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On extensions of higher derivations for algebraic extensions of fields of positive characteristics

Dedicated to Professor M. Nagata on his sixtieth birthday

By

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§0. Introduction

In the present paper, we will present systematical arguments on extensions of higher derivations for algebraic field extensions of positive characteristics. Arguments on ordinary derivations are included as special cases.

Assume that a higher derivation d is given in a field K of positive characteristic. Let L be an algebraic extension of K. If d is extended to higher derivations of L, we denote one of them by d'.

In §2, we make basic considerations on relationship among constant fields and value domains, of d and d'. In §3, we seek conditions that d can be extended to higher derivations of L. In the case where d is an iterative higher derivation of finite rank we get a conclusion successfully and then we get a criteria for L to be maximal in the set of algebraic extensions of K to which d can be extended. In §4 and §5, after we discuss conditions that the extension of d is unique and conditions that the extension of d keeps the property of being iterative when d is iterative, we show that in the case where d is a higher derivation of infinite rank, there exists the largest algebraic extension of K to which d can be extended. Finally in §6, we discuss non-integrable elements. Actually, the corollary to Theorem 6-1 about this matter for ordinary derivations, has given the author a motivation to start this work. The author has tried to find a literature in which it is explicitly stated. But he has not been able to find one so far, except for that R. Baer in his paper [1] touched upon it under some restricted conditions. We conclude §6 in proving that if a higher derivation d of K is iterative and of infinite rank, then each non-integrable element of an arbitrary order in K is non-inrtegrable for every extension d' of d on an algebraic extension of K, as long as the index of d' equals the index of d. (For definitions of a non-integrable element and the index, see §1.) This may correspond to the fact that an integration of a rationally non-integrable element is transcendental, in the case of characteristic 0.

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§1. Preliminaries

Let K be a field. We denote by p the characteristic of K. Throughout this paper we assume that $p > 0$. Let $d = \{d_i\}_{0 \leqslant i < m+1}$ be a higher derivation of K in the sense of H. Hasse and F. K. Schmidt. We do not assume in general that d is iterative, unless otherwise stated. We call m the rank of d, where m may be ∞. We always assume that $d_0 = id_K$ and $d_i(K) \subset K$ for every i. The first positive integer g such that $d_g \neq 0$ is called the index of d and denoted by $I(d)$. By d being iterative we just mean that besides conditions for a higher derivation it satisfies conditions $d_i d_j = \binom{i+j}{i} d_{i+j}$ for $i + j < m + 1$. By d being a higher derivation of rank 1, we just mean that d is an ordinary derivation. In that case, we use d itself for d_1, and d is also considered even as an iterative higher derivation of rank 1. For a positive inrteger t such that $t < m + 1$, we denote by $d^{(t)}$ the induced higher derivation $\{d_i\}_{0 \leqslant i \leqslant t}$, which is called the t-th section of d. We denote by $K_{c,d}$ the subfield of constants of d. Then if $i < j$ then $K_{c,d^{(i)}} \supset K_{c,d^{(j)}}$ and $K_{c,d} = \bigcap_{0 < i < m+1} K_{c,d^{(i)}}$. The value domain $d_i(K)$ is $K_{c,d^{(i)}}$-module. An element z of K is called non-integrable of order i in K, if $z \notin d_i(K)$. If d is an ordinary derivation, a non-integrable element of order 1 is simply called non-integrable.

Hereafter, throughout this paper, we use the following notations and assumptions for the simplicity. (1) K is a field with a higher derivation d. (2) $\bar{K}$ is an algebraic closure of K and every algebraic extension of K is considered to be contained in $\bar{K}$. (3) L is an algebraic field extension of K, and d' is one of extensions of d to L if they exist, that is, d' is a higher derivation of L *whose restriction on* K is d. (4) We write as $K_i = K_{c,d^{(i)}}$ for every i, $K_c = K_{c,d}$ and $K_0 = K$, and similarly as $L_i = L_{c,d'^{(i)}}$, $L_c = L_{c,d'}$ and $L_0 = L$.

We denote by E the isomorphism of Taylor expansion of K into $K[[U]]$ or into $K[U]/(U^{m+1})$ defined by $E(a) = a + d_1(a)U + d_2(a)U^2 + \cdots$, where U is an analytically independent element.

To check whether a higher derivation over a subfield is iterative or not, it is enough to check that property on generators. This assertion may be proved in various ways but the following formulation by Miyanishi, [7], may prove it in the simplest way.

Lemma 1-1. *Let S be a subfield of K. Let d be a higher derivation over S. Put $R = S[[U]]$ if $m = \infty$ and $R = S[U]/(U^{m+1})$ if $m < \infty$. We define a ring homomorphism $\Delta\colon R \to R \mathbin{\widehat{\otimes}_S} R$* by $\Delta(U) = U \otimes 1 + 1 \otimes U$. *Then the condition that d is iterative is equivalent to the condition that the following diagram is comutative.*

$$\begin{tikzcd} K \arrow[r, "E"] \arrow[d, "E"'] & R \mathbin{\widehat{\otimes}_S} K \arrow[d, "\Delta \widehat{\otimes} id"] \\ R \mathbin{\widehat{\otimes}_S} K \arrow[r, "id \widehat{\otimes} E"] & R \mathbin{\widehat{\otimes}_S} R \mathbin{\widehat{\otimes}_S} K . \end{tikzcd}$$

If M is an overfield of K_c, $L = K \otimes_{K_c} M$ and a higher derivation d^* of M over K_c is given, then we can compose d and d^*, and get a higher derivation d' of L. It follows from the general construction by Kawahara-Yokoyama [6] or Berger [2]. If d^* is trivial, then it holds that $d'_1 = d_i \otimes M$, and we say that d' is a coefficient-field extension of d and denote $d' = d \otimes M$ (or denote $d' = d \otimes_{K_c} M$ if necessary). It is easy to see that in this case the field of constants of d' is just M.

