. Thus we have proved that for any pair (V_0, V_1) in Lemma 5.14, the corresponding representation (π, V), $V=V_0+V_1$, admits an invariant inner product.

Summarizing these results, we get the following

Theorem 5.15. *Let* (ρ, V_0) *be an irreducible* $(\mathfrak{g}_0, K_0)$*-module of* $\mathfrak{g}_0=\mathfrak{sp}(2;\boldsymbol{R})$ *which admits a (non-degenerate) invariant inner product.*

(i) *An irreducible extension* (π, V), $V=V_0+V_1$, *of* (ρ, V_0) *to* $\mathfrak{g}=\mathfrak{osp}(2/1)$ *admits an invariant inner product if and only if* ρ *does not belong to the unitary continuous* *principal series with regular infinitesimal characters, i.e.,* $\rho\not\cong\mathcal{D}_{i\eta,\nu}$ *for any* $\eta\in\boldsymbol{R}$, $\neq 0$, $\nu=0, 1$. *Or equivalently,* ρ *is equivalent to one of those in Lemma* 5.14.

(ii) *Assume that* ρ *is listed up in Lemma* 5.14. *Then any of its irreducible extensions admits an invariant inner product which is given by* (5.41)–(5.43). *The associated constant* $j^2=\varepsilon i$ *is determined by* (5.45) *for an* (*or any*) $m\in\Omega$ *such that* $m+1\in\Omega(V_1)$.

The associated constant $j^2=\varepsilon i$ is changed to $-\varepsilon i$ if we multiply by -1 the inner product on V_1. Therefore, in case the index $(n_+(V_1), n_-(V_1))$ of the inner product on V_1 is equal to (∞, ∞), there is no apriori standard to determine the sign ε.

From the above result for extensions, we get the following classification of irreducible representations with invariant inner products.

Theorem 5.16. *Any irreducible representation* (π, V), $V=V_0+V_1$, *with* (*non-degenerate*) *invariant inner product is equivalent, up to exchange of the roles of* V_0 *and* V_1, *to one of those corresponding to the following pairs* (V_0, V_1) *of* $\mathfrak{g}_0$*-modules.*

V_0 *as* $\mathfrak{g}_0$*-module*	V_1 *as* $\mathfrak{g}_0$*-module*
$\mathcal{D}_{c,\nu}$ $(c\in\boldsymbol{R},\ \nu=0, 1,\ c\not\equiv\nu+1(2))$	$\mathcal{D}_{-c+1,\nu+1}$
D^{α}_{μ} $(\mu\in(1/2)\boldsymbol{Z},\ \geqq 1/2)$	$D^{\alpha}_{\mu-1/2}$
F_N $(N\in\boldsymbol{Z},\ \geqq 1)$	F_{N-1} $(F_0=(0))$

Remark 5.17. Let (ρ, V_0) be in the complementary series, that is, $\rho\cong\mathcal{D}_{c,0}$ with $0<c<1$. Then ρ is unitary but has not any irreducible unitary extension to $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1$. However it has two irreducible extensions (π, V), $V=V_0+V_1$ with $V_1=\mathcal{D}_{c+1,1}$ or $\mathcal{D}_{-c+1,1}$ (as $\mathfrak{g}_0$-module), and both of them admit invariant inner products. Note that the index $(n_+(V_1), n_-(V_1))$ of the inner product on V_1 is equal to (∞, ∞) in both cases. If we define an invariant inner product on V_1 in such a manner that $\langle[w_1], [w_1]\rangle>0$ for the weight vector $[w_1]$ with weight 1, then we have always $j^2=i$ for (π, V) with $V_1=\mathcal{D}_{\gamma c+1,1}$ $(\gamma=\pm 1)$.

For $\rho=\mathcal{D}_{0,0}$ and its unique irreducible extension (π, V) with $V_1=\mathcal{D}_{1,1}$, similar statements are true.

§6. Irreducible unitary extensions for type $A(1, 0)$, Part I.

In this section we take the complex Lie superalgebra of type $A(1, 0)$, and also its real form as the Lie superalgebra $\mathfrak{g}$ in Problem 2, and determine all the irreducible unitary extensions of irreducible representations of the even part $\mathfrak{g}_0$. But, as is shown in Example 4.8, when we take $A(1, 0)=\mathfrak{sl}(2, 1)$ as $\mathfrak{g}$ in the unitary extension problem, there exists no irreducible unitary extensions (=IUEs) except for the trivial representation which has the trivial extension. Thus we study irreducible unitary extensions for each real form $\mathfrak{g}$ (cf. [3]).

6.1. Definitions for $A(m, n)$**.** First we define the Lie superalgebra of type

$A(m, n)$. We denote by $M(p, q; K)$ the set of all matrices of type $p\times q$ with entries in a field K. Let $\mathfrak{v}=M(m+n, m+n; \boldsymbol{C})$, and let $E_{i,j}$, $1\leqq i, j\leqq m+n$, be an element of $\mathfrak{v}$ with components 1 at (i, j) and 0 elsewhere. Let $\mathfrak{v}_0$ be a complex subspace of $\mathfrak{v}$ generated by

$$\{E_{i,j};\ 1\leqq i, j\leqq m\} \cup \{E_{i,j};\ m+1\leqq i, j\leqq m+n\}\ .$$

Further let $\mathfrak{v}_{1,+}$(resp. $\mathfrak{v}_{1,-}$) be a complex subspace of $\mathfrak{v}$ generated by

$$\{E_{i,j};\ 1\leqq i\leqq m,\ m+1\leqq j\leqq m+n\}\ ,$$
$$(\text{resp. } \{E_{i,j};\ m+1\leqq i\leqq m+n,\ 1\leqq j\leqq m\})\ ,$$

and put $\mathfrak{v}_1=\mathfrak{v}_{1,+}+\mathfrak{v}_{1,-}$.

The bracket product

$$[X, Y] = XY-(-1)^{st}\, YX \qquad \text{for} \quad X\in\mathfrak{v}_s,\ Y\in\mathfrak{v}_t\ ,$$

where s, t are 0 or 1, makes $\mathfrak{v}$ a Lie superalgebra, denoted by $\mathfrak{l}(m, n)$, where $\mathfrak{l}(m, n)_s=\mathfrak{v}_s$ (Example 4.7). We put $\mathfrak{l}(m, n)_{1,\pm}=\mathfrak{v}_{1,\pm}$. On $\mathfrak{l}(m, n)$, there defined the supertrace str, a linear form on $\mathfrak{l}(m, n)$, in (4.8). We defined $\mathfrak{sl}(m, n)$ as

$$\mathfrak{sl}(m, n) = \{X\in\mathfrak{l}(m, n);\ \operatorname{str} X = 0\}\ .$$

This is a subalgebra in $\mathfrak{l}(m, n)$ of codimension 1. In case $m=n$, $\mathfrak{sl}(n, n)$ has one-dimensional center $\mathfrak{z}$ consisting of scalar matrices $\lambda\cdot I_{2n}$ ($\lambda\in\boldsymbol{C}$). We set

$$A(m, n) = \mathfrak{sl}(m+1, n+1) \qquad \text{for} \quad m, n\geqq 0,\ m\neq n\ ,$$
$$A(n, n) = \mathfrak{sl}(n+1, n+1)/\mathfrak{z} \qquad \text{for} \quad n>0\ .$$

We denote by $\mathfrak{g}_{\boldsymbol{C}}$ the complex algebra $A(1, 0)$, keeping the symbol $\mathfrak{g}$ to its real form. For later use, we give two kinds of basis of a Cartan subalgebra $\mathfrak{h}_{\boldsymbol{C}}$ of $\mathfrak{g}_{\boldsymbol{C}}$:

$$H_{1,1} = E_{1,1}+E_{3,3}\ ,\quad H_{2,2} = E_{2,2}+E_{3,3}\ , \tag{6.1}$$

and

$$H = E_{1,1}-E_{2,2}\ ,\quad C = E_{1,1}+E_{2,2}+2E_{3,3}\ . \tag{6.2}$$

6.2. Real forms of $A(1, 0)$**.** Here we list up real forms $\mathfrak{g}$ of $\mathfrak{g}_{\boldsymbol{C}}=A(1, 0)$ (cf. [8, §5]). We define two types of real subalgebras of a Lie superalgebra $\mathfrak{g}_{\boldsymbol{C}}$. A real subalgebra of first type is

$$\mathfrak{sl}(2, 1; \boldsymbol{R}) = \mathfrak{sl}(2, 1)\cap M(3, 3; \boldsymbol{R})\ .$$

Real subalgebras of second type are defined as follows. Let $p\in\{0, 1, 2\}$ and $q\in\{0, 1\}$. Put for $s=0, 1$,

$$\mathfrak{su}(2, 1; p, q)_s = \{X\in\mathfrak{sl}(2, 1)_s;\ J_{p,q}\, X+{}^t\bar{X}\, J_{p,q}\, J_0^{\,s} = 0\}\ ,$$

where ${}^tX$ is the transposed matrix of X, and

$$J_{p,q} = \mathrm{diag}(a, b, -(-1)^q\sqrt{-1}), \quad J_0 = \mathrm{diag}(-1, -1, 1),$$

with $(a, b)=(-1, -1)$ for $p=0$, $(a, b)=(1, -1)$ for $p=1$, $(a, b)=(1, 1)$ for $p=2$, where diag $(\cdot, \cdot, \cdot)$ denotes a diagonal matrix. Then $\mathfrak{su}(2, 1; p, q)=\mathfrak{su}(2, 1; p, q)_0 \oplus\mathfrak{su}(2, 1; p, q)_1$ is a real Lie superalgebra for each (p, q).

Proposition 6.1 (cf. [8, §5]). *Real forms of* $A(1, 0)$ *are isomorphic, up to transition to their duals, to one of the following three types:*

$$(a)\ \ \mathfrak{sl}(2, 1; \boldsymbol{R}); \qquad (b)\ \ \mathfrak{su}(2, 1; 2, 1); \qquad (c)\ \ \mathfrak{su}(2, 1; 1, 1).$$

6.3. Extension problem for the Case (a): $\mathfrak{g}=\mathfrak{sl}(2, 1; \boldsymbol{R})$. Let $\mathfrak{g}=\mathfrak{sl}(2, 1; \boldsymbol{R})$. Then there exist no IUEs except the case of the trivial representation which has a trivial extension.

More generally, for this type of real form $\mathfrak{sl}(m, n; \boldsymbol{R})$ of $\mathfrak{sl}(m, n)$, we have a similar situation as above, as shown in the next subsection.

6.4. Extension problem for $\mathfrak{sl}(m, n; \boldsymbol{R})$. Put $\mathfrak{g}=\mathfrak{sl}(m, n; \boldsymbol{R})\equiv\mathfrak{sl}(m, n)\cap M(m+n, m+n; \boldsymbol{R})$, and $\mathfrak{g}_{1,\pm}=\mathfrak{g}\cap\mathfrak{l}(m, n)_{1,\pm}$.

Theorem 6.2. *Let* $\mathfrak{g}=\mathfrak{sl}(m, n; \boldsymbol{R})$, $m, n\geqq 1$. *Then it has only a unique irreducible unitary representation, the trivial one.*

Proof. Let π be an irreducible unitary representation of $\mathfrak{g}$ on $V=V_0+V_1$, and put $\rho=\pi(\mathfrak{g}_0)\,|\,V_0$. Let $B(\cdot, \cdot)$ be the bilinear map $\mathfrak{g}_1\times\mathfrak{g}_1\to\mathfrak{gl}(V_0)$, associated with π.

We examine four conditions (EXT1)–(EXT3) and (UNI). The condition (EXT2) implies that

$$(6.3)\qquad B(E_{i,j}, E_{i,j}) = 0 \qquad \text{for} \quad E_{i,j}\in\mathfrak{g}_1,$$

$$(6.4)\qquad B(E_{i,j}, E_{k,l})+B(E_{k,l}, E_{i,j}) = 0 \qquad \text{for} \quad E_{i,j}, E_{k,l}\in\mathfrak{g}_{1,+} \quad (\text{or} \quad E_{i,j}, E_{k,l}\in\mathfrak{g}_{1,-}),$$

$$(6.5)\qquad B(E_{i,j}, E_{j,i})+B(E_{j,i}, E_{i,j}) = \rho(E_{i,i}+E_{j,j}) \qquad \text{for} \quad E_{i,j}\in\mathfrak{g}_1,$$

$$(6.6)\qquad B(E_{i,j}, E_{k,l})+B(E_{k,l}, E_{i,j}) = \rho(\delta_{i,l}\, E_{k,j}+\delta_{k,j}\, E_{i,l}) \qquad \text{for} \quad E_{i,j}\in\mathfrak{g}_{1,+} \quad \text{and} \quad E_{k,l}\in\mathfrak{g}_{1,-},$$

where $\delta_{i,j}$ denotes Kronecker's δ. Now apply the condition

$$(\mathrm{UNI}')\qquad j^2B(\xi, \xi)\geqq 0 \quad (j^4 = -1)$$

for $\xi=E_{i,j}\pm E_{j,i}\in\mathfrak{g}_1$, and use (6.3) and (6.6), then

$$j^2\rho(E_{i,i}+E_{j,j})\geqq 0 \quad \text{and} \quad -j^2\rho(E_{i,i}+E_{j,j})\geqq 0.$$

Therefore

$$(6.7)\qquad \rho(E_{i,i}+E_{j,j}) = 0 \qquad \text{for} \quad 1\leqq i\leqq m;\ m+1\leqq j\leqq m+n.$$

Similarly as above, apply (UNI′) to

$$\xi = E_{i,m+1} \pm E_{m+1,j} \quad (1 \leqq i, j \leqq m,\ i \neq j),$$

and use (6.3) and (6.6), then

$$\rho(E_{i,j}) = 0 \qquad \text{for} \quad 1 \leqq i, j \leqq m,\ i \neq j. \tag{6.8}$$

Similarly, using

$$\xi = E_{1,j} \pm E_{i,1} \quad (m+1 \leqq i, j \leqq m+n,\ i \neq j),$$

and (6.3) and (6.6), we get

$$\rho(E_{i,j}) = 0 \qquad \text{for} \quad m+1 \leqq i, j \leqq m+n,\ i \neq j. \tag{6.9}$$

Equations (6.7)–(6.9) imply that $\rho=0$, or dim V_0-multiple of the one-dimensional trivial representation of $\mathfrak{g}_0$.

We now prove that $\rho=0$ has an IUE, the trivial representation, if and only if dim $V_0=1$. We show $B(\xi, \eta)=0$ for any $\xi, \eta \in \mathfrak{g}_1$, case by case.

Case 1. Put

$$C = n \sum_{i=1}^{m} E_{i,i} + m \sum_{j=m+1}^{m+n} E_{j,j},$$

then

$$[C, \xi] = \pm(n-m)\, \xi \qquad \text{for} \quad \xi \in \mathfrak{g}_{1,\pm}. \tag{6.10}$$

We apply the condition (EXT1) for $X=C$, $\xi, \eta \in \mathfrak{g}_{1,+}$ (or $\xi, \eta \in \mathfrak{g}_{1,-}$), and use (6.10), then,

$$(n-m)\, B(\xi, \eta) + (n-m)\, B(\xi, \eta) = 0.$$

Hence

$$B(\xi, \eta) = 0 \qquad \text{for} \quad \xi, \eta \in \mathfrak{g}_{1,+} \quad (\text{or} \quad \xi, \eta \in \mathfrak{g}_{1,-}), \tag{6.11}$$

for $m \neq n$. Even when $m=n$, we can see that (6.11) still holds.

Case 2. Apply (EXT1) for $X=E_{i,i}-E_{j,j}$ $(i \neq j)$, $\xi=E_{i,k} \in \mathfrak{g}_{1,+}$, $\eta=E_{l,j} \in \mathfrak{g}_{1,-}$ $(1 \leqq i, j \leqq m;\ m+1 \leqq k, l \leqq m+n)$, then

$$B(E_{i,k}, E_{l,j}) + B(E_{i,k}, E_{l,j}) = 0.$$

Hence

$$B(E_{i,k}, E_{l,j}) = 0 \qquad \text{for} \quad E_{i,k} \in \mathfrak{g}_{1,+},\ E_{l,j} \in \mathfrak{g}_{1,-},\ i \neq j.$$

Case 3. Apply (EXT1) for $X=E_{k,k}-E_{l,l}(k \neq l)$, $\xi=E_{i,k} \in \mathfrak{g}_{1,+}$, $\eta=E_{l,j} \in \mathfrak{g}_{1,-}$ $(1 \leqq i, j \leqq m;\ m+1 \leqq k, l \leqq m+n)$, then

$$B(E_{i,k}, E_{l,j}) + B(E_{i,k}, E_{l,j}) = 0,$$

whence

$$B(E_{i,k}, E_{l,j}) = 0 \quad \text{for} \quad E_{i,k} \in \mathfrak{g}_{1,+}, E_{l,j} \in \mathfrak{g}_{1,-}, k \neq l.$$

Case 4. We apply (EXT3) for $\tau = E_{i,k}$, $\xi = E_{j,l}$, $\eta = E_{l,j}$, $\xi = E_{k,i}$ $(1 \leqq i, j \leqq m;$ $m+1 \leqq k, l \leqq m+n$, $i=j$ or $k=l$, $(i, k) \neq (j, l))$, then

$$(6.12) \quad B(E_{i,k}, E_{j,l})\, B(E_{l,j}, E_{k,i}) + B(E_{i,k}, E_{l,j})\, B(E_{j,l}, E_{k,i}) = B(E_{i,k}, E_{k,i}).$$

When $m+n \geqq 3$, for any (i, k) there is at least one pair (j, l) satisfying the above conditions, and so we get from (6.12) and (6.11)

$$B(E_{i,k}, E_{k,i}) = 0 \quad \text{for all} \quad E_{i,k} \in \mathfrak{g}_{1,+}, E_{k,i} \in \mathfrak{g}_{1,-}.$$

When $m=n=1$, this also holds.