§2. Fields of constants and value domains

In this § we always assume that d' exists on L.

Proposition 2-1. *Let i be an integer such that $0 < i < m + 1$. Assume that $g' = I(d')$ and t is a positive integer such that $i < g'p^t$. Then we have the following.*

(1) $L_i \supset L^{p^t}$.

(2) *K and L_i are linearly disjoint over K_i.*

(3) $L_i \cap K(L^{p^t}) = K_i(L^{p^t})$.

(4) *$K(L^{p^t})$ and L_i are linearly disjoint over $K_i(L^{p^t})$.*

Proof. For an element y of L. We consider Taylor expansion $E(y) = y + d_{g'}(y)U^{g'} + d_{g'+1}(y)U^{g'+1} + \cdots$. Then $E(y^{p^t}) = y^{p^t} + (d_{g'}(y))^{p^t}U^{g'p^t} + \cdots$. Hence $d'_i(y^{p^t}) = 0$, which proves (1). Assume that (2) is not true. Let $\{m_1, \ldots, m_h\}$ be the shortest sequence of elements in L_i such that they are linearly independent over K_i and linearly dependent over K. Then there exist elements $k_2, \ldots, k_h$ in K such that $m_1 + k_2m_2 + \cdots + k_hm_h = 0$. Since $d'_j(m_q) = 0$ for $q = 1, \ldots, h$, we have $d_j(k_2)m_2 + \cdots + d_j(k_h)m_h = 0$ for $j = 1, \ldots, i$. This contradicts the fact that $\{m_1, \ldots, m_h\}$ is the shortest sequence, unless all the $d_j(k_q) = 0$. However in that case $k_q \in K_i$ for $q = 2, \ldots, h$, which contradicts the linear independence of $m_1, \ldots, m_h$ over K_i. Next we prove (3). Let $a \in L_i \cap K(L^{p^t})$. Then we have an expression $a = k_1m_1 + \cdots + k_hm_h$, where $k_q \in K$ and $m_q \in L^{p^t}$ for $q = 1, \ldots, h$ and the m_q are linearly independent over K. Since $d'_j(m_q) = 0$ for $q = 1, \ldots, h$, we have $0 = d_j(k_1)m_1 + \cdots + d_j(k_h)m_h$ and hence $d_j(k_1) = \cdots = d_j(k_h) = 0$ for $j = 1, \ldots, i$. Therefore $k_q \in K_i$ for $q = 1, \ldots, h$, which means that $a \in K_i(L^{p^t})$. Finally, we prove (4). Assume that (4) is not true. Then take the shortest sequence $c'_1, \ldots, c'_f \in L_i$ which are linearly independent over $K_i(L^{p^t})$ and linearly dependent over $K(L^{p^t})$. Take a linear relation $c'_1 + g_2c'_2 + \cdots + g_fc'_f = 0$, with the $g_q \in K(L^{p^t})$. Then in the same way as above, we can prove that the $g_q \in L_i$. Hence by (3) $g_q \in K_i(L^{p^t})$, which is a contradiction ■

Lemma 2-1. *Let ∂ be an ordinary derivation of K. Let ∂' be an extension of ∂ to L. Then for every element $w \in L$ such that $w \notin K(L_{c,\partial'})$, we have $\partial'w \notin L_{c,\partial'}\partial(K)$.*

Proof. Take an element $w \in L$ such that $w \notin K(L_{c,\partial'})$. Assume that $\partial'w \in L_{c,\partial'}\partial(K)$. Then there exists an element $u \in K(L_{c,\partial'})$ such that $\partial'u = \partial'w$. Put $v = w - u$. Then $v \notin K(L_{c,\partial'})$. On the other hand, $\partial'v = 0$ means that $v \in L_{c,\partial'}$. A contradiction. ■

Lemma 2-2. *Let i be an integer such that $0 < i < m + 1$. Then we have $L_i d_i(K) \cap K = d_i(K)$.*

Proof. This follows from Proposition 2-1, (2). ∎

Proposition 2-2. *Let i be a positive integer such that $0\ i < m + 1$. Then the following three conditions are equivalent.*

(1) $L = K \otimes_{K_i} L_i$.

(2) $L_{q-1} = K_{q-1} \otimes_{K_q} L_q$ *for* $q = 1, 2, \dots, i$.

(3) $L = K \otimes_{K_q} L_q$ *for* $q = 0, 1, \dots, i$.

If these conditions are satisfied, we have the following.

(4) $d'^{(i)} = d^{(i)} \otimes_{K_i} L_i$.

(5) $d'_q(L) = L_q d_q(K)$ *for* $q = 1, 2, \dots, i$.

Proof. First we prove (2)$\Rightarrow$(3). (3) is true for $q = 0$. Assume that $L = K \otimes_{K_q} L_q$ for $q < i$. Then by (2) it holds that $L = K \otimes_{K_q}(K_q \otimes_{K_{q+1}} L_{q+1}) = K \otimes_{K_{q+1}} L_{q+1}$. (3)$\Rightarrow$(1) is trivial. We prove (1)$\Rightarrow$(2). Assume that $L_{q-1} \supsetneq K_{q-1} \otimes_{K_q} L_q$ for some q with $0 < q \leqslant i$. Then $L \supset K \otimes_{K_{q-1}} L_{q-1} \supsetneq K \otimes_{K_{q-1}}(K_{q-1} \otimes_{K_q} L_q) = K \otimes_{K_q} L_q \supset K \otimes_{K_q}(K_q \otimes_{K_i} L_i) = K \otimes_{k_i} L_i$, which contradicts (1). Since $d'^{(1)}$ is trivial on L_i, (1)$\Rightarrow$(4). The last assertion is clear. ∎