We see from Cases 1~4,

$$B(E_{i,j}, E_{k,l}) = 0 \quad \text{for all} \quad E_{i,j}, E_{k,l} \in \mathfrak{g}_1.$$

Therefore the subalgebra $\mathfrak{m}$ in $W = \mathfrak{g}_{1,C} \otimes_C V_0$ is equal to W itself. Hence $V_1 = W/\mathfrak{m} = (0)$ and the extension π is trivial. Q.E.D.

6.5. The conditions (EXT1)–(EXT3) for a real form of $\mathfrak{sl}(2, 1)$. Before examining Cases (b) and (c), we write down the conditions (EXT1)–(EXT3) using $\{E_{i,j}\}$, the basis of $\mathfrak{g}_C$. Then we see that for any real form, they have the same form.

For $i, j \in \{1, 2\}$, put

$$B_{i,j} = B(E_{i,3}, E_{j,3}), \quad B_{-i,-j} = B(E_{3,i}, E_{3,j});$$
$$B_{i,-j} = B(E_{i,3}, E_{3,j}), \quad B_{-i,j} = B(E_{3,i}, E_{j,3});$$

and for $k, l \in \{\pm 1, \pm 2\}$, put

$$(6.13) \quad A_{k,l} = B_{k,l} - B_{l,k}.$$

Lemma 6.3. *For $i, j \in \{1, 2\}$,*

$$B_{i,j} = B_{-i,-j} = 0; \quad A_{i,j} = A_{-i,-j} = 0.$$

Proof. Let (π, V), $V = V_0 + V_1$, be an extension of (ρ, V_0). Decompose V into eigenspaces for $C = E_{1,1} + E_{2,2} + 2E_{3,3}$, an element of the center of $\mathfrak{g}_{0,C}$. Then the irreducibility of V_0 implies that V_0 is in a unique eigenspace for C.

On the other hand, we get from (EXT1),

$$(6.14) \quad [\rho(C), B_{i,j}] = -2B_{i,j}, \quad [\rho(C), B_{-i,-j}] = 2B_{-i,-j},$$

for $i, j \in \{1, 2\}$. It follows from this that $B_{i,j} = B_{-i,-j} = 0$. In fact, assume $B_{i,j} \neq 0$. Then there is a $v \in V_0$ such that $B_{i,j} v \neq 0$. We see from (6.14) that C-eigenvalues of v and $B_{i,j} v \in V_0$ are different. This contradiction gives $B_{i,j} = 0$. Similarly we get $B_{-i,-j} = 0$.

The assertion $A_{i,j}=A_{-i,-j}=0$ is implied from the above result through the defintion (6.13). Q.E.D.

Consider $\mathfrak{g}_{1,C}$ as a $\mathfrak{g}_{1,C}$-module. Then the condition (EXT1) is equivalent to the condition that the map B: $\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\to\mathfrak{gl}(V_0)$ is a $\mathfrak{g}_{0,C}$-homomorphism. This $\mathfrak{g}_{0,C}$-equivariance is very important to simplify the condition (EXT3), as seen in the proof of the following proposition.

Proposition 6.4. *Suppose that a map* B: $\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\to\mathfrak{gl}(V_0)$, *is* $\mathfrak{g}_{0,C}$*-equivariant and that the condition* (EXT2) *holds. Then the condition* (EXT3) *is equivalent to the following conditions* (EXT3.1)–(EXT3.8) *modulo* $\mathfrak{g}_{0,C}$*-equivariance in the sense in* §3.6:

(EXT3.1) $\quad B_{1,-1}B_{1,-1}=B_{1,-1}\rho(H_{1,1})$,

(EXT3.2) $\quad B_{1,-1}B_{2,-2}=-B_{1,-1}+B_{1,-2}\rho(E_{2,1})$,

(EXT3.3) $\quad B_{1,-1}B_{1,-2}=B_{1,-2}+B_{1,-2}\rho(H_{1,1})$,

(EXT3.4) $\quad B_{1,-1}B_{2,-1}=B_{1,-1}\rho(E_{2,1})$,

(EXT3.5) $\quad B_{-1,1}B_{-1,1}=B_{-1,1}\rho(H_{1,1})$,

(EXT3.6) $\quad B_{-1,1}B_{-2,2}=B_{-1,1}+B_{-1,2}\rho(E_{1,2})$,

(EXT3.7) $\quad B_{-1,1}B_{-2,1}=B_{-1,1}\rho(E_{1,2})$,

(EXT3.8) $\quad B_{-1,1}B_{-1,2}=-B_{-1,2}+B_{-1,2}\rho(H_{1,1})$.

Proof. Since the condition (EXT3) is multilinear in τ, ξ, η, $\zeta\in\mathfrak{g}_{1,C}$, it can be considered as a condition for $\tau\otimes\xi\otimes\eta\otimes\zeta$ in

$$\mathfrak{g}_1^{(4)}=\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\otimes\mathfrak{g}_{1,C}\,.$$

Taking into account the $\mathfrak{g}_{0,C}$-equivariance of (EXT3), we study the structure of $\mathfrak{g}_1^{(4)}$ as a $\mathfrak{g}_{0,C}$-module. The space $\mathfrak{g}_1^{(4)}$ is decomposed into 16 invariant subspaces

$$\mathfrak{g}_1^{(4)}(*,*,*,*)=\mathfrak{g}_{1,*}\otimes\mathfrak{g}_{1,*}\otimes\mathfrak{g}_{1,*}\otimes\mathfrak{g}_{1,*}\,,$$

where each $*$ denotes $+$ or $-$ and $\mathfrak{g}_{1,\pm}=\mathfrak{g}_{1,C}\cap\mathfrak{l}(2,1)_{1,\pm}$. It is sufficient to consider the condition (EXT3) on each subspace.

On the other hand, $B(\xi,\eta)=0$ and $[\xi,\eta]=0$ for ξ, $\eta\in\mathfrak{g}_{1,+}$ or ξ, $\eta\in\mathfrak{g}_{1,-}$. Therefore the condition (EXT3) is trivial on the subspace $\mathfrak{g}_1^{(4)}(*,*,*,*)$ with

$$\begin{aligned}(*,*,*,*)=\;&(+,+,+,+),(-,-,-,-),(+,+,+,-),(-,-,-,+),\\&(+,+,-,+),(-,-,+,-),(+,-,+,+),(-,+,-,-),\\&(-,+,+,+),(+,-,-,-)\,.\end{aligned}$$

Moreover, since (EXT3) is symmetric with respect to the second variable ξ and the third one η, the condition (EXT3) on each of the following subspaces are mutually equivalent:

$$\mathfrak{g}_1^{(4)}(+,-,+,-) \quad \text{and} \quad \mathfrak{g}_1^{(4)}(+,+,-,-),$$
$$(\text{resp.} \quad \mathfrak{g}_1^{(4)}(-,+,-,+) \quad \text{and} \quad \mathfrak{g}_1^{(4)}(-,-,+,+)).$$

Now the condition (EXT3) on the subspaces $\mathfrak{g}_1^{(4)}(+,-,-,+)$ and $\mathfrak{g}_1^{(4)}(-,+,+,-)$ is induced in total from that on the subspaces $\mathfrak{g}_1^{(4)}(+,-,+,-)$ and $\mathfrak{g}_1^{(4)}(-,+,-,+)$, using (EXT2). So it is sufficient to consider (EXT3) on the following two subspaces:

$$\mathfrak{g}_1^{(4)}(+,-,+,-) = \mathfrak{g}_{1,+}\otimes\mathfrak{g}_{1,-}\otimes\mathfrak{g}_{1,+}\otimes\mathfrak{g}_{1,-},$$
$$\mathfrak{g}_1^{(4)}(-,+,-,+) = \mathfrak{g}_{1,-}\otimes\mathfrak{g}_{1,+}\otimes\mathfrak{g}_{1,-}\otimes\mathfrak{g}_{1,+}.$$

The invariant subspace $\mathfrak{g}_1^{(4)}(+,-,+,-)$ is generated, as $\mathfrak{g}_{0,C}$-module, by the following four elements:

$$E_{1,3}\otimes E_{3,1}\otimes E_{1,3}\otimes E_{3,1}, \quad E_{1,3}\otimes E_{3,1}\otimes E_{2,3}\otimes E_{3,2},$$
$$E_{1,3}\otimes E_{3,1}\otimes E_{1,3}\otimes E_{3,2}, \quad E_{1,3}\otimes E_{3,1}\otimes E_{2,3}\otimes E_{3,1}.$$

Therefore (EXT3) on the subspace $\mathfrak{g}_1^{(4)}(+,-,+,-)$ is equivalent to the conditions (EXT3.1)–(EXT3.4).

Similarly we get (EXT3.5)–(EXT3.8) from the condition (EXT3) on the subspace $\mathfrak{g}_1^{(4)}(-,+,-,+)$. Q.E.D.

§7. Irreducible unitary extensions for type $A(1, 0)$, Part II.

7.1. Extension problem for the Case (b): $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$. The even part $\mathfrak{g}_0\cong\mathfrak{u}(2)$ and the odd part $\mathfrak{g}_1$ of $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$ are spanned respectively by

$$\{\sqrt{-1}\,H_{1,1}, \sqrt{-1}\,H_{2,2}, \sqrt{-1}\,E_{1,2}+\sqrt{-1}\,E_{2,1}, E_{2,1}-E_{1,2}\},$$

and

$$\{E_{1,3}+\sqrt{-1}\,E_{3,1}, \sqrt{-1}\,E_{1,3}+E_{3,1}, E_{2,3}+\sqrt{-1}\,E_{3,2}, \sqrt{-1}\,E_{2,3}+E_{3,2}\}.$$

The conditions (EXT1)–(EXT3) can be considered for $\mathfrak{g}_C$ instead of $\mathfrak{g}$ through complex linearity. Since $\mathfrak{g}_0\cong\mathfrak{u}(2)$, an irreducible unitarizable $\mathfrak{g}_0$-module (ρ, V_0) is finite-dimensional, and it has a highest weight $\Lambda\in\mathfrak{h}_C^*$, and is of dimension $n=\Lambda(H)+1$, where $H=E_{1,1}-E_{2,2}$. Choosing appropriately an orthonormal basis $\{v_1, \cdots, v_n\}$ of V_0 such that each v_k is a weight vector with weight $\Lambda-(k-1)\alpha$, where α is the positive root of $[\mathfrak{g}_{0,C}, \mathfrak{g}_{0,C}]=\mathfrak{sl}(2; \boldsymbol{C})$, we have

$$\begin{aligned} \rho(H)\,v_k &= (n+1-2k)\,v_k, \\ \rho(E_{1,2})\,v_k &= \sqrt{(k-1)(n+1-k)}\;v_{k-1}, \\ \rho(E_{2,1})\,v_k &= \sqrt{k(n-k)}\;v_{k+1}, \end{aligned} \tag{7.1}$$

for $1\leqq k\leqq n$, where $v_0=v_{n+1}=0$. Put $m=\Lambda(C)$, then $\rho(C)=m\cdot I_{V_0}$, where I_{V_0} denotes the identity operator on V_0.

The results in Case (b) are summarized in the following theorem.

Theorem 7.1. *Let $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$, and (ρ, V_0) be the finite-dimensional irreducible unitary $\mathfrak{g}_0$-module with highest weight Λ. Put $n=\Lambda(H)+1=\dim V_0$, and $m=\Lambda(C)$. Then there exist its irreducible unitary extensions (=IUEs) if and only if one of the following three conditions holds*:

(i) $n=1$ *and* $m=-2, 0, 2$;

(ii) $n=2$ *and* $m\in\boldsymbol{R}$, $|m|\geqq 1$;

(iii) $n\geqq 3$ *and* $m=\pm(n-1), \pm(n+1)$.

Moreover IUEs are unique up to isomorphism, except the cases $n=2$ and $m=\pm 3$. In these exceptional cases there exist exactly two IUEs up to isomorphism.

We prepare three lemmas for the proof of this theorem.

Lemma 7.2. *The operator $A=A_{1,-1}$ determines $B_{i,j}$, $i, j\in\{\pm 1, \pm 2\}$, as follows*:

(7.2) $$B_{1,-1}=\frac{1}{2}(\rho(H_{1,1})+A),$$

(7.3) $$B_{1,-2}=-[\rho(E_{1,2}), B_{1,-1}],$$

(7.4) $$B_{2,-1}=[\rho(E_{2,1}), B_{1,-1}],$$

(7.5) $$B_{2,-2}=[\rho(E_{2,1}), B_{1,-2}]+B_{1,-1},$$

(7.6) $$B_{-i,j}=\rho(E_{j,i}+\delta_{i,j}E_{3,3})-B_{j,-i},$$

(7.7) $$B_{i,j}=B_{-i,-j}=0 \quad \text{for} \quad i, j\in\{1, 2\}.$$

Proof. The equation (7.2) follows from the definition of $A_{i,j}$. (EXT1) implies (7.3)–(7.5). (7.6) is from (EXT2), and (7.7) was shown in Lemma 6.3. Q.E.D.

These $B_{i,j}$ are well-defined when (EXT1*) is satisfied for A.

Lemma 7.3. *The operator $A=A_{1,-1}$ is diagonal. Moreover*

(7.8) $$Av_k=\varepsilon_k\rho(H_{1,1})v_k \quad (\varepsilon_k=\pm 1).$$

Proof. Now (EXT1) gives $[\rho(H), A]=0$. So each H-eigenspace is A-invariant. As we see from (7.1), each H-eigenspace is one-dimensional. Hence A is diagonal with respect to the basis $\{v_k\}$.

From the equation (EXT3.1) and (7.2),

$$A^2=(\rho(H_{1,1}))^2.$$

On the other hand, $\rho(H_{1,1})=\frac{1}{2}(\rho(C)+\rho(H))$ is also diagonal. So we get (7.8). Q.E.D.

Lemma 7.4. *$A_{1,-1}+A_{2,-2}$ is a scalar operator on V_0.*

Proof. From the equation (EXT1), we get

$$[\rho(X), A_{1,-1}+A_{2,-2}] = 0 \qquad \text{for any} \quad X \in \mathfrak{g}_{0,C} .$$

Therefore $A_{1,-1}+A_{2,-2}$ is a scalar operator. Q.E.D.

Proof of Theorem 7.1. (1°) Similarly as in Lemma 7.3, we see that $A_{2,-2}$ is also a diagonal operator and

$$A_{2,-2}\, v_k = \varepsilon'_k \rho(H_{2,2})\, v_k ,$$

where $\varepsilon'_k = \pm 1$. Therefore from (7.1) and $\rho(C) = m \cdot I_{V_0}$,

$$\begin{aligned}(A_{1,-1}+A_{2,-2})\, v_k &= \varepsilon_k \rho(H_{1,1})\, v_k + \varepsilon'_k \rho(H_{2,2})\, v_k \\ &= \frac{1}{2} \{\varepsilon_k(m+n+1-2k) + \varepsilon'_k(m-n-1+2k)\}\, v_k .\end{aligned}$$

On the other hand, $A_{1,-1}+A_{2,-2}$ is a scalar operator by Lemma 7.4. Hence

$$\begin{aligned}d_k &= \varepsilon_k(m+n+1-2k) + \varepsilon'_k(m-n-1+2k) \\ &= (\varepsilon_k+\varepsilon'_k)\, m + (\varepsilon_k-\varepsilon'_k)\,(n+1-2k)\end{aligned}$$

should be a constant independent of k.

(2°) From this criterion, we deduce a necessary condition for (n, m), $n = \Lambda(H)+1$, $m = \Lambda(C)$, to have an IUE. Let us discuss case by case.

Case I: $m \neq 0$ and $n \neq 1, 2$. In this case we can take $\varepsilon_k = \varepsilon'_k$ for all k, and then they are all equal to 1 or -1 at the same time. Therefore

$$A = \rho(H_{1,1}) \quad \text{or} \quad A = -\rho(H_{1,1}) .$$

Case II: $m = 0$ and $n \neq 1, 2$. In this case $d_k = (\varepsilon_k - \varepsilon'_k)\,(n+1-2k)$. We see first $\varepsilon_k = \varepsilon'_k$ for all k, whence $d_k \equiv 0$. So

$$A_{1,-1}+A_{2,-2} = 0 . \tag{7.9}$$

Equation (EXT1) together with (7.9) gives

$$[\rho(E_{2,1}), [\rho(E_{1,2}), A]] - 2A = 0 .$$

Apply the left hand side to the vector v_k, then

$$\{[\rho(E_{2,1}), [\rho(E_{1,2}), A]] - 2A\}\, v_k = c_k\, v_k = 0 ,$$

where

$$c_k = (n-2k+3)\,(k-1)\,(n-k+1)\,(\varepsilon_k - \varepsilon_{k-1}) + (n-2k-1)\, k(n-k)\,(\varepsilon_k - \varepsilon_{k+1}) .$$

Suppose $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_l$, then we obtain from $c_l = 0$

$$l(n-l)\,(n-2l-1)\,(\varepsilon_l - \varepsilon_{l+1}) = 0 .$$

When $n \neq l$, $2l+1$, we have clearly $\varepsilon_l = \varepsilon_{l+1}$. When $n = l$, the number ε_{l+1} and the equation itself do not exist. Let $n = 2l+1$, then ε_{l+1} appears only in $Av_{l+1} =$

$\varepsilon_{l+1}(n-2l-1)\, v_{l+1}=0$, so we can set $\varepsilon_{l+1}=\varepsilon_l$.