Proposition 2-3. *If d is an ordinary derivation, then we have the following.*

$$L = K \otimes_{K_c} L_c \text{ if and only if } d'(L) = L_c d(K).$$

Proof. This follows from Lemma 2-1. ∎

§3. Extensibility of higher derivations of finite rank

Lemma 3-1. *Assume that rank m of d is finite. If there exists a subfield M of L, containing K_c, such that $L = K \otimes_{K_c} M$, then d is extended to d' of L such that $I(d') = I(d) = g$. In this case, $M \supset K_c(L^{p^r})$, where r is a positive integer such that $gp^{r-1} \leqslant m < gp^r$.*

Proof. For the extensibility, we have only to consider the coefficient-field extension from K_c to M. Since for every $x \in K$ we have $x^{p^r} \in K_c$, $L^{p^r} = K^{p^r}(M^{p^r}) \subset K_c(M) = M$. ∎

Hereafter in this § we treat exclusively ordinary derivations or iterative higher derivations. We start with summarizing known facts about iterative higher derivations, which will be used in the forthcoming discussions. They mostly appear in Weisfeld [13]. For the simplicity, we use the notation $K = K_{p^{-1}}$.

1*. For a positive integer $j < m + 1$ and the p-adic expansion $j = j_0 + j_1 p + \cdots + j_q p^q$, we have $d_j = \dfrac{1}{j_0! j_1! \dots j_q!}(d_1)^{j_0}(d_p)^{j_1} \dots (d_{p^q})^{j_q}$. (Follows from the iterativity condition).

2*. Let $p^f = I(d)$. Then, $K = K_{p^{-1}} = \cdots = K_{p^f - 1} \supset K_{p^f} = K_{p^f + 1} = \cdots = K_{p^{f+1} - 1} \supset K_{p^{f+1}} = \cdots$. (Deduced from 1*.)

3*. $K_{p^{i-1}}$ is purely inseparable over K_{p^i} for $p^i < m + 1$. (General property for higher derivation.)

4*. If $m \geqslant p^{r-1}$, then $[K_{p^{i-1}} : K_{p^i}] = p$ for $i = f, f + 1, \ldots, r - 1$. (Follows from Wronskian argument.)

5*. For every choice of $x \in K$ such that $x \notin K_{p^f}$, it holds that $K_{p^{i-1}} = K_{p^i}(x^{p^{i-f}})$ for i such that $i \geqslant f$ and $p^{i+1} < m + 1$. (Follows from 4*.)

In 6* and 7*, we assume that $m < \infty$ and $p^{r-1} \leqslant m < p^r$.

6*. Fix an element x as in 5*. We can take as a p-independent base of $K_{p^{r-2}}$ over K_c a set of elements of the form $\{x^{p^{r-f-1}}\} + A$, where $+$ denotes the set theoretical sum. (Follows from the fact that $d_{p^{r-1}}|_{K_{p^{r-2}}}$ is an ordinary derivation.)

7*. With elements in 6*, $K = K_c(x) \otimes_{K_c} (\bigotimes_{K_c, a \in A} K_c(a))$. (Deduced from 1* $\sim$ 6*.)

Lemma 3-2. *Let d be an iterative higher derivation of finite rank m such that $p^{r-1} \leqslant m < p^r$. If i is an integer such that $-1 \leqslant i \leqslant r - 1$, then $K_{p^i} = \{y \in K \mid y^{p^{r-i-1}} \in K_c\}$.*

Proof. We consider the Taylor expansion $E: K \to K[[U]]/(U^{m+1})$. If $y \in K_{p^i}$, then $E(y) = y + (d_{p^{i+1}}(y))U^{p^{r+1}} + \cdots$. Since $p^{i+1}p^{r-i-1} = p^r > m$, $E(y^{p^{r-i-1}}) = E(y)^{p^{r-i-1}} = y^{p^{r-i-1}}$. Hence $y^{p^{r-i-1}} \in K_c$. Conversely, assume that $y \in K$ and $y^{p^{r-i-1}} \in K_c$. Then $y^{p^{r-i-1}} = E(y^{p^{r-i-1}}) = (E(y))^{p^{r-i-1}} = y^{p^{r-i-1}} + (d_1(y))^{p^{r-i-1}}U^{p^{r-i-1}} + \cdots + (d_j(y))^{p^{r-i-1}}U^{jp^{r-i-1}} + \cdots$. Hence if $jp^{r-i-1} \leqslant m$, $d_j(y) = 0$. On the other hand, if $j \leqslant p^i$, $jp^{r-i-1} \leqslant p^{r-1} \leqslant m$. Hence $y \in K_{p^i}$. ∎

Proposition 3-1. *Assume that d is iterative and $m < \infty$. Then d can be extended to d' of L such that d' is iterative and $I(d') = I(d) = p^f$, if and only if there exists a subfield M of L, containing K_c such that $L = K \otimes_{K_c} M$. We can choose M so that $M \supset L_c$. M always contains $K_c(L^{p^r})$, where r is a positive integer such that $m < p^{r+f}$.*

Proof. We have only to prove the only if part and the last two assertions. We have a commutative diagram:

$$\begin{array}{ccccc} K & \longrightarrow & K(L_c) & \longrightarrow & L \\ \uparrow & & \uparrow & & \\ K_c & \longrightarrow & L_c & & \end{array},$$

where K and L_c are linearly disjoint over K_c. We take x and A as in 5* and 6*. $x^{p^{r-f-1}}$ and A are p-independent over L_c. Therefore, we can choose a p-independent basis for $L_{p^{r-2}}$, of the form $\{x^{p^{r-f-1}}\} + A + B$. If $B = \varnothing$, $L = L_c(x) \otimes_{L_c} \left(\bigotimes_{L_c, a \in A} L_c(a) \right) = K \otimes_{K_c} L_c$. Hence we can put $M = L_c$. Otherwise, $L = L_c(x) \otimes_{L_c} \left(\bigotimes_{L_c, a \in A} L_c(a) \right) \otimes_{L_c} \left(\bigotimes_{L_c, b \in B} L_c(b) \right) = (K \otimes_{K_c} L_c) \otimes_{L_c} \left(\bigotimes_{L_c, b \in B} L_c(b) \right) = K \otimes_{K_c} \left(\bigotimes_{L_c, b \in B} L_c(b) \right)$. Therefore we can put $M = \bigotimes_{L_c, b \in B} L_c(b)$. Since $K^{p^r} \subset K_c$, $M \supset K_c(K^{p^r}(M^{p^r})) = K_c(L^{p^r})$ for every M. ∎