Altogether we get

$$A=\rho(H_{1,1}) \quad \text{or} \quad A=-\rho(H_{1,1}).$$

Case III: $n=1$. Here ρ is one-dimensional. Hence

$$A=\rho(H_{1,1}) \quad \text{or} \quad A=-\rho(H_{1,1}).$$

Case IV: $n=2$. This case is exceptional at the point that there may happen

$$A\neq\pm\rho(H_{1,1}).$$

Except the cases $A=\pm\rho(H_{1,1})$, we have $\varepsilon_1=-\varepsilon_2=-\varepsilon'_1=\varepsilon'_2$, since $A_{1,-1}+A_{2,-2}$ is a scalar operator.

Until now, we studied case by case the operator $A=A_{1,-1}$ and found that there exist three cases $A=\rho(H_{1,1})$, $A=-\rho(H_{1,1})$, and one exceptional case.

(3°) As the second step, we apply Lemma 7.2 to these A's.

Case $A=\rho(H_{1,1})$. In this case

$$B_{i,-j}=\rho(E_{i,j}+\delta_{i,j}E_{3,3}), \quad B_{-i,j}=0, \tag{7.10}$$

for $i, j\in\{1, 2\}$. They altogether satisfy the conditions (EXT3.1)–(EXT3.8) except (EXT3.2). We insert (7.10) into (EXT3.2), then obtain

$$\rho(H_{1,1})\,\rho(H_{2,2})+\rho(H_{1,1})-\rho(E_{1,2})\,\rho(E_{2,1})=0.$$

We apply this to the vector v_k, and get

$$\begin{aligned}0&=\{\rho(H_{1,1})\,\rho(H_{2,2})+\rho(H_{1,1})-\rho(E_{1,2})\,\rho(E_{2,1})\}\,v_k\\&=\frac{1}{4}(m+n+1)(m-n+1)\,v_k.\end{aligned}$$

Therefore $m=\pm n-1$.

Case $A=-\rho(H_{1,1})$. In this case

$$B_{i,-j}=0, \quad B_{-i,j}=\rho(E_{j,i}+\delta_{j,i}E_{3,3}), \tag{7.11}$$

for all $i, j\in\{1, 2\}$. They satisfy the conditions (EXT3.1)–(EXT3.8) except (EXT3.6). Insert (7.11) into (EXT3.6), then we get

$$\rho(H_{1,1})\,\rho(H_{2,2})-\rho(H_{1,1})-\rho(E_{2,1})\,\rho(E_{1,2})=0. \tag{7.12}$$

Apply this to v_k, then

$$\begin{aligned}0&=\{\rho(H_{1,1})\,\rho(H_{2,2})-\rho(H_{1,1})-\rho(E_{2,1})\,\rho(E_{1,2})\}\,v_k\\&=\frac{1}{4}(m+n-1)(m-n-1)\,v_k.\end{aligned}$$

Hence we get $m=\pm n+1$.

EXCEPTIONAL CASE $n=2$ and $A\neq\pm\rho(H_{1,1})$. When $\varepsilon_1=1$, all the conditions (EXT3.1)–(EXT3.8) hold for $B_{i,j}$. Their exact form is simple but not written down here. When $\varepsilon_1=-1$, there is no m, for which $B_{i,j}$'s satisfy the conditions (EXT3.2) and (EXT3.6).

(4°) Finally we check the condition (UNI) in each case. This holds in the case $A=\pm\rho(H_{1,1})$. In the exceptional case $n=2$ and $A\neq\pm\rho(H_{1,1})$, the condition (UNI) implies that $m\in\boldsymbol{R}$, $|m|>1$. (If $|m|=1$, we have $A=\pm\rho(H_{1,1})$, necessarily.)

This completes the proof of Theorem 7.1. Q.E.D.

7.2. Extension problem for the Case (c): $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$. The even part $\mathfrak{g}_0$ and the odd part $\mathfrak{g}_1$ of $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$ are spanned respectively by

$$\{\sqrt{-1}H_{1,1},\ \sqrt{-1}H_{2,2},\ \sqrt{-1}E_{2,1}-\sqrt{-1}E_{1,2},\ E_{1,2}+E_{2,1}\}\ ,$$

and

$$\{E_{1,3}+\sqrt{-1}E_{3,1},\ \sqrt{-1}E_{1,3}+E_{3,1},\ E_{2,3}-\sqrt{-1}E_{3,2},\ -\sqrt{-1}E_{2,3}+E_{3,2}\}\ .$$

Note that $\mathfrak{g}_0\cong\mathfrak{u}(1, 1)$, and that $\mathfrak{u}(1, 1)$ is isomorphic to $\mathfrak{sl}(2; \boldsymbol{R})$ plus one-dimensional center. From the classification of irreducible Harish-Chandra modules for $\mathfrak{sl}(2; \boldsymbol{R})$, we may take as (ρ, V_0) the unitarizable $(\mathfrak{g}_0, K_0)$-modules listed up below. Notations here follow those in [15, Chap. V].

(T) trivial representation;

(PCS) principal continuous series $(V^{l,s}, H)$, where

$$l=0,\ 1/2\ ,\quad s\in\boldsymbol{C}\ ,\quad Re(s)=1/2\ ,\quad (l, s)\neq(1/2, 1/2);$$

(LDS) limit of discrete series $(V^{1/2,1/2}|H^{\pm}, H^{\pm})$;

(DS) discrete series (U^n, H_n), where $n\in(1/2)\boldsymbol{Z}$, $|n|\geqq 1$;

(CS) complementary series (V^s, H_s), $1/2<s<1$.

For convenience to treat the limit of discrete series together with discrete series, we introduce new notation for the former:

$$(U^{-1/2}, H_{-1/2})=(V^{1/2,1/2}|H^{+}, H^{+})\ ,$$
$$(U^{1/2}, H_{1/2})\quad=(V^{1/2,1/2}|H^{-}, H^{-})\ .$$

For details of the actions of $\mathfrak{g}_0$ on these modules, we refer the book [loc. cit.], however, for our later calculations, we list up some of them.

Case (PCS). Let $\{f_p;\ p\in\boldsymbol{Z}\}$ be the standard orthonormal basis of H given in [15, p. 216], then

$$V^{l,s}(\sqrt{-1}H)f_p=-2\sqrt{-1}(p+l)f_p\ .$$

Case (LDS) and (DS). Let $\{f_p^n;\ p\in\boldsymbol{Z}, p\geqq 0\}$ and $\{\bar{f}_p^n;\ p\in\boldsymbol{Z}, p\geqq 0\}$ be the standard orthonormal bases of H_n and H_{-n} respectively (cf. [15, p. 237]), then

$$U^n(\sqrt{-1}H)f_p^n = -2\sqrt{-1}(n+p)f_p^n,$$
$$U^n(\sqrt{-1}E_{2,1}-\sqrt{-1}E_{1,2})f_p^n$$
$$= \sqrt{-1}\sqrt{(2n+p)(p+1)}f_{p+1}^n+\sqrt{-1}\sqrt{(2n+p-1)p}f_{p-1}^n,$$
$$U^n(E_{1,2}+E_{2,1})f_p^n = \sqrt{(2n+p)(p+1)}f_{p+1}^n-\sqrt{(2n+p-1)p}f_{p-1}^n,$$
$$U^{-n}(\sqrt{-1}H)\bar{f}_p^n = 2\sqrt{-1}(n+p)\bar{f}_p^n,$$
$$U^{-n}(\sqrt{-1}E_{2,1}-\sqrt{-1}E_{1,2})\bar{f}_p^n$$
$$= -\sqrt{-1}\sqrt{(2n+p)(p+1)}\bar{f}_{p+1}^n-\sqrt{-1}\sqrt{(2n+p-1)p}\bar{f}_{p-1}^n,$$
$$U^{-n}(E_{1,2}+E_{2,1})\bar{f}_p^n = \sqrt{(2n+p)(p+1)}\bar{f}_{p+1}^n-\sqrt{(2n+p-1)p}\bar{f}_{p-1}^n.$$

Case (CS). Let $\{f_p;\, p\in \boldsymbol{Z}\}$ be the standard orthonormal basis of H_s [15, p. 243], then

$$V^s(\sqrt{-1}H)f_p = 2\sqrt{-1}pf_p.$$

Case (T). One-dimensional representation, i.e.

$$\rho(H) = \rho(E_{1,2}) = \rho(E_{2,1}) = 0.$$

For each (ρ, V_0), let $\rho(C)=m\cdot I_{V_0}$. Because of the (infinitesimal) unitarity of ρ, m must be a real number.