Lemma 3-3. *Assume that T is a field of characteristic $p > 0$ and Q is an infinite subfield of T. Let y be an element of an algebraic extension of T. Let y_1 be an element such that $y_1 \in T(y)$ and $y_1 \notin T(y^p)$. Then there exists an element $c \in Q$ such that if we put $y_2 = y_1 - cy^p$ then it holds that $T(y) = T(y_2)$.*

Proof. Let e be the inseparability exponent of y over T. Let $h=[T(y^{p^e}):T]$. Then there exists h different isomorphisms $\sigma_1, \ldots, \sigma_h$ of $T(y^{p^e})$ into its algebraic closure over T. Then $\sigma_i(y^{p^e}) \neq \sigma_j(y^{p^e})$ for $i \neq j$. Each σ_i is uniquely extended to an isomorphism of $T(y)$. We denote it by the same symbol for each i. Then $\sigma_i(y^p) \neq \sigma_j(y^p)$ for $i \neq j$. Since Q is infinite, there exists an element $c \in Q$ such that $\sigma_i(y_1) - \sigma_j(y_1) \neq c(\sigma_i(y^p) - \sigma_j(y^p))$ for $i \neq j$. We put $y_2 = y_1 - cy^p$. Then $\sigma_1(y_2), \ldots, \sigma_h(y_2)$ are all different. Hence $\sigma_1(y_2^{p^e}), \ldots, \sigma_h(y_2^{p^e})$ are all different. Since $y_2^{p^e} \in T(y^{p^e})$, it holds that $T(y^{p^e}) = T(y_2^{p^e})$. We prove the assertion that $T(y^{p^r}) = T(y_2^{p^r})$ for $r = e, \ldots, 0$, using induction. The assertion is true for e. Assume that it is true for an $r > 0$. Since $T(y) = T(y^p, y_2)$, we have $T(y^{p^{r-1}}) = T(y^{p^r}, y_2^{p^{r-1}})$. Since $y^{p^r} \in T(y_2^{p^r}) \subset T(y_2^{p^{r-1}})$, it holds that $T(y^{p^{r-1}}) = T(y_2^{p^{r-1}})$. ∎

Proposition 3-2. *Assume that d is an ordinary derivation.*

(1) *d can be extended to L if and only if there exists a subfield M of L, containing K_c, such that $L = K \otimes_{K_c} M$.*

(2) *Assume that L is a simple extension of K. Then, for every choice of M in (1), M is a simple extension of K_c.*

Proof. (1) is a special case of Proposition 3-1. We prove (2). Assume that $L = K(y)$. We have a commutative diagram:

$$\begin{array}{ccccc} K & \longrightarrow & K(y^p) & \longrightarrow & K(y) \\ \uparrow & & \uparrow & & \uparrow \\ K_c & \longrightarrow & K_c(y^p) & \longrightarrow & K_c(y) \end{array} \quad ,$$

where $K_c(y^p) = K_c(L^p)$ because $K_c \supset K^p$. In case $K(y) = K(y^p)$, that is, in case y is separable over K, $M = K_c(y^p)$ which is a simple extension of K_c. Assume that $K(y) \supsetneqq K(y^p)$. Then for every choice of M, $M \supsetneqq K_c(y^p)$, because otherwise $L = K(M) = K(y^p)$, contradicting our assumption. In this case $[M : K_c(y^p)] = [K(y): K(y^p)] = p$. We take an element $y_1 \in M$ such that $y_1 \notin K_c(y^p)$. Then, we have $M = K_c(y^p, y_1)$ and $y_1 \notin K(y^p)$ because $K(y^p)$ and M are linearly disjoint over $K_c(y^p)$. Here, we apply Lemma 3-3. We take K for T and K_c for Q in that lemma. Then there is an element $y_2 \in L$ such that $y_2 = y_1 - cy^p$, with $c \in K_c$ such that $K(y) = K(y_2)$. Then $K_c(y^p) = K_c(y_2^p)$, because both are $L_c \cap K(L^p)$ by Proposition 2-1, (3). Since $[M : K_c(y_2^p)] = p$ and $y_2 \notin K_c(y_2^p)$, $M = K_c(y_2^p, y_2) = K_c(y_2)$, which is a simple extension of K_c. ∎

By Zorn's Lemma, it is easy to see that there exist maximal subfields L of $\bar{K}$, containing K, to which d is extended, keeping the index. If d is iterative, then we also see the existence of maximal fields L, where extended derivations keep the iterativity conditions. Later in §5 we will show even the existence of the largest extension, in the case $m = \infty$. However, generally speaking, it is not easy to describe, in a concrete way, the condition that L is maximal. But, owing to Proposition 3-1, if d is iterative and of finite rank, we have the following characterization.

Theorem 3-1. *Assume that d is an iterative higher derivation with rank $m < \infty$. Let L be a subfield of $\bar{K}$, containing K, to which d can be extended, keeping the iterativity condition and the index. Let r be an integer such that $p^{r-1} \leqslant m < p^r$. Then the following three conditions are equivalent.*

(1) *L is maximal in the set of algebraic extensions of K, to which d can be extended, keeping the iterativity condition and the index.*

(2) *There is no nontrivial field extension M of L_c in $\bar{K}$ such that L and M are linearly disjoint over L_c.*

(3) *L contains the separably algebraic closure K_s of K in $\bar{K}$ and $L_{p^r-2} = L_c^{p^{-1}}$.*

Proof. (1)$\Leftrightarrow$(2) follows from Proposition 3-1. Assume that (1) and (2) hold. And assume that $L_{p^r-2} \subsetneq L_c^{p^{-1}}$. Then take $y \in L_c^{p^{-1}}$ such that $y \notin L_{p^r-2}$. Since $L_{p^r-2} = L_c^{p^{-1}} \cap L$ by Lemma 3-2, $y \notin L$. Hence L and $L_c(y)$ are linearly disjoint over L_c, contradicting (2). Hence $L_{p^r-2} = L_c^{p^{-1}}$. Assume that T is a separable extension of L. Then L and T^p are linearly disjoint over L^p. Hence $L^{p^{q-1}}$ and T^{p^q} are linearly disjoint over L^{p^q} for $q = 1, 2, \ldots, r$. Hence L and T^{p^r} are linearly disjoint over L^{p^r}. Since $L_c \supset L^{p^r}$, L and $L_c(T^{p^r})$ are linearly disjoint over L_c. Hence by (2), $L_c = L_c(T^{p^r})$. Hence $L = L(T^{p^r}) = T$. Therefore, L contains K_s. Conversely, assume that (3) holds. If L is not maximal, then there exists a nontrivial overfield M of L_c such that L and M are linearly disjoint over L_c. Then by our assumption every element in M is purely inseparable over L, hence over L_c. Therefore, there exists an element z of M such that $z \notin L_c$ and $z^p \in L_c$, contradicting our assumption. ∎

Corollary 1. *Assume that d is an iterative higher derivation with rank $m < \infty$ and $d_1 \neq 0$. Let L be a subfield of $\bar{K}$, containing K, to which d can be extended, keeping the iterativity condition. Let r be an integer such that $p^{r-1} \leqslant m < p^r$. Then the following three conditions are equivalent.*

(1) *L is maximal in the set of algebraic extensions of K, to which d can be extended, keeping the iterativity condition.*

(2) *There is no nontrivial field-extension of M of L_c in $\bar{K}$ such that L and M are linearly disjoint over L_c.*

(3) *L contains the separably algebraic closure K_s of K in $\bar{K}$ and $L_{p^r-2} = L_c^{p^{-1}}$.*

Proof. Since $d_1 \neq 0$, $d_1' \neq 0$ for every extension d' of d. Hence our assertion follows from the theorem. ∎

Corollary 2. *Assume that d is an ordinary derivation. Let L be a subifield of $\bar{K}$, containing K, to which d can be extended. Then the following three conditions are equivalent.*

(1) *L is maximal in the set of algebraic extensions of K, to which d can be extended.*

(2) *There is no nontrivial field-extension M of L_c in $\bar{K}$ such that L and M are linearly disjoint over L_c.*

(3) *L contains the separably algebraic closure K_s of K in $\bar{K}$ and $L = L_c^{p^{-1}}$.*

§4. Uniqueness of extensions

Lemma 4-1. *Assume that there is a sequence $\{G_i\}_{0<i<m+1}$ of subfields of L such that:*

(1) *the following diagram is commutative:*

$$\begin{tikzcd}
L & G_1 \arrow[l] & G_2 \arrow[l] & \cdots \arrow[l] & G_i \arrow[l] & \cdots \arrow[l] \\
K \arrow[u] & K_1 \arrow[l] \arrow[u] & K_2 \arrow[l] \arrow[u] & \cdots \arrow[l] & K_i \arrow[l] \arrow[u] & \cdots, \arrow[l]
\end{tikzcd}$$

where each arrow represents an inclusion map, and

(2) *for every i with $0 < i < m + 1$, $L = K \otimes_{K_i} G_i$.*

Then $G_i = K_i \otimes_{K_{i+1}} G_{i+1}$ for i with $0 < i < m$ and if we put $d^{*\{i\}} = d^{(i)} \otimes_{K_i} G_i$ for each i, then there exists an extension d^* of d on L such that $d^{*(i)} = d^{*\{i\}}$. In this case if d is iterative then d^* is iterative.

Proof. If $G_i \supsetneqq K_i \otimes_{K_{i+1}} G_{i+1}$, then by the linear disjointness $L = K \otimes_{K_i} G_i \supsetneqq K \otimes_{K_i} (K_i \otimes_{K_{i+1}} G_{i+1}) = K \otimes_{K_{i+1}} G_{i+1}$, contradicting our assumption. Next, that $d^{*\{i\}} = d^{(i)} \otimes_{K_i} G_i$ implies that $d_j^{*\{i\}} = d_j \otimes_{K_i} G_i$ for each j with $0 < j \leqslant i$. On the other hand, d_j is a 0-map on K_j. Hence $d_j \otimes_{K_i} G_i = d_j \otimes_{K_j} (K_j \otimes_{K_i} G_i) = d_j \otimes_{K_j} G_j$. Hence $d_j^{*\{i\}}$ is independent of the choice of i with $i \geqslant j$. Hence we can define d^* by putting $d_j^* = d_j^{*\{i\}}$ with $i \geqslant j$. The last assertion is clear. ∎

Lemma 4-2. *For every integer i such that $0 < i < m + 1$, we put $G_i = K_i(L^{p^{t(i)}})$, where $t(i)$ is an integer with $p^{t(i)-1} \leqslant i < p^{t(i)}$, and we assume that $L = K \otimes_{K_i} G_i$. Then the following assertions hold.*