Lemma 7.5. *If the condition (UNI) holds, then one of the following cases occurs*:

(i) $\rho(H)\geqq |m|\cdot I_{V_0}$ *and* $j^2=\sqrt{-1}$,

(ii) $\rho(H)\leqq -|m|\cdot I_{V_0}$ *and* $j^2=-\sqrt{-1}$,

where $C\geqq D$ *for* C, $D\in\mathfrak{gl}(V_0)$ *means that* C-D *is positive definite.*

Proof. From the condition (UNI), we have

$$j^2\langle B(\xi, \xi)v, v\rangle_0\geqq 0, \tag{7.13}$$

for all $v\in V_0$, where $\xi=E_{1,3}+\sqrt{-1}E_{3,1}$ and $j^4=-1$. Using (EXT2), we rewrite the above operator $B(\xi, \xi)$ as

$$\begin{aligned}B(\xi, \xi) &= B(E_{1,3}+\sqrt{-1}E_{3,1},\, E_{1,3}+\sqrt{-1}E_{3,1})\\ &= \frac{1}{2}\rho([E_{1,3}+\sqrt{-1}E_{3,1},\, E_{1,3}+\sqrt{-1}E_{3,1}])\\ &= \sqrt{-1}\rho(H_{1,1}) = \sqrt{-1}\,\frac{1}{2}\rho(H+C).\end{aligned}$$

Therefore (7.13) is rewritten as

$$j^2\sqrt{-1}(\rho(H)+m\cdot I_{V_0})\geqq 0.$$

Similarly as above, we get from the inequality for $B(\xi, \xi)$, $\xi=E_{2,3}-\sqrt{-1}E_{3,2}$,

$$j^2\sqrt{-1}(\rho(H)-m\cdot I_{V_0})\geqq 0.$$

From these two inequalities, we arrive at the case (i) or (ii) in the lemma. Q.E.D.

Comparing the range of the weights of ρ with the conditions in the above lemma, we get

Proposition 7.6. (i) *If* (ρ, V_0) *is in* (*PCS*) *or in* (*CS*), *then there does not exist any IUEs.*

(ii) *If* (ρ, V_0) *is* (*T*), *then there exists an IUE if and only if* $m=0$. *Actually an IUE is given by the trivial representation of* $\mathfrak{g}$.

In case (ρ, V_0) is in (LDS) or in (DS), we get the following result.

Theorem 7.7. *Let* (ρ, V_0) *be* $(U^{\pm n}, H_{\pm n})$, *where* $n\in(1/2)\boldsymbol{Z}$, $n>0$. *Then there exist IUEs if and only if one of the following conditions holds*:

(i) $n=1/2$ *and* $m=\pm 1$;

(ii) $n=1$ *and* $m=0, \pm 2$;

(iii) $n\geqq 3/2$ *and* $m=\pm 2n, \pm 2(n-1)$.

Moreover IUEs are unique up to isomorphism except the case $n=1$ *and* $m=0$. *In the exceptional case there exist exactly two IUEs up to isomorphism.*

Remark 7.8. When ρ is U^n $(n>0)$, ρ is a highest weight representation of $\mathfrak{g}_0$ with highest weight Λ, where $\Lambda(H)=-2n\in\boldsymbol{Z}, <0$, and $\Lambda(C)=m$.

Proof of Theorem 7.7. Lemmas 7.2, 7.3 and 7.4 are also true in this case. Similarly as in the proof of Theorem 7.1, we can conclude that $A=\rho(H_{1,1})$ or $A=-\rho(H_{1,1})$, this time without exception because V_0 is infinite-dimensional.

CASE $A=\rho(H_{1,1})$. In this case, we get

$$B_{i,-j}=\rho(E_{i,j}+\delta_{i,j}E_{3,3}), \quad B_{-i,j}=0, \tag{7.14}$$

for all $i, j\in\{1, 2\}$. These operators satisfy the conditions (EXT3.1)–(EXT3.8) except (EXT3.2). We insert (7.14) into (EXT3.2) to get

$$\rho(H_{1,1})\rho(H_{2,2})+\rho(H_{1,1})-\rho(E_{1,2})\rho(E_{2,1})=0. \tag{7.15}$$

In case of U^n, $n>0$, apply the above operator to the vector f_p^n, then

$$\begin{aligned} 0 &= \{\rho(H_{1,1})\rho(H_{2,2})+\rho(H_{1,1})-\rho(E_{1,2})\rho(E_{2,1})\}f_p^n \\ &= \frac{1}{4}(m+2n)(m-2n+2)f_p^n. \end{aligned}$$

Therefore, there should be $m=2n-2, -2n$.

In case of U^{-n}, $n>0$, we apply the operator in (7.15) to $\bar{f}_p^n$, and get

$$\frac{1}{4}(m+2n)(m-2n+2)=0,$$

whence $m=2n-2, -2n$.

CASE $A=-\rho(H_{1,1})$. In this case, we have

(7.16) $$B_{i,-j}=0\,,\quad B_{-i,j}=\rho(E_{j,i}+\delta_{j,i}E_{3,3})\,,$$

for all $i, j\in\{1, 2\}$. These operators satisfy the conditions (EXT3.1)–(EXT3.8) except (EXT3.6). We insert (7.16) into (EXT3.6) to get

$$\rho(H_{1,1})\rho(H_{2,2})-\rho(H_{1,1})-\rho(E_{2,1})\rho(E_{1,2})=0\,.$$

Apply this to $f=f_p^n$ or to $f=\bar{f}_p^n$, according to $\rho=U^n$ or U^{-n}, $n>0$, then

$$\begin{aligned}0&=\{\rho(H_{1,1})\rho(H_{2,2})-\rho(H_{1,1})-\rho(E_{2,1})\rho(E_{1,2})\}f\\&=\frac{1}{4}(m-2n)(m+2n-2)f\,.\end{aligned}$$

Thus, we get $m=-2n+2,\ 2n$.

The condition (UNI) is satisfied in any case. So we get the theorem. Q.E.D.

§ 8. Explicit construction of irreducible unitary extensions.

In this section we give explicit realizations of IUEs classified in § 7 for real forms $\mathfrak{g}$ of $\mathfrak{sl}(2, 1)$.

Denote by $L(\Lambda)$ an irreducible highest weight representation of $\mathfrak{g}_{0,C}$ with highest weight $\Lambda\in\mathfrak{h}_C^*$. Two positive roots β, $\gamma\in\mathfrak{h}_C^*$ of $\mathfrak{g}$, other than the positive root α of $[\mathfrak{g}_{0,C}, \mathfrak{g}_{0,C}]$, are given by

$$\beta(H)=\beta(C)=-1\,,\quad\text{and}\quad\gamma(H)=1\,,\quad\gamma(C)=-1\,,$$

where $H, C\in\mathfrak{h}_C$ are as in (6.2).

Let v_1 be a non-zero highest weight vector of $L(\Lambda)$. When $L(\Lambda)$ is finite-dimensional, we define $\{v_k\}_{1\leq k\leq n}$, a standard orthonormal basis of V_0, starting from v_1 as

$$\sqrt{k(n-k)}\ v_{k+1}=\rho(E_{2,1})v_k\qquad\text{for}\quad 1\leqq k\leqq n-1\,,$$

where $n=\dim L(\Lambda)=\Lambda(H)+1$. When $L(\Lambda)$ is infinite-dimensional, we set $v_k=f_{k+1}^n$ in 7.2.

In the following, we construct the odd part V_1 from the even part $V_0\cong L(\Lambda)$ and thus realize (π, V), $V=V_0+V_1$, explicitly case by case. An orthonormal basis $\{w_k;\ k\in I\}$, I an index set, of V_1 will be given canonically in the following. We can check (EXT1*) each time when $A=A_{1,-1}$, a generator of $B_{i,j}$, is given.

Since IUEs of $\mathfrak{su}(2, 1; 1, 1)$ can be realized in the same way as those of $\mathfrak{su}(2, 1; 2, 1)$, we give these realizations at the same time for convenience. In case of $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$, we have $\dim V_0<\infty$, and this is devided into Cases A, B, E, F, I and J. In case of $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$, we have $\dim V_0=\infty$, and this is devided into Cases C, D, G and H. In 8.4, the results for these cases are summarized in two tables.

8.1. Case of π for which $A_{1,1}=\rho(H_{1,1})$. In this case we have already shown in the proof of Theorem 7.1 that

$$B_{i,-j} = \rho(E_{i,j}+\delta_{i,j}E_{3,3}), \quad B_{-i,j} = 0,$$

for all $i, j \in \{1, 2\}$, and that, if $V_0 \cong L(\Lambda)$ has extensions, then

$$\Lambda(C) = \Lambda(H) \quad \text{or} \quad \Lambda(C) = -\Lambda(H)-2.$$

This and $B_{-i,j}=0$ show

$$\pi(E_{i,3})V_0 = (0) \quad \text{and} \quad \pi(E_{3,i})V_1 = (0) \qquad \text{for} \quad i = 1, 2. \tag{8.1}$$

We construct V_1 in each of the following four cases, Cases A~D, classified by whether $\Lambda(C)=\Lambda(H)$ or $-\Lambda(H)-2$ and whether dim V_0 is finite or infinite. Elements of $V_1 \cong W/\mathfrak{m}$ is expressed by linear combinations of $[x\otimes v]$, $x\otimes v \in W = \mathfrak{g}_{1,C}\otimes_C V_0$.

Case A: $\Lambda(C)=\Lambda(H)$ and V_0 is finite-dimensional.

In this case, we give a basis $\{w_k;\ k\in I\}$ of V_1 by

$$\sqrt{n-k}\, w_k = [E_{3,1}\otimes v_k] \qquad \text{for} \quad k\in I = \{k;\ 1\leqq k\leqq n-1\},$$

and so, as $\mathfrak{g}_{0,C}$-modules, $V_1 \cong L(\Lambda-\gamma)$, using the results in [3] on the subalgebra $\mathfrak{m}$ in 3.2.

The action of $\mathfrak{g}_{1,C}$ is given by (8.1) and (8.2) below:

$$\begin{aligned} &\pi(E_{3,1})v_k = \sqrt{n-k}\, w_k, \quad \pi(E_{3,2})v_k = \sqrt{k-1}\, w_{k-1}, \\ &\pi(E_{1,3})w_k = \sqrt{n-k}\, v_k, \quad \pi(E_{2,3})w_k = \sqrt{k}\, v_{k+1}. \end{aligned} \tag{8.2}$$

Case B: $\Lambda(C)=-\Lambda(H)-2$ and V_0 is finite-dimensional.

In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{n}\, w_1 = -[E_{3,2}\otimes v_1], \quad \sqrt{k}\, w_{k+1} = [E_{3,1}\otimes v_k]$$
$$\text{for} \quad 1\leqq k\leqq n, \quad I = \{k;\ 1\leqq k\leqq n+1\},$$

and thus, as $\mathfrak{g}_{0,C}$-modules, $V_1 \cong L(\Lambda-\beta)$, and the action of $\mathfrak{g}_{1,C}$ is given by (8.1) and (8.3) below:

$$\begin{aligned} &\pi(E_{3,1})v_k = \sqrt{k}\, w_{k+1}, \qquad \pi(E_{3,2})v_k = -\sqrt{n+1-k}\, w_k, \\ &\pi(E_{1,3})w_k = -\sqrt{k-1}\, v_{k-1}, \quad \pi(E_{2,3})w_k = \sqrt{n+1-k}\, v_k. \end{aligned} \tag{8.3}$$

Case C: $\Lambda(C)=\Lambda(H)$ and V_0 is infinite-dimensional.

In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{2n+k-1}\, w_k = [E_{3,1}\otimes v_k] \qquad \text{for} \quad k\in I = \{1, 2, 3, \cdots\},$$

and so, as $\mathfrak{g}_{0,C}$-modules, $V_1 \cong L(\Lambda-\gamma)$.

The action of $\mathfrak{g}_{1,C}$ is given by (8.1) and (8.4) below:

$$\begin{aligned} &\pi(E_{3,1})v_k = \sqrt{2n+k-1}\, w_k, \qquad \pi(E_{3,2})v_k = \sqrt{k-1}\, w_{k-1}, \\ &\pi(E_{1,3})w_k = -\sqrt{2n+k-1}\, v_k, \quad \pi(E_{2,3})w_k = \sqrt{k}\, v_{k+1}. \end{aligned} \tag{8.4}$$

Case D: $\Lambda(C)=-\Lambda(H)-2$ and V_1 is infinite-dimensional ($\Lambda(H)\leqq-2$).
In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{2n+k-2}\, w_k=[E_{3,2}\otimes v_k] \qquad \text{for} \quad I=\{1,2,3,\cdots\},$$

and thus, as $\mathfrak{g}_{0,C}$-modules, $V_1\cong L(\Lambda-\beta)$, and the action of $\mathfrak{g}_{1,C}$ is given by (8.1) and (8.5) below:

$$\text{(8.5)}\quad \begin{aligned} &\pi(E_{3,1})v_k=\sqrt{k}\, w_{k+1}, && \pi(E_{3,2})v_k=\sqrt{2n+k-2}\, w_k, \\ &\pi(E_{1,3})w_k=-\sqrt{k-1}\, v_{k-1}, && \pi(E_{2,3})w_k=\sqrt{2n+k-2}\, v_k. \end{aligned}$$

Remark 8.1. When $V_0\cong L(\Lambda)$ is infinite-dimensional, we have $\Lambda(H)\in\boldsymbol{Z}$ and $\Lambda(H)\leqq-1$. And when $\Lambda(H)=-1$ in Case D, this is already contained in Case C.

8.2. Case of π for which $\Lambda_{1,-1}=-\rho(H_{1,1})$. In this case we have shown in the proof of Theorem 7.1 that

$$B_{i,-j}=0, \quad B_{-i,j}=\rho(E_{j,i}+\delta_{j,i}E_{3,3}),$$

for $i,j\in\{1,2\}$, and that if there exist extensions, then

$$\Lambda(C)=\Lambda(H)+2 \quad \text{or} \quad \Lambda(C)=-\Lambda(H).$$

Therefore we get

$$\text{(8.6)}\qquad \pi(E_{3,i})V_0=(0) \quad \text{and} \quad \pi(E_{i,3})V_1=(0) \qquad \text{for} \quad i=1,2.$$

We construct V_1 in each of the following four cases, Cases E∼H, classified by whether $\Lambda(C)=-\Lambda(H)$ or $\Lambda(H)+2$ and whether dim V_0 is finite or infinite.

Case E: $\Lambda(C)=-\Lambda(H)$ and V_0 is finite-dimensional.
In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{n-k}\, w_k=[E_{2,3}\otimes v_k] \qquad \text{for} \quad k\in I=\{k;\ 1\leqq k\leqq n-1\},$$

and so, as $\mathfrak{g}_{0,C}$-modules, $V_1\cong L(\Lambda+\beta)$.
The action of $\mathfrak{g}_{1,C}$ is given by (8.6) and (8.7) below:

$$\text{(8.7)}\quad \begin{aligned} &\pi(E_{1,3})v_k=-\sqrt{k-1}\, w_{k-1}, && \pi(E_{2,3})v_k=\sqrt{n-k}\, w_k, \\ &\pi(E_{3,1})w_k=\sqrt{k}\, v_{k+1}, && \pi(E_{3,2})w_k=-\sqrt{n-k}\, v_k. \end{aligned}$$

Case F: $\Lambda(C)=\Lambda(H)+2$ and V_0 is finite-dimensional.
In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{n}\, w_1=[E_{1,3}\, v_1], \quad \sqrt{k}\, w_{k+1}=[E_{2,3}\otimes v_k] \quad \text{for} \quad 1\leqq k\leqq n,\ I=\{k;\ 1\leqq k\leqq n+1\},$$

and thus, as $\mathfrak{g}_{0,C}$-modules, $V_1\cong L(\Lambda+\gamma)$.
The action of $\mathfrak{g}_{1,C}$ is given by (8.6) and (8.8) below:

(8.8)
$$\pi(E_{1,3})v_k = \sqrt{n+1-k}\, w_k\,, \quad \pi(E_{2,3})v_k = \sqrt{k}\, w_{k+1}\,,$$
$$\pi(E_{3,1})w_k = \sqrt{n+1-k}\, v_k\,, \quad \pi(E_{3,2})w_k = \sqrt{k-1}\, v_{k-1}\,.$$

Case G: $\Lambda(C) = -\Lambda(H)$ and V_0 is infinite-dimensional.

In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{2n+k-1}\, w_k = [E_{2,3}\otimes v_k] \qquad \text{for} \quad \{I = 1, 2, 3, \cdots\}\,,$$

and so, as $\mathfrak{g}_{0,C}$-modules, $V_1 \simeq L(\Lambda+\beta)$.

The action of $\mathfrak{g}_{0,C}$ is given by (8.6) and (8.9) below:

(8.9)
$$\pi(E_{1,3})v_k = -\sqrt{k-1}\, w_{k-1}\,, \quad \pi(E_{2,3})v_k = \sqrt{2n+k-1}\, w_k\,,$$
$$\pi(E_{3,1})w_k = \sqrt{k}\, v_{k+1}\,, \qquad \pi(E_{3,2})w_k = \sqrt{2n+k-1}\, v_k\,.$$

Case H: $\Lambda(C) = \Lambda(H)+2$ and V_0 is infinite-dimensional ($\Lambda(H) \leqq -2$).

In this case, $\{w_k\}_{k\in I}$ is given by

$$\sqrt{2n+k-2}\, w_k = [E_{1,3}\otimes v_k] \qquad \text{for} \quad k\in I = \{1, 2, 3, \cdots\}\,,$$

and so, as $\mathfrak{g}_{0,C}$-modules, $V_1 \simeq L(\Lambda+\gamma)$.

The action of $\mathfrak{g}_{1,C}$ is given by (8.6) and (8.10) below:

(8.10)
$$\pi(E_{1,3})v_k = \sqrt{2n+k-2}\, w_k\,, \quad \pi(E_{2,3})v_k = -\sqrt{k}\, w_{k+1}\,,$$
$$\pi(E_{3,1})w_k = -\sqrt{2n+k-2}\, v_k\,, \quad \pi(E_{3,2})w_k = -\sqrt{k-1}\, v_{k-1}\,.$$

8.3. The Exceptional Case: dim $V_0 = 2$ and $\Lambda \neq \pm\rho(H_{1,1})$. In this case we have already shown that $m = \Lambda(C) \in \boldsymbol{R}$, $|m| > 1$. We construct V_1 in each case of $m = \Lambda(C) > 1$ and $m = \Lambda(C) < -1$.