(1) *d' exists uniquely and $I(d') = I(d)$.*

(2) *If there exists an overfield L^* of L such that d is extended to a higher derivation d^* of L^*, then L is closed under d^*, that is, $d_i^*(L) \subset L$ for each i.*

(3) *If d is iterative, then d' is iterative.*

Proof. The existence of d' is due to Lemma 4-1. Now for every extension d', we have $L_i = K_i(L^{p^{t(i)}})$ by Proposition 2-1. Hence for each i $d'^{(i)} = d^{(i)} \otimes_{K_i} L_i$, which is uniquely determined. Hence d' is unique. Since by Proposition 2-1 $L^*_{c,d^{*(i)}} \supset L^{*p^{t(i)}} \supset L^{p^{t(i)}}$, d_i^* is trivial on $L^{p^{t(i)}}$. Since $L = K(L^{p^{t(i)}})$, $d_i^*(L) \subset (d_i(K))L^{p^{t(i)}} \subset L$. Other assertions are clear. ∎

Proposition 4-1. *If $L = K(L^p)$ and there exists an algebraic overfield L^* of L*

such that d is extended to d^* *of* L^*, *then* d^* *has a restriction* d' *on L and* d' *is a unique extension of d to L and* $I(d') = I(d)$. *If d is iterative then* d' *is iterative.*

Proof. By Proposition 2-1 $L^*_{c,d^{*(i)}} \supset L^{*p^t} \supset L^{p^t}$ with $p^{t-1} \leqslant i < p^t$. Since K and $L^*_{c,d^{*(i)}}$ are linearly disjoint over K_i, K and $K_i(L^{p^t})$ are linearly disjoint over K_i. Since $L = K(L^p) = \cdots = K(L^{p^t})$, $L = K(K_i(L^{p^t})) = K \otimes_{K_i} K_i(L^{P^t})$. Hence by Lemma 4-2, we get our assertions. ■

Proposition 4-2. *Assume that L is a separable extension of K. Then* d' *exists uniquely,* $I(d') = I(d)$ *and every extension of d to an overfield of L has the restriction* d' *on L. If d is iterative then* d' *is iterative.*

Proof. Since L is separable over K, in the same way as in the proof of Theorem 3-1, we see that K and $K_i(L^{p^t})$ are linearly disjoint over K_i, where $p^{t-1} \leqslant i < p^t$. Hence $L = K(L^{p^t}) = K \otimes_{K_i} K_i(L^{p^t})$. Hence by Lemma 4-2, we get our assertions. ■

Theorem 4-1. *Assume that L has a finite exponent over K, that is, the exponents of orders of inseparability of elements in L have an upper bound. If d is an iterative higher derivation of finite rank, then the following conditions are equivalent.*

(1) *L is separable over K.*

(2) *d extends uniquely to an iterative higher derivation* d' *which satisfies necessarily* $I(d) = I(d')$.

Proof. (1)$\Rightarrow$(2) is in Proposition 4-2. We prove (2)$\Rightarrow$(1). Assume that L is not separable over K. Then take separably algebraic closure K' of K in L. Then by Proposition 4-2 $d_i'(K') \subset K'$. Therefore, we may assume that $K = K'$ and L is purely inseparable over K. Since d' exists, there exists an overfield M of K_c such that $L = K \otimes_{K_c} M$. Then M is purely inseparable over K_c. Put $M' = K_c(M^p)$. Then $M' \neq M$; for otherwise $M = K_c(M^p)$ and hence, $L = K(L^p) = K(L^{p^2}) = \cdots$, which means that $L = K$, because L has a finite exponent over K, contradicting our assumption. We consider the coefficient-field extension of d by M' and we may assume that $K = K(M')$. Then $M^p \subset K_c$. Let $\{z_j\}_{j \in A}$, be a p-independent base of M over K_c. Let $\{b_j\}_{j \in A}$ be an arbitrary set of elements of K_c. Put $d_i^*(z_j) = 0$ for $i \neq p^{r-1}$ and $d^*_{p^{r-1}}(z_j) = b_j$ for $j \in A$, where $p^{r-1} \leqslant m < p^r$. This induces an iterative higher derivation $d^* = \{d_i^*\}_{0 \leqslant i \leqslant m}$ of M. Take the composite d' of d and d^*. It is easy to see that d' is iterative, if we check that d' is iterative on generators. $I(d') = I(d)$ is clear. Hence d' is not unique. ■

Corollary. *Assume that L has finite exponent over K. If d is an ordinary derivation of K, then the following conditions are equivalent.*

(1) *L is separable over K.*

(2) *d extends uniquely to an ordinary derivation* d' *on L. In this case,* $L_t = K_c(L^p)$.

§5. Higher derivations of infinite rank

We first remark that for an iterative higher derivation of infinite rank, the condition that $[K_{p^{s-1}} : K_{p^s}] = p$ for every integer $s \geqslant f$ with $p^f = I(d)$, is kept because $d^{(k)}$ is iterative for every positive integer k.

We start with a typical feature of iterative higher derivations of infinite rank.