Case I: $\Lambda(C) > 1$.

In this case, $\{w_k\}_{k\in I} = \{w_+, w_-\}$, $I = \{+, -\}$, is given by

$$\sqrt{m-1}\, w_+ = \sqrt{2}\,[E_{1,3}\otimes v_2]\,, \quad \sqrt{m+1}\, w_- = \sqrt{2}\,[E_{3,1}\otimes v_1]\,,$$

and thus, as $\mathfrak{g}_{0,C}$-modules, $V_1 \simeq L(\Lambda+\beta) \oplus L(\Lambda-\gamma)$.

We represent the action of $\mathfrak{g}_{1,C}$ by matrices with respect to the basis $\{v_1, v_2; w_-, w_+\}$ of V:

$$\pi(E_{1,3}) = \begin{pmatrix} 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 \end{pmatrix}, \quad \pi(E_{2,3}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 \end{pmatrix},$$
$$\pi(E_{3,1}) = {}^t\pi(E_{1,3})\,, \qquad \pi(E_{3,2}) = {}^t\pi(E_{2,3})\,,$$

where $p = \dfrac{\sqrt{2m+2}}{2}$, $q = \dfrac{\sqrt{2m-2}}{2}$ and ${}^tX$ denotes the transposed of a matrix X.

Case J: $\Lambda(C) < -1$.

In this case, $\{w_k\}_{k\in I} = \{w_+, w_-\}$, $I = \{+, -\}$, is given by

$$\sqrt{-m+1}\, w_+ = \sqrt{2}[E_{1,3}\otimes v_2]\,, \quad \sqrt{-m-1}\, w_- = \sqrt{2}\,[E_{3,1}\otimes v_1]\,,$$

and so, as $\mathfrak{g}_{0,C}$-modules, $V_1 \cong L(\Lambda+\beta)\oplus L(\Lambda-\gamma)$.

We represent the action of $\mathfrak{g}_{1,C}$ by matrices with respect to the basis $\{v_1, v_2; w_-, w_+\}$ of V:

$$\pi(E_{1,3}) = \begin{pmatrix} 0 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 \end{pmatrix}, \quad \pi(E_{2,3}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 \end{pmatrix},$$

$$\pi(E_{3,1}) = -{}^t\pi(E_{1,2})\,, \qquad \pi(E_{3,2}) = -{}^t\pi(E_{3,2})\,,$$

where $r=\dfrac{\sqrt{-2m-2}}{2}$, $s=\dfrac{\sqrt{-2m+2}}{2}$.

8.4. Summary. At the end, in each Case of real forms $\mathfrak{su}(2, 1; 2, 1)$ and $\mathfrak{su}(2, 1; 1, 1)$, we list up, for $V_0 \cong L(\Lambda)$, (1) the condition on Λ for existence of an IUE, (2) V_1 as $\mathfrak{g}_0$-module, and (3) the operator $A=A_{1,-1}$ which determines the $\mathfrak{g}_1$-action on $V=V_0+V_1$.

(1°) $\mathfrak{g}=\mathfrak{su}(2, 1; 2, 1)$

Cases	the value of $\Lambda(C)$	V_1 (as $\mathfrak{g}_0$-module)	the operator $A=A_{1,-1}$
A	$\Lambda(H)$	$L(\Lambda-\gamma)$	$\rho(H_{1,1})$
B	$-\Lambda(H)-2$	$L(\Lambda-\beta)$	$\rho(H_{1,1})$
E	$-\Lambda(H)$	$L(\Lambda+\beta)$	$-\rho(H_{1,1})$
F	$\Lambda(H)+2$	$L(\Lambda+\gamma)$	$-\rho(H_{1,1})$
I, J	(*1)	$L(\Lambda+\beta)\oplus L(\Lambda+\gamma)$	(*2)

(*1) In this case, $\Lambda(C)\in \boldsymbol{R}$, $|\Lambda(C)|>1$.

(*2) In this case, $A=\dfrac{1}{2}\begin{pmatrix} 1+m & 0 \\ 0 & 1-m \end{pmatrix}$, where $m=\Lambda(C)$.

dim $V_0=2$ for Cases I and J.

(2°) $\mathfrak{g}=\mathfrak{su}(2, 1; 1, 1)$

Cases	the value of $\Lambda(C)$	V_1 (as $\mathfrak{g}_0$-module)	the operator $A=A_{1,-1}$
C	$\Lambda(H)$	$L(\Lambda-\gamma)$	$\rho(H_{1,1})$
D	$-\Lambda(H)-2$	$L(\Lambda-\beta)$	$\rho(H_{1,1})$
G	$-\Lambda(H)$	$L(\Lambda+\beta)$	$-\rho(H_{1,1})$
H	$\Lambda(H)+2$	$L(\Lambda+\gamma)$	$-\rho(H_{1,1})$

Remark 8.2. As a $\mathfrak{g}$-module, we can exchange the roles of V_0 and V_1, so each $\mathfrak{g}$-module is counted twice except the Cases I and J, in the above lists of extensions.

8.5. Concluding Remarks. We solved Problem 2 completely for the case $\mathfrak{g}$ is $\mathfrak{sl}(2, 1)$ itself or a real form of it. In the case of a real form, for each irreducible highest weight representation, there exists at least one irreducible extension when the value for the center is suitably chosen. But this phenomenon is rather special from a general point of view. In fact, when we consider a real form of $\mathfrak{sl}(n, 1)$ for $n \geqq 3$, there are few irreducible unitary representations ρ of $\mathfrak{g}_0$ which can be extended to those of $\mathfrak{g}$. For finite-dimensional ρ's, a part of them have unique extensions, and for infinite-dimensional highest weight representations ρ, they have no extensions in general.

In this way, we are naturally forced to extend the problem of irreducible unitary extensions to the case where ρ is not necessarily irreducible (Problem 2bis). Note that the adjoint representation of $\mathfrak{g}$ itself is already in such a case. Solving this generalized problem, Problem 2bis, we can classify all the irreducible unitary representation of $\mathfrak{g}$ completely. In a forthcoming paper, we give a complete results in the case of real forms of $\mathfrak{sl}(2, 1)$ (cf. [4]) and $\mathfrak{sl}(3, 1)$.

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References

[1] W. Casselman and D. Miličić, Asymptotic behavior of matrix coefficients of admissible representations, Duke Math. J., **49** (1982), 869–930.
[2] C. Fronsdal and T. Hirai, Unitary representations of supergroups, in Essays on Supersymmetry, Math. Physics Studies, Vol. 8, pp. 16–67, D. Reidel Publishing Company, 1986.
[3] H. Furutsu, On unitary representations of real forms of Lie superalgebra of type $A(1, 0)$, Master Thesis in Japanese, Kyoto University, 1985.
[4] H. Furutsu, Classification of Irreducible unitary representations of real forms of Lie superalgebra $sl(2, 1)$, in RIMS-Kôkyûroku, to appear, RIMS, Kyoto University.
[5] M. I. Graev, Unitary representations of real simple Lie groups, Trudy Moskov Matem. Ob., **7** (1958), 335–389 (in Russian) (English translation in AMS Translations, Ser. 2, Vol. 66).
[6] T. Hirai, Unitary representations of a Lie superalgebra, in Proc. 14th ICGTMP held in 1985 at Seoul, pp. 218–221, World Scientific, 1986.
[7] T. Hirai, Invariant eigendistributions of Laplace operators on real simple Lie groups, II, Japanese J. Math., **2** (1976), 27–89.
[8] V. G. Kac, Lie superalgebras, Adv. in Math., **26** (1977), 8–96.
[9] V. G. Kac, Representations of classical Lie superalgebras, in Lecture Notes in Math., Vol. 676, pp. 597–626, Springer Verlag, 1978.
[10] V. G. Kac, Characters of typical representations of classical Lie superalgebras, Comm. in Algebra, **5** (1977), 889–897.
[11] I. Kaplansky, Graded Lie algebras, I and II, Univ. of Chicago, preprint, 1975.
[12] B. Kostant, Graded manifolds, graded Lie theory, and prequantization, in Lecture Notes in Math., Vol. 570, pp. 177–306, Springer Verlag, 1977.
[13] E. Nelson, Analytic vectors, Ann. Math., **78** (1959), 572–615.

[14] S. Sternberg, Some recent results on supersymmetry, in Lecture Notes in Math., Vol. 570, pp. 145–177, Springer Verlag, 1977.
[15] M. Sugiura, Unitary representations and Harmonic analysis, Kodansha, Tokyo, 1975.
[16] M. Wakimoto, A certain series of unitarizable representations of Lie superalgebras $gl(p/q)$, preprint.

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