Proposition 5-1. *Let d be an iterative higher derivation of infinite rank. Assume that d is extended to an iterative higher derivation d' of L, keeping the condition that $I(d') = I(d)$. Then for every positive integer t, $d'^{(t)} = d^{(t)} \otimes_{K_t} L_t$ and especially, $L = K \otimes_{K_t} L_t$.*

Proof. We put $p^f = I(d)$. In the commutative diagram:

$$\begin{tikzcd}
L = L_{p^{-1}} = \cdots = L_{p^{f-1}} & L_{p^f} \arrow[l] & L_{p^{f+1}} \arrow[l] & \cdots L_{p^s} \arrow[l] & \cdots \arrow[l] \\
K = K_{p^{-1}} = \cdots = K_{p^{f-1}} \arrow[u] & K_{p^f} \arrow[l] \arrow[u] & K_{p^{f+1}} \arrow[l] \arrow[u] & \cdots K_{p^s} \arrow[l] \arrow[u] & \cdots, \arrow[l]
\end{tikzcd}$$

$[L_{p^{s-1}} : L_{p^s}] = [K_{p^{s-1}} : K_{p^s}] = p$ and $K_{p^{s-1}}$ and L_{p^s} are linearly disjoint over K_{p^s} for $s \geqslant f$. *Hence* $L_{p^{s-1}} = K_{p^{s-1}} \otimes_{K_{p^s}} L_{p^s}$ for $s \geqslant 0$ and $L = K \otimes_{K_{p^s}} L_{p^s}$. Hence it follows from Proposition 2-2 that $d'^{(t)} = d^{(t)} \otimes_{K_t} L_t$ and $L = K \otimes_{K_t} L_t$ for every positive integer t. ∎

Corollary. *Let d be an interative higher derivation of infinite rank. Assume that d is extended to an iterative higher derivation d' of L and $d'_1 \neq 0$. Then for every positive integer t, $d'^{(t)} = d^{(t)} \otimes_{K_t} L_t$.*

Lemma 5-1. *Assume that L is purely inseparable over K. Let d be a higher derivation of infinite rank. Then if d' exists, d' is unique.*

Proof. Let $x \in L$. Then there exists a positive integer n such that $x^{p^n} \in K$. Consider Taylor expansions $E: K \to K[[U]] \subset L[[U]]$ and $E': L \to L[[U]]$. Then E' is an extension of E. $E'(x)$ is a p^n-th root of $E(x^{p^n}) = E'(x^{p^n})$. Hence $E'(x)$ is unique, because in $L[[U]]$ a p^n-th root of an element is unique if it exists. ∎

Proposition 5-2. *Let d be a higher derivation of infinite rank. Assume that d' exists. Then d' is unique. If d is iterative, then d' is iterative.*

Proof. We denote by K' the separably algebraic closure of K in L. Then by Proposition 4-2, K' is closed under d' and the restriction of d' on K' is iterative if d is iterative. Replacing K by K', we may assume that L is purely inseparable over K. Then by Lemma 5-1 d' is unique. Assume that d is iterative. Then apply Lemma 1-1, where L stands for K, the Taylor expansion E' of L stands for E and K_c stands for S. By the iterativity of d, $(\Delta \hat{\otimes} id) \circ E'$ and $(id \hat{\otimes} E') \circ E'$ coincides on K. Since $R \hat{\otimes}_{K_c} R \hat{\otimes}_{K_c} L$ is a power series ring of two variables over L, then two isomorphisms have to coincide on L by the same reason as in the proof of Lemma 5-1. Hence d' is iterative. ∎

We denote by K_s the separable closure of K in $\bar{K}$. As was stated in §3, there exist maximal subfields L of $\bar{K}$, containing K, to which d is extensible. By Proposition 4-2, each L has to contain K_s. However if d is of infinite rank, we can prove the existence of the largest subfield of $\bar{K}$ with that property. This is an extenion of Shikishima's result for iterative higher derivations in [9].

To prove it, we start with a definition.

Definition. Let d be a higher derivation of infinite rank. For each element w of K and for each natural number e, we define a subset $A_{w,e}^{K,d}$ of K in the following way. Let a be an element of K. Then $a \in A_{w,e}^{K,d}$ if and only if there exist a sequence of elements $a_0, a_1, \dots, a_t$ of K, where $a_0 = w$ and $a_t = a$, and a sequence of natural numbers $i_1, i_2, \dots, i_t$ such that $a_j = d_{i_j p^e}(a_{j-1})$ for $j = 1, 2, \dots, t$.

Proposition 5-3. *Let d be a higher derivation of infinite rank. For $x \in \bar{K}$ such that $x^{p^e} = w \in K$ there exists an overfield L of K contained in $\bar{K}$ such that $x \in L$ and d is extended to L, if and only if for every $y \in A_{w,e}^{K,d}$ it holds that $d_j(y) = 0$ for $j \not\equiv 0$ (p^e). If these conditions are satisfied the least L with above properties is $K((A_{w,e}^{K,d})^{p^{-e}})$.*

Proof. First we prove the only if part. If $y \in K$ is such that $y = z^{p^e}$ with $z \in L$, then for every natural number i we have $d_{ip^e}(y) = (d'_i(z))^{p^e}$. Hence if we put $u = d_{ip^e}(y)$ and $v = d'_i(z)$, then $u^{p^{-e}} = v \in L$. Therefore $d_j(u) = d'_j(v^{p^e}) = 0$ for $j \not\equiv 0$ (p^e). This fact shows our requirement for $A_{w,e}^{K,d}$ and also shows that $L \supset (A_{w,e}^{K,d})^{p^{-e}}$. Conversely, assume that the requirement for $A_{w,e}^{K,d}$ is satisfied. Put $L = K((A_{w,e}^{K,d})^{p^{-e}})$. Then $L^{p^e} = K^{p^e}(A_{w,e}^{K,d})$. It is easily seen that for $a, b \in K$ such that $d_j(a) = d_j(b) = 0$ for $j \not\equiv 0$ (p^e), we have $d_j(ab) = 0$ for $j \not\equiv 0$ (p^e). Therefore, for every element c in L^{p^e}, we have $d_j(c) = 0$ for $j \not\equiv 0$ (p^e). Moreover from this fact and by the definition of $A_{w,e}^{K,d}$, it is easily seen that L^{p^e} is closed under d and actually only the d_{jp^e} work on L^{p^e} non-trivially. Therefore we can define a higher derivation $d^* = \{d_j^*\}_{j=0,1,\dots}$ of L^{p^e}, putting $d_j^* = d_{jp^e}$ for $j = 0, 1, 2, \dots$. Let $\Psi\colon L^{p^e} \to L$ be an isomorphism such that $\Psi(a) = a^{p^{-e}}$ for $a \in L^{p^e}$. Then d^* induces a higher derivation d' on L through Ψ. Then $\Psi \circ d^* = d' \circ \Psi$. Take an element $a \in K$. Then $d'_i(a) = d'_i(\Psi(a^{p^e})) = \Psi(d_i^*(a^{p^e}) = \Psi(d_{ip^e}(a^{p^e})) = \Psi((d_i(a))^{p^e}) = d_i(a)$. Hence d' is an extension of d. ■

Theorem 5-1. *If d is a higher derivation of infinite rank, then there is the largest subfield L of $\bar{K}$, containing K and to which d is extended.*

Proof. We have only to prove a maximal subfield with the above property is the largest. Take such a maximal subfield L. Since L contains K_s, we may assume that $K = K_s$. Assume that there exists a maximal subfield L_1 with the above property, different from L. Then there exists an element $x \in L_1$ such that $x \notin L$. Then there exists a positive integer e such that $w = x^{p^e} \in K$. Then $A_{w,e}^{K,d}$ satisfies the condition in Proposition 5-3. However, since $L \supset K$ and d is the restriction of d' on K, $A_{w,e}^{K,d} = A_{w,e}^{L,d'}$ and the requirement to it with respect to d' is the same as that with respect to d. Hence d' on L can be extended to a subfield of $\bar{K}$, containing x keeping the said property, which contradicts the maximality of L. ■

§6. Non-integrable elements and order of inseparability

Theorem 6-1. *Let d be a higher derivation. Let i be a positive integer such that $i < m+1$. Let z be an element in K such that $z \notin d_i(K)$. Assume that there exists an element y in L such that $d'_i(y) = z$. Then we have the following.*

(1) *y is inseparable over K.*

Assume that m is large enough compared with i. Then,

(2) *there exists a positive integer t such that $z^{p^t} \in d_{ip^t}(K)$*

and

(3) *The exponent of the order of inseparability of y over K is greater than or equals the least t satisfying* (2).

Proof. Let q be the exponent of the order of inseparability of y over K. Let $g' = I(d')$. Assume that $i < g'p^s$. Since y^{p^q} is separable over K, we have $y^{p^q} \in K(y^{p^{q+s}}) \subset K(L^{p^{s+q}})$, where $s \geqslant 1$. Since $ip^q < g'p^{s+q}$, $L^{p^{q+s}} \subset L_{ip^q}$ by Proposition 2-1, if $ip^q < m+1$. Then $z^{p^q} = (d'_i(y))^{p^q} = d'_{ip^q}(y^{p^q}) \in d'_{ip^q}(K(L^{p^{q+s}})) \subset d'_{ip^q}(K(L_{ip^q})) = L_{ip^q}d_{ip^q}(K)$. Hence $z^{p^q} \in K \cap L_{ip^q}d_{ip^q}(K) = d_{ip^q}(K)$ by Lemma 2-2. Hence (2) holds. By the choice of q, (3) is also true. If y is separable over K then $q = 0$, $z \in d_i(K)$, contradicting our assumption. Hence (1) holds. ■

Corollary 1. *Let d be an ordinary derivation which is extended to d' on L. If $z \in K$ is a non-integrable element and $d'(y) = z$ for $y \in L$, then y is inseparable over K.*

Corollary 2. *Assume that L is a simple extension of K, generated by y. If d is an ordinary derivation and $z \in K$ is a non-integrable element, then the following conditions are equivalent.*

(1) *d is extended to d' on L such that $d'(y) = z$.*

(2) *y is inseparable over K, and K and $K_c(y)$ are linearly disjount over K_c.*

If these conditions are satisfied, we have $L \supsetneqq K \otimes_{K_c} L_c$.

Proof. Assume that (1) is true. By Proposition 2-1, (2), $L \supset K(L_c) = K \otimes_{K_c} L_c$ and $d'(K(L_c)) \cap K = L_c d(K) \cap K = d(K)$. Since $z \notin d(K)$ and $d'y = z$, we have $y \notin K(L_c)$. On the other hand, it holds that $y^p \in L_c$. Hence $L_c(y)$ and $K(L_c)$ are linearly disjoint over L_c. Hence $L_c(y)$ and K are linearly disjoint over K_c. Especially K and $K_c(y)$ are linearly disjoint over K_c. Hence (1) $\Rightarrow$ (2). Assume that (2) is true. Since y is inseparable over K_c, the module of differential $\Omega = \Omega_{K_c(y)/K_c}$ is a free $K_c(y)$-module generated by $\bar{d}(y)$, where $\bar{d}$ denotes the canonical derivation of $K_c(y)$ to Ω. Hence there exists a derivation d^* of $K_c(y)$ over K_c such that $d^*(y) = z$. Since $L = K \otimes_{K_c} K_c(y)$, we can compose d and d^*, and prove (2) $\Rightarrow$ (1). Assume that (1) and (2) are true. Since $z \in d'(L)$ and $z \notin L_c d(K)$ by Lemma 2-2, $d'(L) \supsetneqq L_c d(K)$. Hence $L \supsetneqq K \otimes_{K_c} L_c$ by Proposition 2-3. ■

Theorem 6-2. *Assume that d is an iterative higher derivation of infinite rank which is extended to d' of L. If $I(d')=I(d)$, then every non-integrable element of an arbitrary order in K is non-integrable in L of the same order.*

Proof. This is a result of Proposition 5-1. Since for every $i>0$ we have $L=K\otimes_{K_i}L_i$, it holds that $d_i'(L)=L_id_i(K)$. Hence by Lemma 2-2, we have $d_i(K)=L_id_i(K)\cap K=d_i'(L)\cap K$. ∎

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